Linear Regression

Major: All Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

Linear Regression

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What is Regression?

What is regression? Given n data points $(x_1, y_1), (x_2, y_2), \ldots\ldots, (x_n, y_n)$ best fit $y = f(x)$ to the data.

Residual at each point is $E_i = y_i - f(x_i)$

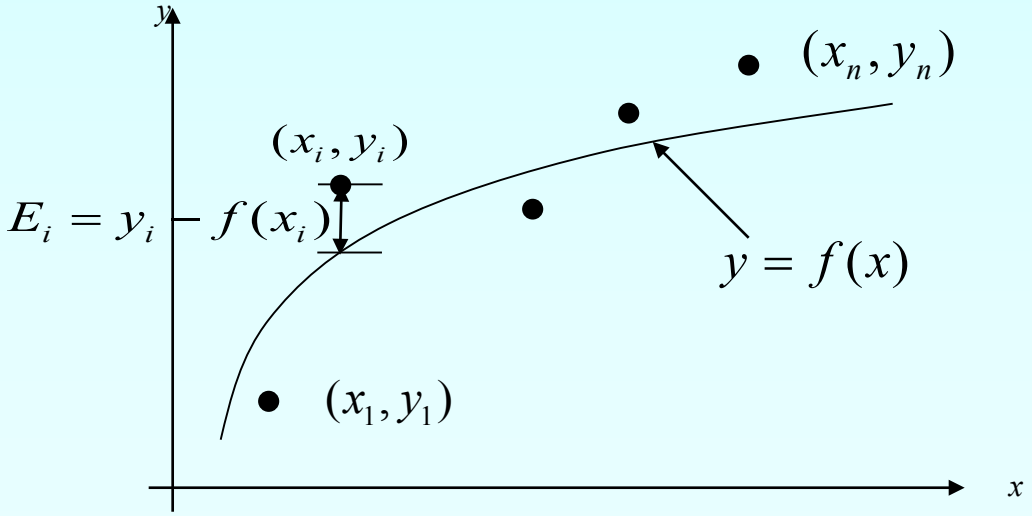


Figure. Basic model for regression

Linear Regression-Criterion#1

Given n data points $(x_1, y_1), (x_2, y_2), \ldots\ldots, (x_n, y_n)$ best fit $y = a_0 + a_1 x$ to the data.

Does minimizing $\sum_{i=1}^{n} E_i$ work as a criterion?

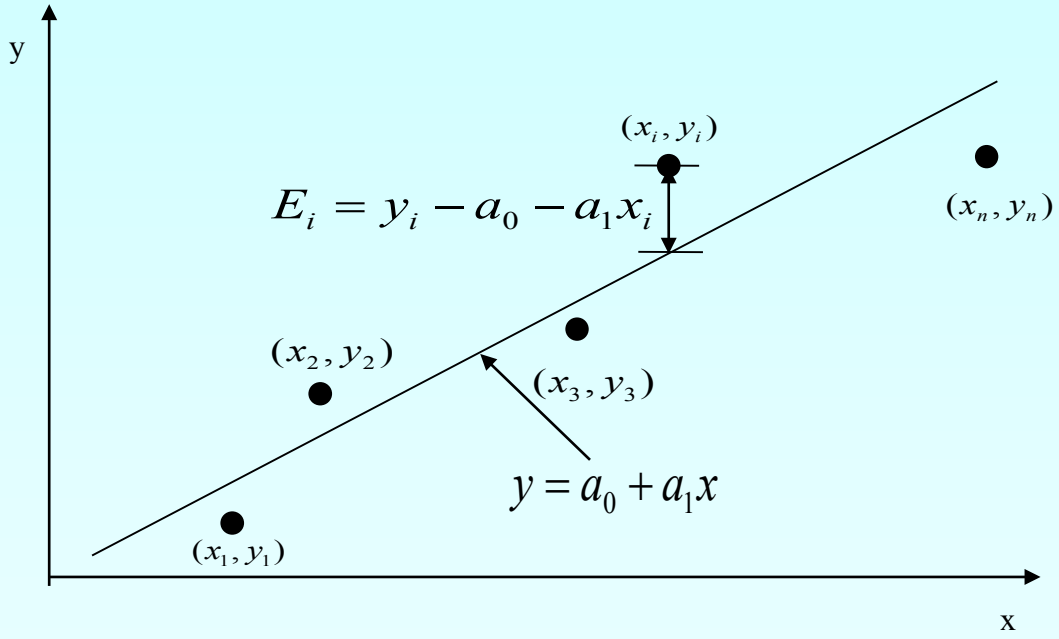


Figure. Linear regression of y vs x data showing residuals at a typical point, x_i.

Example for Criterion#1

Example: Given the data points (2,4), (3,6), (2,6) and (3,8), best fit the data to a straight line using Criterion#1

Minimize $\sum_{i=1}^{n} E_i$

Table. Data Points

x	y
2.0	4.0
3.0	6.0
2.0	6.0
3.0	8.0

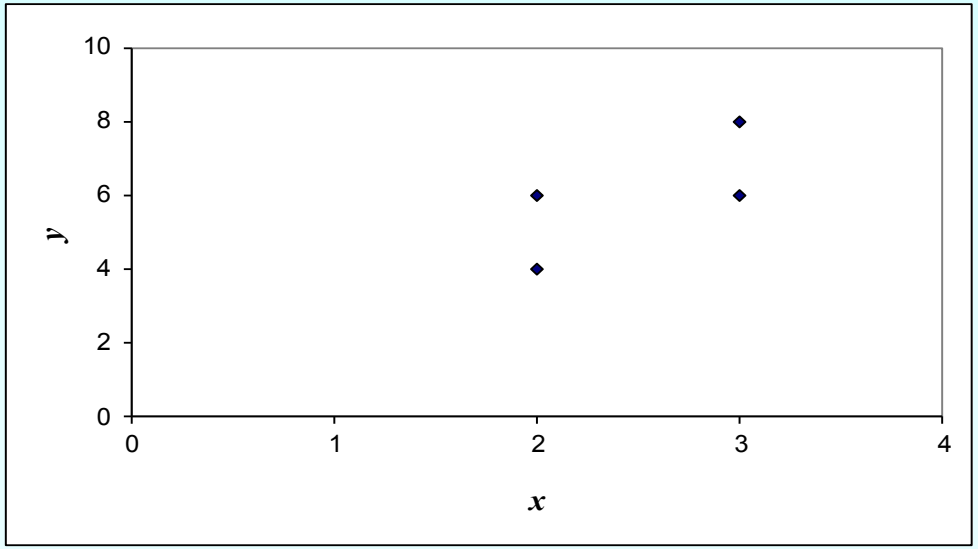


Figure. Data points for y vs x data.

