Chapter 07.01
Primer on Integration

After reading this chapter, you should be able to:

1. *define an integral,*
2. *use Riemann's sum to approximately calculate integrals,*
3. *use Riemann's sum and its limit to find the exact expression of integrals, and*
4. *find exact integrals of different functions such as polynomials, trigonometric function and transcendental functions.*

What is integration?

The dictionary definition of *integration* is combining parts so that they work together or form a whole. Mathematically, integration stands for finding the area under a curve from one point to another. It is represented by

$$\int_a^b f(x)dx$$

where the symbol $\int$ is an integral sign, and a and b are the lower and upper limits of integration, respectively, the function f is the integrand of the integral, and x is the variable of integration. Figure 1 represents a graphical demonstration of the concept.

Riemann Sum

Let f be defined on the closed interval $[a,b]$, and let Δ be an arbitrary partition of $[a,b]$ such as: $a = x_0 < x_1 < x_2 < ..... < x_{n-1} < x_n = b$, where Δx_i is the length of the ith subinterval (Figure 2).

If c_i is any point in the ith subinterval, then the sum

$$\sum_{i=1}^n f(c_i)\Delta x_i, x_{i-1} \le c_i \le x_i$$

is called a Riemann sum of the function f for the partition Δ on the interval $[a,b]$. For a given partition Δ, the length of the longest subinterval is called the norm of the partition. It is denoted by $\|\Delta\|$ (the norm of Δ). The following limit is used to define the definite integral.
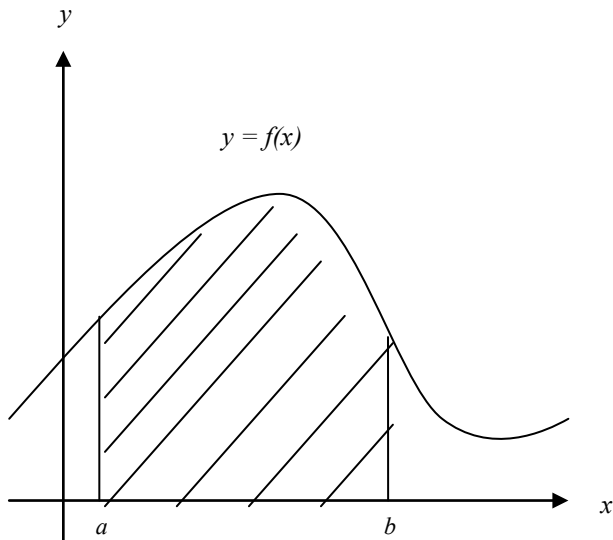
Figure 1 The definite integral as the area of a region under the curve, $Area = \int_{a}^{b} f(x)dx$.

If c_i is any point in the ith subinterval, then the sum

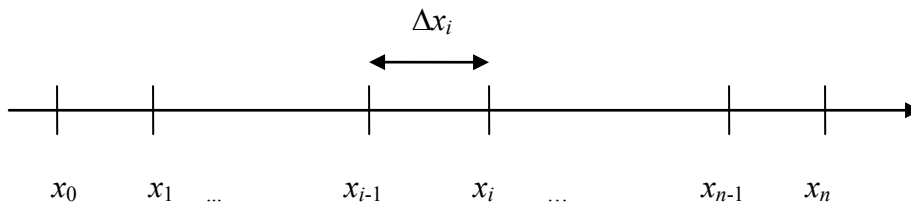$$\sum_{i=1}^{n} f(c_i)\Delta x_i,\; x_{i-1} \le c_i \le x_i$$

Figure 2 Division of interval into n segments.

is called a Riemann sum of the function f for the partition Δ on the interval $[a,b]$. For a given partition Δ, the length of the longest subinterval is called the norm of the partition. It is denoted by $\|\Delta\|$ (the norm of Δ). The following limit is used to define the definite integral.

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = I$$

This limit exists if and only if for any positive number ε, there exists a positive number δ such that for every partition Δ of $[a,b]$ with $\|\Delta\| < \delta$, it follows that

$$\left| I - \sum_{i=1}^{n} f(c_i)\Delta x_i \right| < \varepsilon$$

for any choice of c_i in the ith subinterval of Δ.