Linear Regression-Criteria#1

Using $y=4x-4$ as the regression curve

Table. Residuals at each point for regression model $y=4x-4$

x	y	$y_{predicted}$	$E = y - y_{predicted}$
2.0	4.0	4.0	0.0
3.0	6.0	8.0	-2.0
2.0	6.0	4.0	2.0
3.0	8.0	8.0	0.0
			$\sum_{i=1}^{4} E_i = 0$

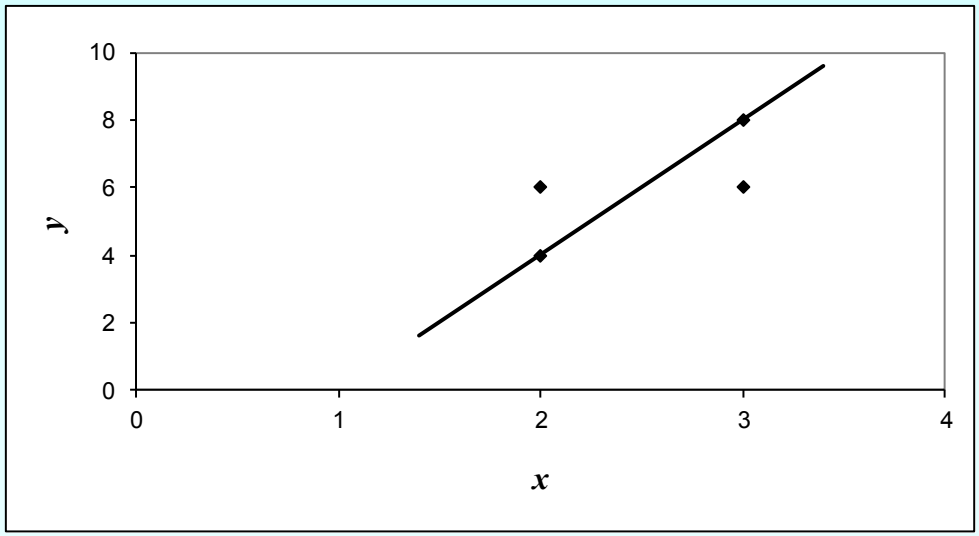


Figure. Regression curve $y=4x-4$ and y vs x data

Linear Regression-Criterion#1

Using y=6 as a regression curve

Table. Residuals at each point for regression model y=6

x	y	$y_{predicted}$	$E = y - y_{predicted}$
2.0	4.0	6.0	-2.0
3.0	6.0	6.0	0.0
2.0	6.0	6.0	0.0
3.0	8.0	6.0	2.0
			$\sum_{i=1}^{4} E_i = 0$

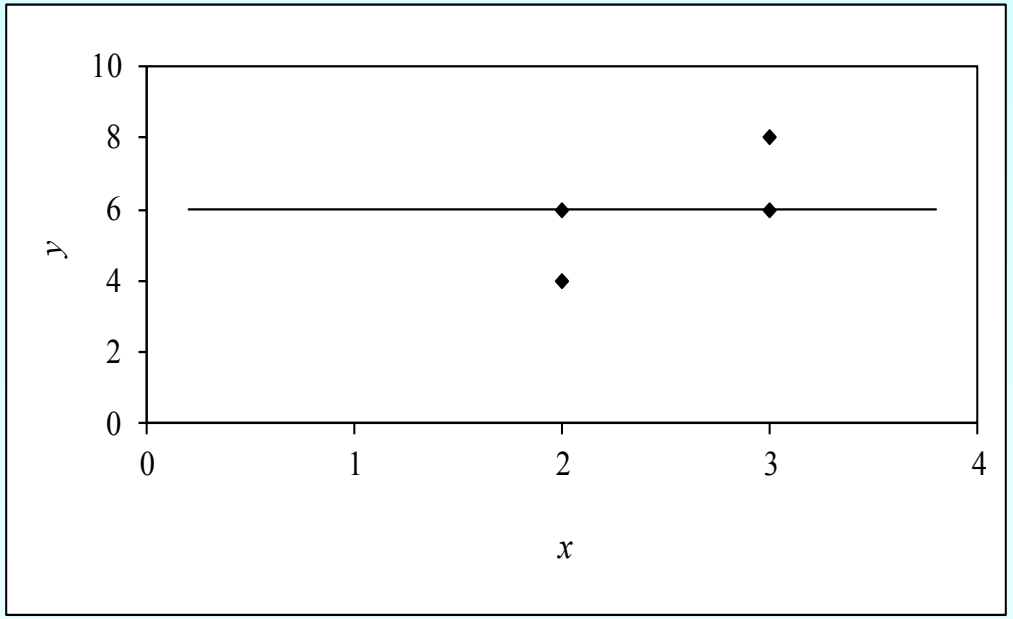


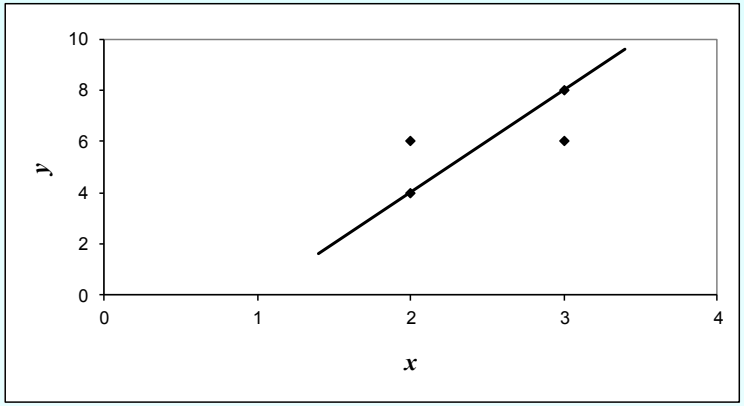
Figure. Regression curve y=6 and y vs x data

Linear Regression – Criterion #1

$\sum_{i=1}^{4} E_i = 0$ for both regression models of $y=4x-4$ and $y=6$

The sum of the residuals is minimized, in this case it is zero, but the regression model is not unique.