If the limit of a Riemann sum of f exists, then the function f is said to be integrable over $[a,b]$ and the Riemann sum of f on $[a,b]$ approaches the number I.

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = I$$

where

$$I = \int_{a}^{b} f(x)dx$$

Example 1

Find the area of the region between the parabola $y = x^2$ and the x-axis on the interval $[0,4.5]$. Use Riemann's sum with four partitions.

Solution

We evaluate the integral for the area as a limit of Riemann sums. We sketch the region (Figure 3), and partition $[0,4.5]$ into four subintervals of length

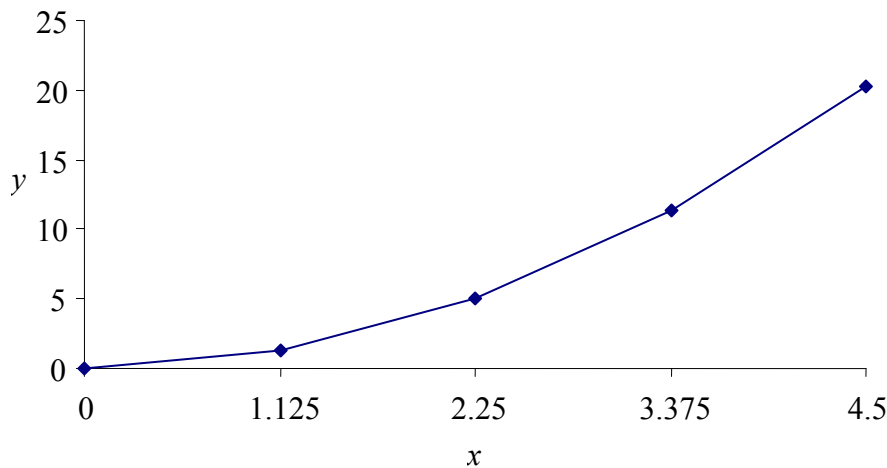$$\Delta x = \frac{4.5 - 0}{4} = 1.125 .$$

Figure 3 Graph of the function $y = x^2$.

The points of partition are
$$x_0 = 0, \ x_1 = 1.125, \ x_2 = 2.25, \ x_3 = 3.375, \ x_4 = 4.5$$

Let's choose c_i's to be right hand endpoint of its subinterval. Thus,
$$c_1 = x_1 = 1.125, \ c_2 = x_2 = 2.25, \ c_3 = x_3 = 3.375, \ c_4 = x_4 = 4.5$$
The rectangles defined by these choices have the following areas:
$$f(c_1)\Delta x = f(1.125) \times (1.125) = (1.125)^2(1.125) = 1.4238$$
$$f(c_2)\Delta x = f(2.25) \times (1.125) = (2.25)^2(1.125) = 5.6953$$
$$f(c_3)\Delta x = f(3.375) \times (1.125) = (3.375)^2(1.125) = 12.814$$
$$f(c_4)\Delta x = f(4.5) \times (1.125) = (4.5)^2(1.125) = 22.781$$
The sum of the areas then is
$$\int_0^{4.5} x^2 dx \approx \sum_{i=1}^{4} f(c_i)\Delta x,$$
$$= 1.4238 + 5.6953 + 12.814 + 22.781$$
$$= 42.715$$

How does this compare with the exact value of the integral $\int_0^{4.5} x^2 dx$?

Example 2

Find the exact area of the region between the parabola $y = x^2$ and the $x-$axis on the interval $[0,b]$. Use Riemann's sum.