Hence the criterion of minimizing the sum of the residuals is a bad criterion.



Linear Regression-Criterion#1

Using $y=4x-4$ as the regression curve

Table. Residuals at each point for regression model $y=4x-4$

x	y	$y_{predicted}$	$E = y - y_{predicted}$
2.0	4.0	4.0	0.0
3.0	6.0	8.0	-2.0
2.0	6.0	4.0	2.0
3.0	8.0	8.0	0.0
			$\sum_{i=1}^{4} E_i = 0$

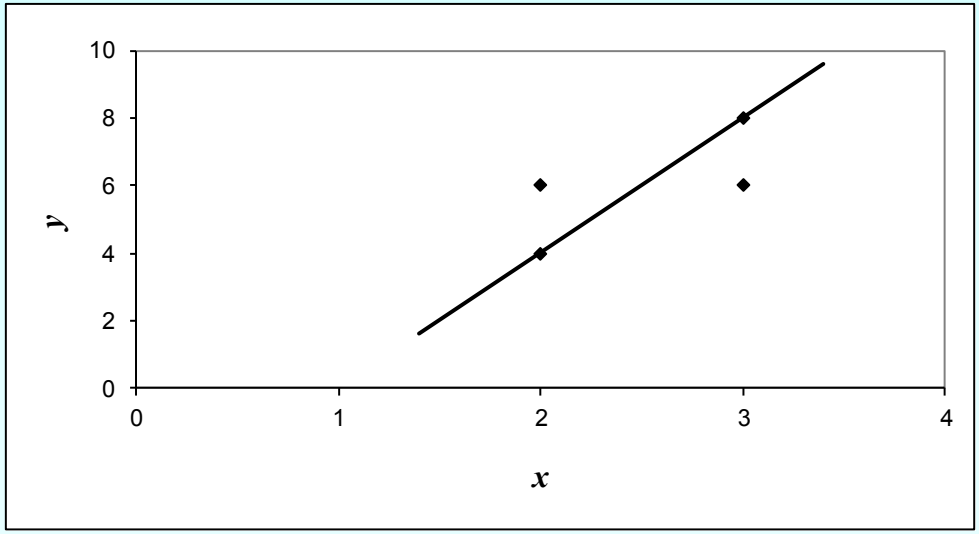


Figure. Regression curve $y=4x-4$ and y vs x data

Linear Regression-Criterion#2

Will minimizing $\sum_{i=1}^{n} |E_i|$ work any better?

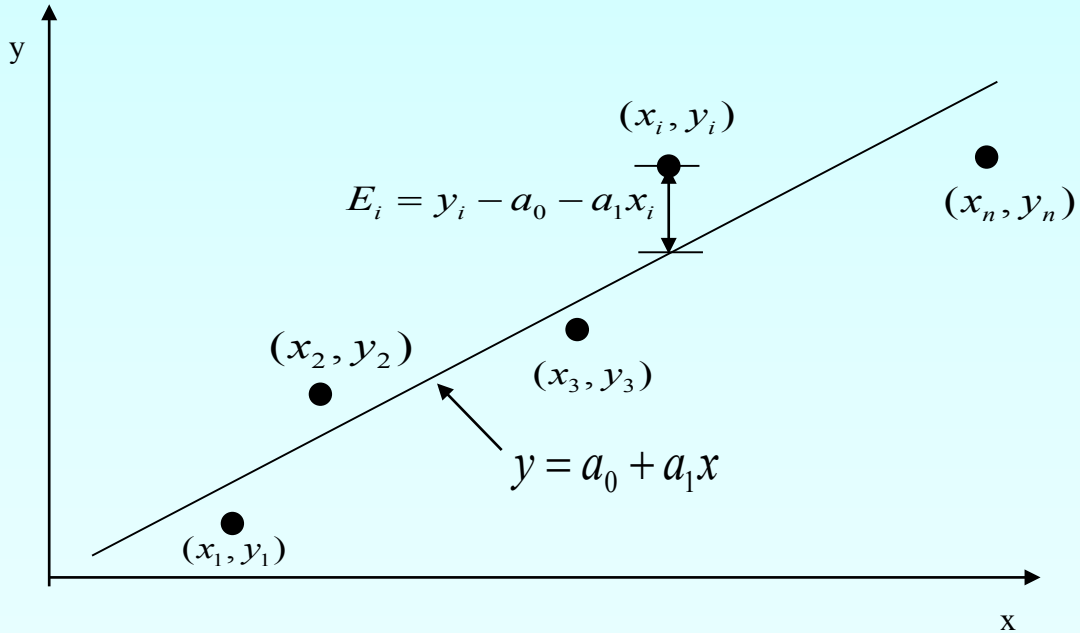


Figure. Linear regression of y vs. x data showing residuals at a typical point, x_i.

Example for Criterion#2

Example: Given the data points (2,4), (3,6), (2,6) and (3,8), best fit the data to a straight line using Criterion#2

Minimize $\sum_{i=1}^{n} | E_i |$

Table. Data Points

x	y
2.0	4.0
3.0	6.0
2.0	6.0
3.0	8.0

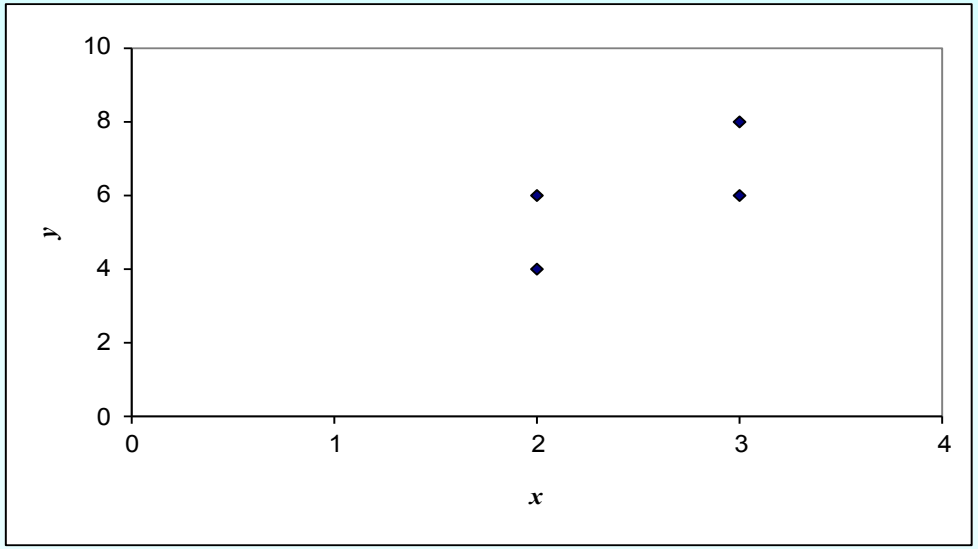


Figure. Data points for y vs. x data.

Linear Regression-Criterion#2

Using $y=4x-4$ as the regression curve

Table. Residuals at each point for regression model $y=4x-4$

x	y	$y_{predicted}$	$E = y - y_{predicted}$
2.0	4.0	4.0	0.0
3.0	6.0	8.0	-2.0
2.0	6.0	4.0	2.0
3.0	8.0	8.0	0.0
			$\sum_{i=1}^{4} \lvert E_i \rvert = 4$

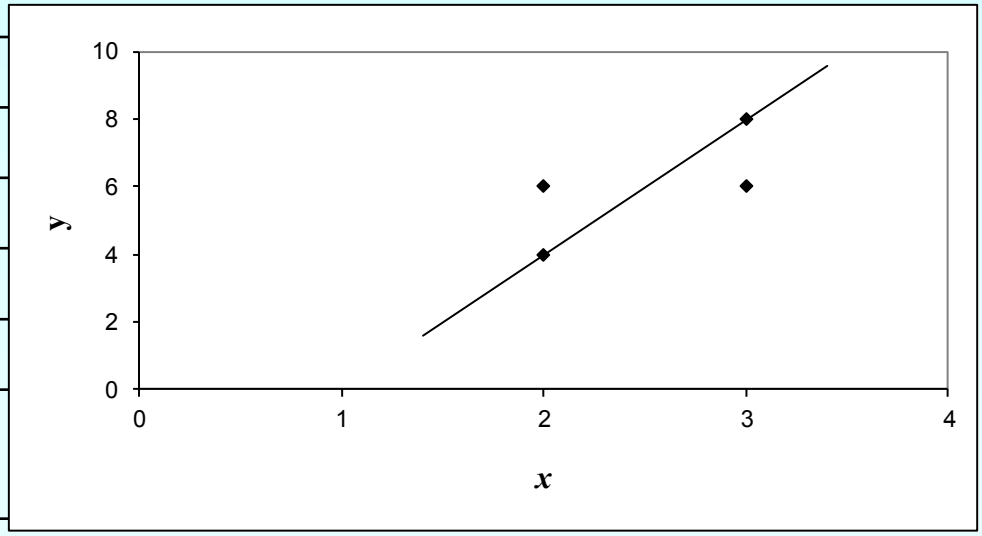


Figure. Regression curve $y=4x-4$ and y vs. x data

Linear Regression-Criterion#2

Using y=6 as a regression curve

Table. Residuals at each point for regression model y=6

x	y	$y_{predicted}$	$E = y - y_{predicted}$
2.0	4.0	6.0	-2.0
3.0	6.0	6.0	0.0
2.0	6.0	6.0	0.0
3.0	8.0	6.0	2.0
			$\sum_{i=1}^{4} \lvert E_i \rvert = 4$

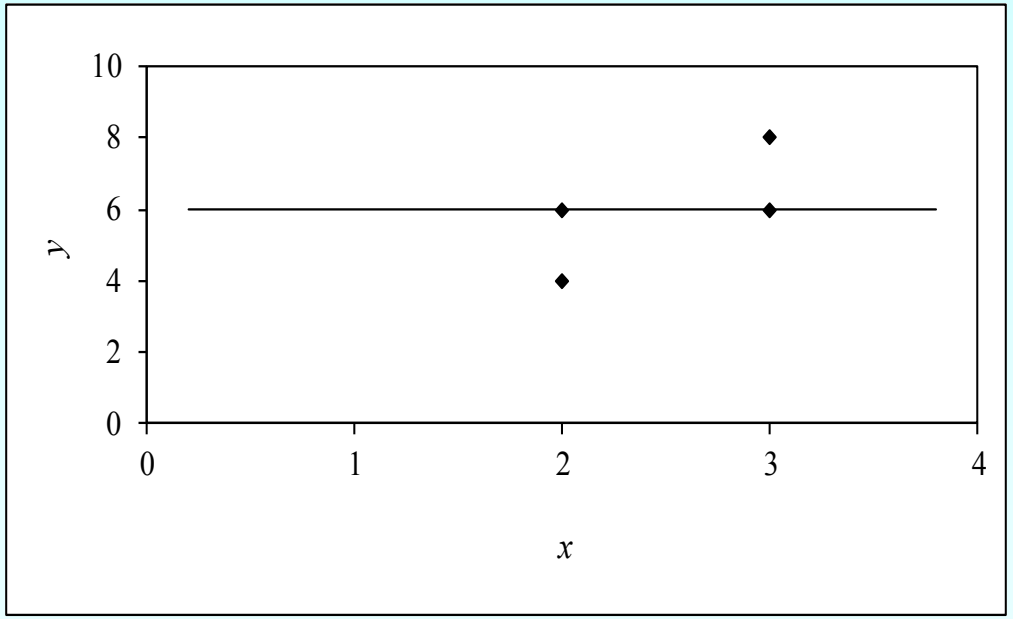


Figure. Regression curve y=6 and y vs x data

Linear Regression-Criterion#2

$\sum_{i=1}^{4}\left|E_i\right| = 4$ for both regression models of $y=4x-4$ and $y=6$.

The sum of the absolute residuals has been made as small as possible, that is 4, but the regression model is not unique.