Solution

Note that in Example 1 for $y = x^2$ that
$$f(c_i)\Delta x = i^2 (\Delta x)^3$$
Thus, the sum of these areas, if the interval is divided into n equal segments is
$$S_n = \sum_{i=1}^{n} f(c_i)\Delta x$$
$$= \sum_{i=1}^{n} i^2 (\Delta x)^3$$
$$= (\Delta x)^3 \sum_{i=1}^{n} i^2$$

Since
$$\Delta x = \frac{b}{n}, \text{ and}$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

then
$$S_n = \frac{b^3}{n^3} \frac{n(n+1)(2n+1)}{6}$$
$$= \frac{b^3}{6} \frac{2n^2 + n + 2n + 1}{n^2}$$

$$= \frac{b^3}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)$$

The definition of a definite integral can now be used

$$\int_a^b f(x)dx = \lim_{\|\Delta x\|\to 0} \sum_{i=1}^n f(c_i)\Delta x$$

To find the area under the parabola from $x = 0$ to $x = b$, we have

$$\int_0^b x^2 dx = \lim_{|\Delta|\to 0} \sum_{i=1}^n f(c_i)\Delta x$$

$$= \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} \frac{b^3}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)$$

$$= \frac{b^3}{6}(2+0+0)$$

$$= \frac{b^3}{3}$$

For the value of $b = 4.5$ as given in Example 1,

$$\int_0^{4.5} x^2 dx = \frac{4.5^3}{3}$$

$$= 30.375$$

The Mean Value Theorem for Integrals

The area of a region under a curve is usually greater than the area of an inscribed rectangle and less than the area of a circumscribed rectangle. The mean value theorem for integrals states that somewhere between these two rectangles, there exists a rectangle whose area is exactly equal to the area of the region under the curve, as shown in Figure 4. Another variation states that if a function f is continuous between a and b, then there is at least one point in $[a,b]$ where the function equals the average value of the function f over $[a,b]$.

Theorem: If the function f is continuous on the closed interval $[a,b]$, then there exists a number c in $[a,b]$ such that:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

Example 3

Graph the function $f(x) = (x-1)^2$, and find its average value over the interval $[0,3]$. At what point in the given interval does the function assume its average value?
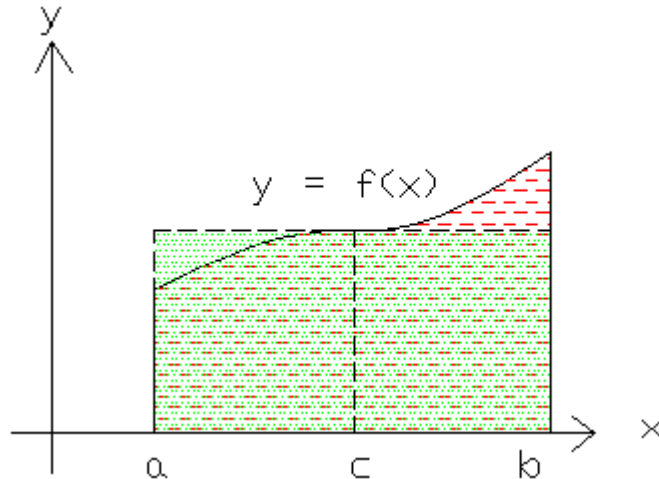
Figure 4   Mean value rectangle.

Solution

$$Average(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

$$= \frac{1}{3-0}\int_0^3 (x-1)^2 dx$$

$$= \frac{1}{3}\int_0^3 (x^2 - 2x + 1)dx$$

$$= \frac{1}{3}\left[\left(\frac{1}{3}\times 27 - 9 + 3\right) - 0\right]$$

$$= 1$$

The average value of the function f over the interval $[0,3]$ is 1. Thus, the function assumes its average value at