Hence the criterion of minimizing the sum of the absolute value of the residuals is also a bad criterion.

Least Squares Criterion

The least squares criterion minimizes the sum of the square of the residuals in the model, and also produces a unique line.

$$S_r = \sum_{i=1}^{n} E_i^{\,2} = \sum_{i=1}^{n} \left(y_i - a_0 - a_1 x_i \right)^2$$

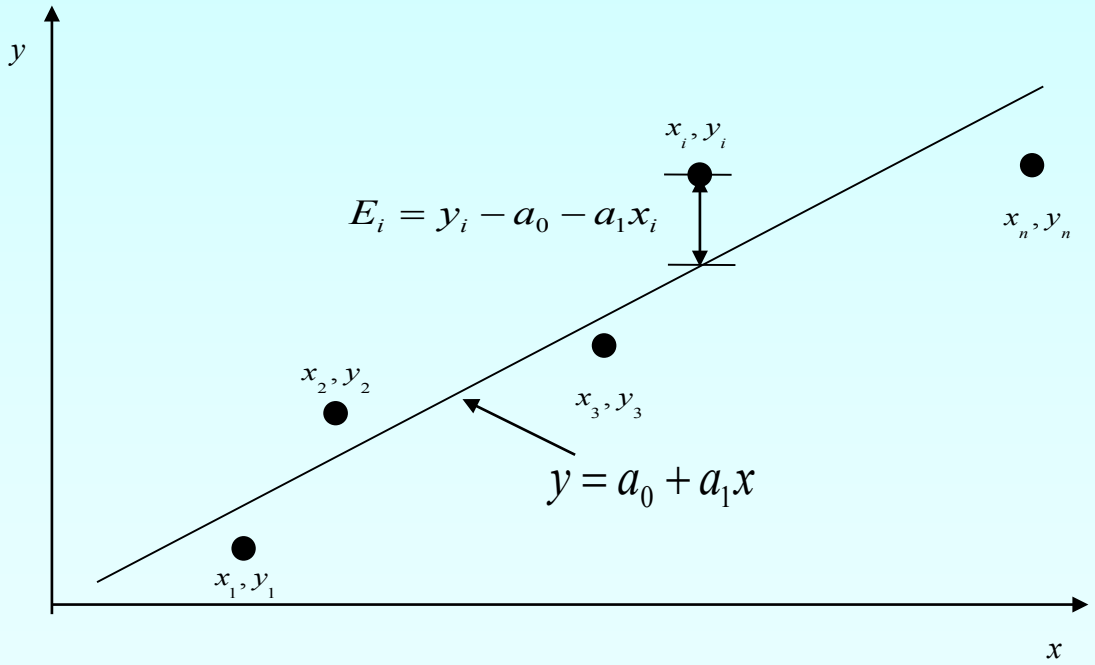


Figure. Linear regression of y vs x data showing residuals at a typical point, x_i.

Finding Constants of Linear Model

Minimize the sum of the square of the residuals: $S_r = \sum_{i=1}^{n} E_i^{\,2} = \sum_{i=1}^{n} \left(y_i - a_0 - a_1 x_i \right)^2$

To find a_0 and a_1 we minimize S_r with respect to a_1 and a_0.

$$\frac{\partial S_r}{\partial a_0} = -2 \sum_{i=1}^{n} \left(y_i - a_0 - a_1 x_i \right)(-1) = 0$$

$$\frac{\partial S_r}{\partial a_1} = -2 \sum_{i=1}^{n} \left(y_i - a_0 - a_1 x_i \right)(-x_i) = 0$$

giving

$$\sum_{i=1}^{n} a_0 + \sum_{i=1}^{n} a_1 x_i = \sum_{i=1}^{n} y_i$$

$$\sum_{i=1}^{n} a_0 x_i + \sum_{i=1}^{n} a_1 x_i^{\,2} = \sum_{i=1}^{n} y_i x_i$$

Finding Constants of Linear Model

Solving for a_0 and a_1 directly yields,

$$a_1 = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

and

$$a_0 = \frac{\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$a_0 = \bar{y} - a_1 \bar{x}$$

Example 1

The torque, T needed to turn the torsion spring of a mousetrap through an angle, is given below. Find the constants for the model given by

$$T = k_1 + k_2\theta$$

Table: Torque vs Angle for a torsional spring

Angle, θ	**Torque, T**
Radians	*N-m*
0.698132	0.188224
0.959931	0.209138
1.134464	0.230052
1.570796	0.250965
1.919862	0.313707

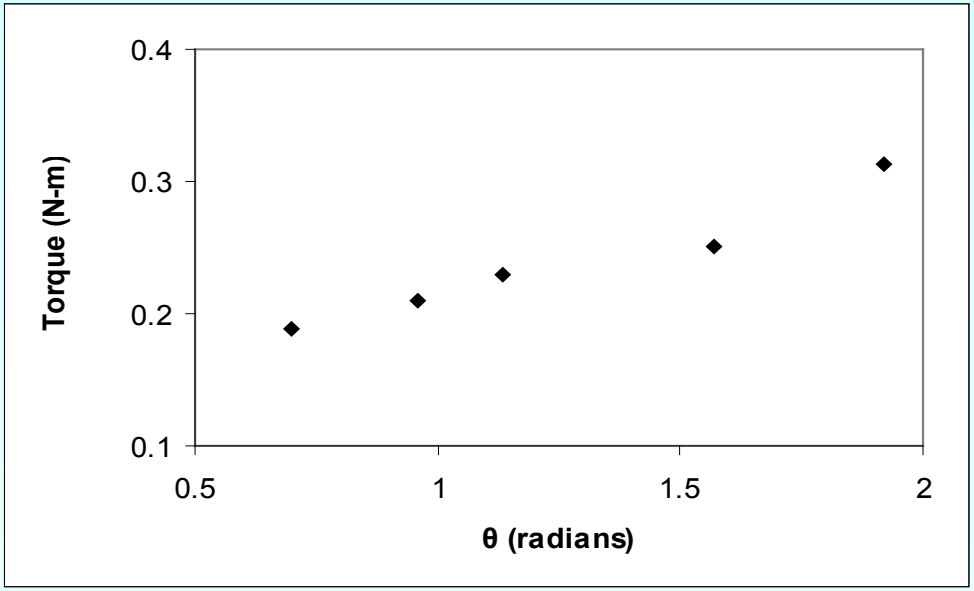


Figure. Data points for Torque vs Angle data

Example 1 cont.

The following table shows the summations needed for the calculations of the constants in the regression model.

Table. Tabulation of data for calculation of important summations

θ	T	θ^2	$T\theta$
Radians	*N-m*	*Radians²*	*N-m-Radians*
0.698132	0.188224	0.487388	0.131405
0.959931	0.209138	0.921468	0.200758
1.134464	0.230052	1.2870	0.260986
1.570796	0.250965	2.4674	0.394215
1.919862	0.313707	3.6859	0.602274
$\sum_{i=1}^{5} =$ 6.2831	1.1921	8.8491	1.5896

Using equations described for a_0 and a_1 with $n = 5$

$$k_2 = \frac{n\sum_{i=1}^{5}\theta_i T_i - \sum_{i=1}^{5}\theta_i \sum_{i=1}^{5} T_i}{n\sum_{i=1}^{5}\theta_i^2 - \left(\sum_{i=1}^{5}\theta_i\right)^2}$$

$$= \frac{5(1.5896) - (6.2831)(1.1921)}{5(8.8491) - (6.2831)^2}$$

$$= 9.6091 \times 10^{-2} \text{ N-m/rad}$$

Example 1 cont.

Use the average torque and average angle to calculate k_1

$$\bar{T} = \frac{\sum_{i=1}^{5} T_i}{n} = \frac{1.1921}{5} = 2.3842 \times 10^{-1}$$

$$\bar{\theta} = \frac{\sum_{i=1}^{5} \theta_i}{n} = \frac{6.2831}{5} = 1.2566$$

Using,

$$k_1 = \bar{T} - k_2 \bar{\theta}$$

$$= 2.3842 \times 10^{-1} - (9.6091 \times 10^{-2})(1.2566)$$

$$= 1.1767 \times 10^{-1} \text{ N-m}$$

Example 1 Results

Using linear regression, a trend line is found from the data

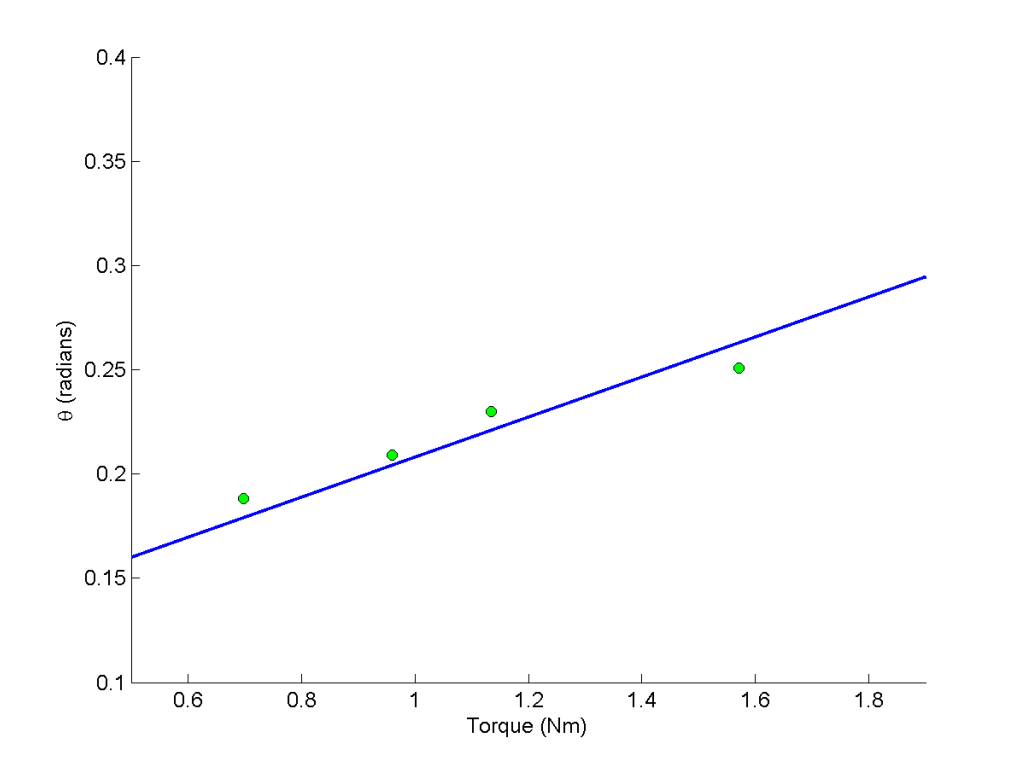


Figure. Linear regression of Torque versus Angle data

Can you find the energy in the spring if it is twisted from 0 to 180 degrees?

Linear Regression (special case)

Given

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

best fit

$$y = a_1 x$$

to the data.

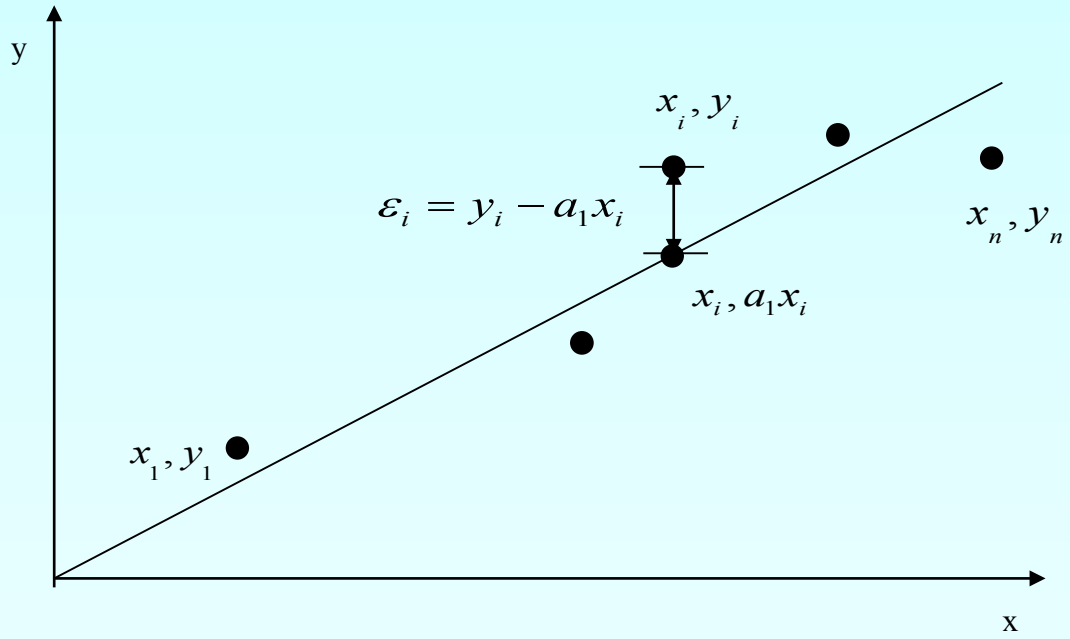
Linear Regression (special case cont.)