$$f(c) = 1$$
$$(c-1)^2 = 1$$
$$c = 0,\ 2$$

The connection between integrals and area can be exploited in two ways. When a formula for the area of the region between the x-axis and the graph of a continuous function is known, it can be used to evaluate the integral of the function. However, if the area of region is not known, the integral of the function can be used to define and calculate the area. Table 1 lists a number of standard indefinite integral forms.
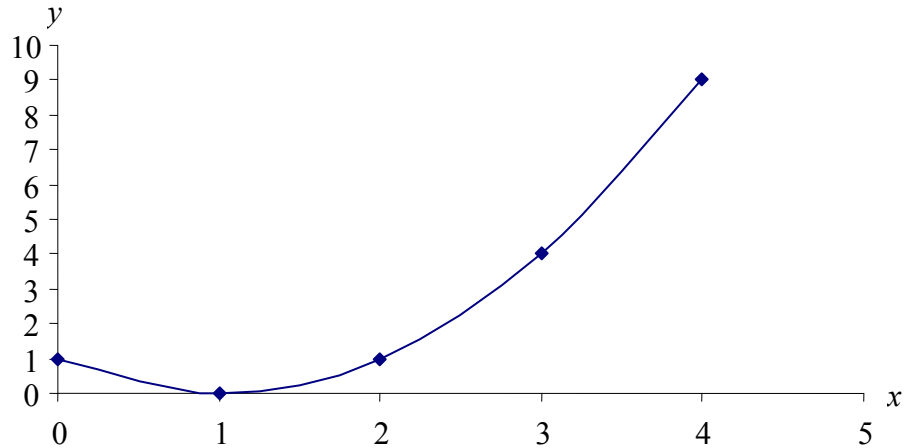
Figure 5 The function $f(x) = (x-1)^2$.

Example 4

Find the area of the region between the circle $x^2 + y^2 = 1$ and the x-axis on the interval $[0,1]$ (the shaded region) in two different ways.
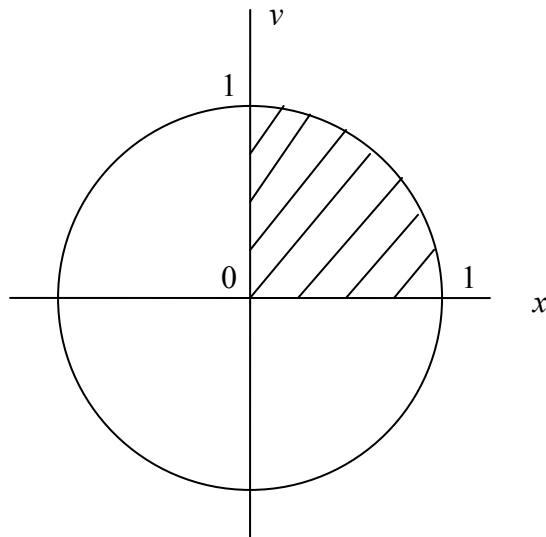
Solution

Figure 6 Graph of the function $x^2 + y^2 = 1$.

The first and easy way to solve this problem is by recognizing that it is a quarter circle. Hence the area of the shaded area is

$$A = \frac{1}{4}\pi r^2$$

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$$= \frac{1}{4}\pi(1)^2$$

$$= \frac{\pi}{4}$$

The second way is to use the integrals and the trigonometric functions. First, let's simplify the function $x^2 + y^2 = 1$.

$$x^2 + y^2 = 1$$

$$y^2 = 1 - x^2$$

$$y = \sqrt{1 - x^2}$$

The area of the shaded region is the equal to

$$A = \int_0^1 \sqrt{1 - x^2}\, dx$$

We set $x = \sin\theta$, $dx = \cos\theta\, d\theta$

$$A = \int_0^1 \sqrt{1 - x^2}\, dx$$

$$= \int_0^{\pi/2} \sqrt{\left(1 - \sin^2\theta\right)}\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{\left(\cos^2\theta\right)}\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \cos^2\theta\, d\theta$$

By using the following formula

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2},$$

we have

$$A = \int_0^{\pi/2} \frac{1 + \cos 2\theta}{2}\, d\theta$$

$$= \int_0^{\pi/2} \left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta$$

$$= \left[\frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= \left(\frac{\pi}{4} + 0\right) - (0 + 0)$$

$$= \frac{\pi}{4}$$

The following are some more examples of exact integration. You can use the brief table of integrals given in Table 1.

Table 1   A brief table of integrals

$\int dx = x + C$	$\int \sin x dx = -\cos x + C$
$\int a f(x) dx = a \int f(x) dx + C$	$\int \cos x dx = \sin x + C$
$\int [u(x) \pm v(x)] dx = \int u(x) dx \pm \int v(x) dx + C$	$\int \tan x dx = -\ln\lvert\cos x\rvert + C = \ln\lvert\sec x\rvert + C$
$\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$	$\int \sec(ax) dx = \dfrac{1}{a}\ln\lvert\sec(ax) + \tan(ax)\rvert + C$
$\int u\, dv = u v - \int v\, du + C$	$\int \cot x dx = -\ln\lvert\csc x\rvert + C = \ln\lvert\sin x\rvert + C$
$\int \dfrac{dx}{ax + b} = \dfrac{1}{a}\ln\lvert ax + b\rvert + C$	$\int \sec^2 ax dx = \dfrac{1}{a}\tan(ax) + C$
$\int a^x dx = \dfrac{a^x}{\ln a} + C$	$\int \sec(x) \tan(x) dx = \sec(x) + C$
$\int e^{ax} dx = \dfrac{e^{ax}}{a} + C$	$\int \csc(x) \cot(x) dx = -\csc(x) + C$

Example 5

Evaluate the following integral

$$\int_0^1 2xe^{-x^2} dx$$

Solution

Let $u = -x^2$, $du = -2xdx$

At $x = 0$, $u = -(0)^2 = 0$

At $x = 1$, $u = -(1)^2 = -1$

$$\int_0^1 2xe^{-x^2}dx = \int_0^1 (-e^{-x^2})(-2xdx)$$

$$= \int_0^{-1} (-e^u)(du)$$

$$= \left[-e^u\right]_0^{-1}$$

$$= -e^{-1} - (-e^0)$$

$$= 0.6321$$

Example 6

Evaluate

$$\int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}dx$$

Solution

$$\int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}dx = \int_0^{\pi/4} \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right)dx$$

$$= \int_0^{\pi/4} (\sec^2 x + \sec x \times \tan x)dx$$

$$= \int_0^{\pi/4} (\sec^2 x)dx + \int_0^{\pi/4} (\sec x)(\tan x)dx$$

$$= \left[\tan x\right]_0^{\pi/4} + \left[\sec x\right]_0^{\pi/4}$$

$$= (1-0) + (\sqrt{2}-1)$$

$$= \sqrt{2}$$

Example 7

Evaluate $\int x\sec^2 x\,dx$

Solution

We use the formula

$$\int u\,dv = uv - \int v\,du$$

Let $u = x$, $du = dx$, and $dv = \sec^2 x\,dx$, $v = \tan x$

So the new integral is

$$\int x \sec^2 x dx = x \tan x - \int \tan x dx$$

$$= x \tan x + \ln|\cos x| + C$$

Example 8

Evaluate

$$\int_1^2 x \ln x dx$$

Solution

Let $u = \ln x,\ du = \dfrac{1}{x} dx$ and $dv = x dx, v = \dfrac{x^2}{2}$

Using the formula $\int u dv = uv - \int v du$, the new integral is

$$\int_1^2 (x)(\ln x) dx = \left[\ln x \times \frac{x^2}{2} \right]_1^2 - \int_1^2 \left(\frac{x^2}{2} \right)\left(\frac{1}{x} dx \right)$$

$$= \left[\ln x \times \frac{x^2}{2} \right]_1^2 - \int_1^2 \frac{x}{2} dx$$

$$= \left[\ln x \times \frac{x^2}{2} \right]_1^2 - \left[\frac{x^2}{4} \right]_1^2$$

$$= \left[\left(\ln 2 \times \frac{2^2}{2} \right) - \left(\ln 1 \times \frac{1^2}{2} \right) \right] - \left[\left(\frac{2^2}{4} \right) - \left(\frac{1^2}{4} \right) \right]$$

$$= \left[(2\ln 2) - \left(\frac{1}{2}\ln 1 \right) \right] - \left[\left(\frac{4}{4} \right) - \left(\frac{1}{4} \right) \right]$$

$$= \left[(2\ln 2) - \left(\frac{1}{2} \times 0 \right) \right] - \left[1 - \frac{1}{4} \right]$$

$$= 0.6362$$

Example 9

Evaluate

$$\int_0^1 \frac{5x}{(4+x^2)^2} dx$$

Solution

We use the formula $\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$, by substituting $u = g(x),\ du = g'(x) dx$

then integrating from $g(a)$ to $g(b)$.