$$y = a_1 x$$

$$a_1 = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

Is this correct?

Linear Regression (special case cont.)



Linear Regression (special case cont.)

Residual at each data point

$$\varepsilon_i = y_i - a_1 x_i$$

Sum of square of residuals

$$S_r = \sum_{i=1}^{n} \varepsilon_i^{\,2}$$

$$= \sum_{i=1}^{n} \left(y_i - a_1 x_i \right)^2$$

Linear Regression (special case cont.)

Differentiate with respect to a_1

$$\frac{dS_r}{da_1} = \sum_{i=1}^{n} 2\left(y_i - a_1 x_i\right)\left(-x_i\right)$$

$$= \sum_{i=1}^{n} \left(-2 y_i x_i + 2 a_1 x_i^{\,2}\right)$$

$$\frac{dS_r}{da_1} = 0$$

gives

$$a_1 = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

Linear Regression (special case cont.)

Does this value of a_1 correspond to a local minima or local maxima?

$$a_1 = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

$$\frac{dS_r}{da_1} = \sum_{i=1}^{n} \left(-2 y_i x_i + 2 a_1 x_i^{\,2} \right)$$

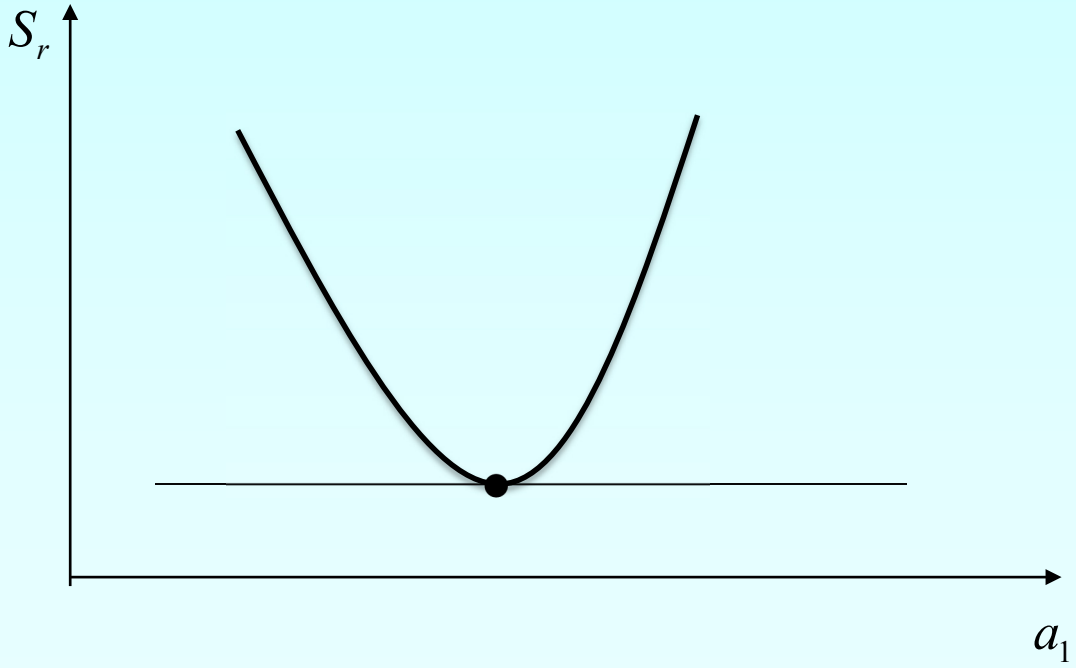
$$\frac{d^2 S_r}{da_1^{\,2}} = \sum_{i=1}^{n} 2 x_i^{\,2} > 0$$

Yes, it corresponds to a local minima.

$$a_1 = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

Linear Regression (special case cont.)

Is this local minima of S_r an absolute minimum of S_r?



Example 2

To find the longitudinal modulus of composite, the following data is collected. Find the longitudinal modulus, E using the regression model $\sigma = E\varepsilon$ and the sum of the square of the residuals.

Table. Stress vs. Strain data

Strain	Stress
(%)	*(MPa)*
0	0
0.183	306
0.36	612
0.5324	917
0.702	1223
0.867	1529
1.0244	1835
1.1774	2140
1.329	2446
1.479	2752
1.5	2767
1.56	2896