Let

$$u = g(x) = 4 + x^2 ,$$

so

$$g(0) = 4, \ g(1) = 5 \text{, and}$$
$$du = (2x)dx$$

The new integral is

$$\int_0^1 \frac{5x}{(4+x^2)^2}dx = \int_0^1 \frac{1}{(4+x^2)^2} \times \frac{5}{2} \times (2x)dx$$

$$= \frac{5}{2}\int_4^5 \frac{1}{u^2}du$$

$$= \frac{5}{2}\left[-\frac{1}{u}\right]_4^5$$

$$= \frac{5}{2}\left[(-\frac{1}{5}) - (-\frac{1}{4})\right]$$

$$= \frac{5}{2} \times \frac{1}{20}$$

$$= 0.125$$

Example 10

Evaluate

$$\int_0^4 |2x - 1|dx$$

Solution

First, let's analyze the expression $|2x - 1|$.

$$|2x - 1| = -(2x - 1), \ x < \frac{1}{2}$$

$$= (2x - 1), \ x \geq \frac{1}{2}$$

$$\int_0^4 |2x - 1|dx = \int_0^{1/2} -(2x - 1)dx + \int_{1/2}^4 (2x - 1)dx$$

$$= \left[-x^2 + x\right]_0^{1/2} + \left[x^2 - x\right]_{1/2}^4$$

$$= \left[\left(-\frac{1}{4} + \frac{1}{2}\right) - 0\right] + \left[(16 - 4) - \left(\frac{1}{4} - \frac{1}{2}\right)\right]$$

$$= 12.5$$

Example 11

Evaluate

$$\int_{-\infty}^{-2} \frac{2}{x^2 - 1} dx$$

Solution

$$\int_{-\infty}^{-2} \frac{2}{x^2 - 1} dx = \int_{-\infty}^{-2} \frac{2}{(x-1)\times(x+1)} dx$$

$$= \int_{-\infty}^{-2} \frac{(x+1)-(x-1)}{(x-1)\times(x+1)} dx$$

$$= \int_{-\infty}^{-2} \frac{x+1}{(x-1)\times(x+1)} - \frac{x-1}{(x-1)\times(x+1)} dx$$

$$= \int_{-\infty}^{-2} \frac{1}{x-1} dx - \int_{-\infty}^{-2} \frac{1}{x+1} dx$$

$$= \lim_{b \to -\infty} \left[\ln|x-1|\right]_b^{-2} - \lim_{b \to -\infty} \left[\ln|x+1|\right]_b^{-2}$$

$$= \lim_{b \to -\infty} \left[\ln\left|\frac{x-1}{x+1}\right|\right]_{-2}^{b}$$

$$= \lim_{b \to -\infty} \left[\ln\left|\frac{-3}{-1}\right| - \ln\left|\frac{b-1}{b+1}\right|\right]$$

$$= \ln(3) - \ln\left(\lim_{b \to -\infty}\left|\frac{b-1}{b+1}\right|\right)$$

$$= \ln(3) - \ln(1)$$

$$= \ln(3)$$

$$= 1.0986$$

Example 12

Graph the function $y = \frac{1}{3}(x^2 + 2)^{3/2}$, and find the length of the curve from $x = 0$ to $x = 3$.