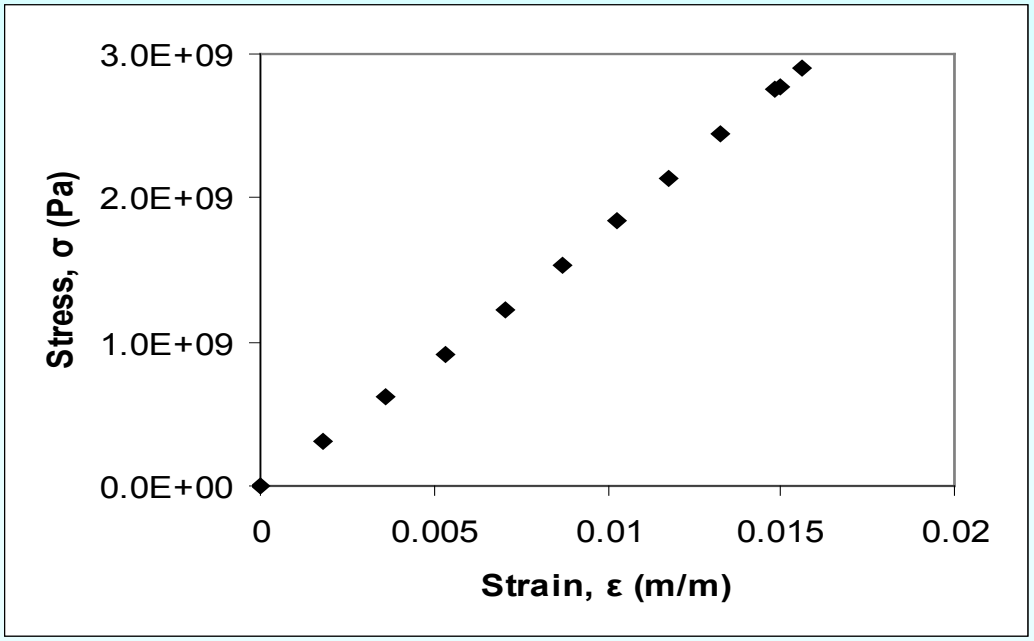


Figure. Data points for Stress vs. Strain data

Example 2 cont.

Table. Summation data for regression model

i	ε	σ	ε^2	$\varepsilon\sigma$
1	0.0000	0.0000	0.0000	0.0000
2	1.8300×10^{-3}	3.0600×10^{8}	3.3489×10^{-6}	5.5998×10^{5}
3	3.6000×10^{-3}	6.1200×10^{8}	1.2960×10^{-5}	2.2032×10^{6}
4	5.3240×10^{-3}	9.1700×10^{8}	2.8345×10^{-5}	4.8821×10^{6}
5	7.0200×10^{-3}	1.2230×10^{9}	4.9280×10^{-5}	8.5855×10^{6}
6	8.6700×10^{-3}	1.5290×10^{9}	7.5169×10^{-5}	1.3256×10^{7}
7	1.0244×10^{-2}	1.8350×10^{9}	1.0494×10^{-4}	1.8798×10^{7}
8	1.1774×10^{-2}	2.1400×10^{9}	1.3863×10^{-4}	2.5196×10^{7}
9	1.3290×10^{-2}	2.4460×10^{9}	1.7662×10^{-4}	3.2507×10^{7}
10	1.4790×10^{-2}	2.7520×10^{9}	2.1874×10^{-4}	4.0702×10^{7}
11	1.5000×10^{-2}	2.7670×10^{9}	2.2500×10^{-4}	4.1505×10^{7}
12	1.5600×10^{-2}	2.8960×10^{9}	2.4336×10^{-4}	4.5178×10^{7}
$\sum_{i=1}^{12}$			1.2764×10^{-3}	2.3337×10^{8}

$$E = \frac{\sum_{i=1}^{n} \sigma_i \varepsilon_i}{\sum_{i=1}^{n} \varepsilon_i^2}$$

$$\sum_{i=1}^{12} \varepsilon_i^2 = 1.2764\times10^{-3}$$

$$\sum_{i=1}^{12} \sigma_i \varepsilon_i = 2.3337\times10^{8}$$

$$E = \frac{\sum_{i=1}^{12} \sigma_i \varepsilon_i}{\sum_{i=1}^{12} \varepsilon_i^2} = \frac{2.3337\times10^{8}}{1.2764\times10^{-3}} = 182.84\ GPa$$

Example 2 Results

The equation $\sigma = 182.84 \times 10^9 \varepsilon$ describes the data.

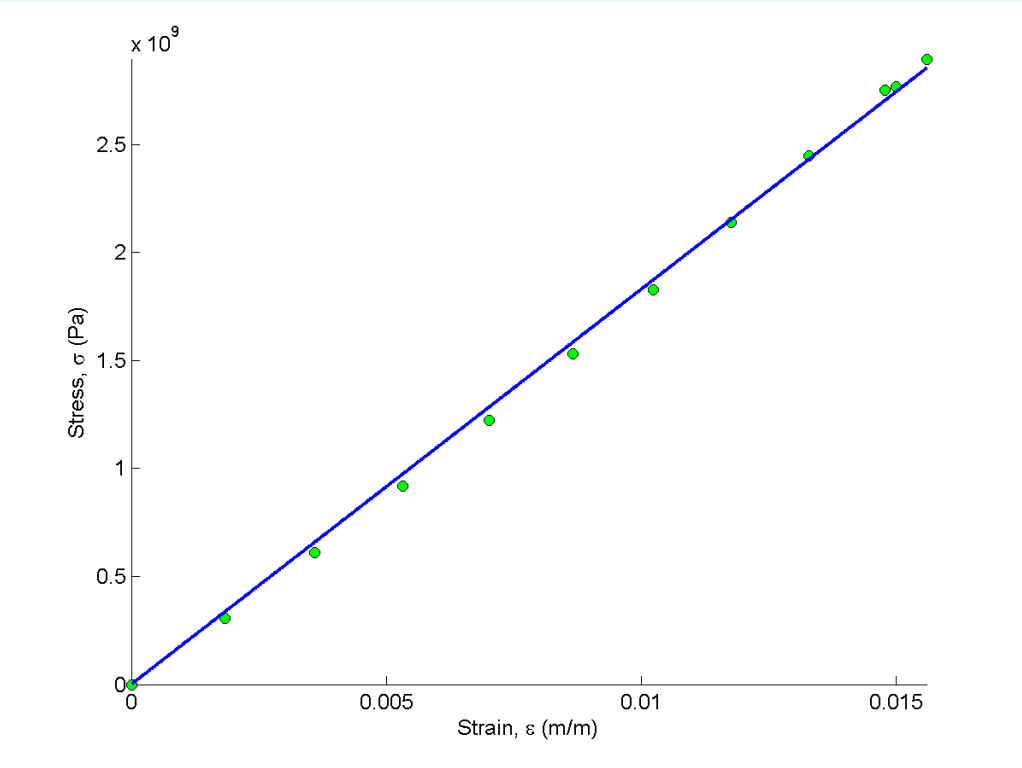


Figure. Linear regression for stress vs. strain data

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/linear_regression.html

THE END

http://numericalmethods.eng.usf.edu