Solution

We use the equation

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx$$

We have:

$$y = \frac{1}{3}(x^2 + 2)^{3/2}$$

So,

$$\frac{dy}{dx} = \left(\frac{1}{3}\right)\times\left(\frac{3}{2}\right)\times\left(x^2 + 2\right)^{3/2-1}\times(2x)$$

$$= x\sqrt{x^2 + 2}$$

$$L = \int_0^3 \sqrt{1 + \left(x\sqrt{x^2+2}\right)^2}\, dx$$

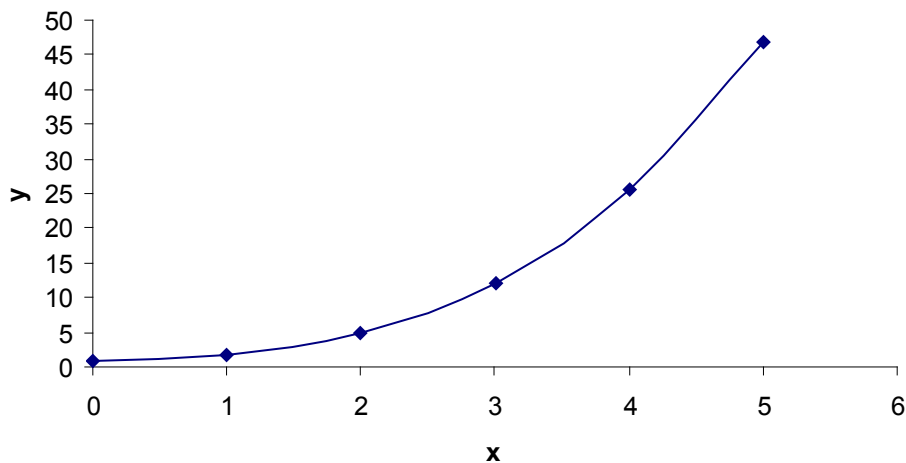**Figure 7**   Graph of the function $y = \dfrac{1}{3}(x^2+2)^{3/2}$

$$= \int_0^3 \sqrt{1 + x^2(x^2+2)}\, dx$$

$$= \int_0^3 \sqrt{1 + x^4 + 2x^2}\, dx$$

$$= \int_0^3 \sqrt{(x^2+1)^2}\, dx$$

$$= \int_0^3 (x^2+1)\, dx$$

$$= \left[\frac{x^3}{3} + x \right]_0^3$$

$$= 12$$

Example 13

Find the area of the shaded region given in Figure 8.
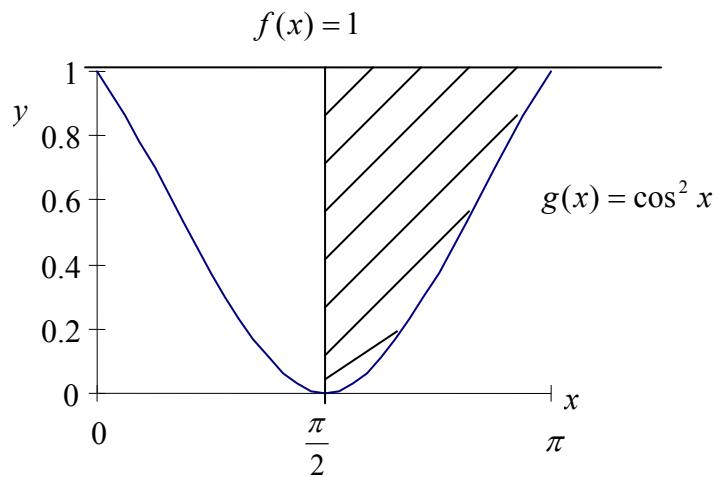
$$f(x) = 1$$

Figure 8   Graph of the function $\cos^2 x$.

Solution

For the sketch given,

$$a = \frac{\pi}{2}, \, b = \pi, \text{ and}$$

$$f(x) - g(x) = 1 - \cos^2 x = \sin^2 x$$

$$A = \int_{\pi/2}^{\pi} \sin^2(x)\,dx$$

$$= \int_{\pi/2}^{\pi} \frac{1 - \cos 2x}{2}\,dx$$

$$= \int_{\pi/2}^{\pi} \left[\frac{1}{2} - \frac{\cos 2x}{2} \right] dx$$

$$= \left[\frac{x}{2} - \frac{\sin 2x}{4} \right]_{\pi/2}^{\pi}$$

$$= \left[\left(\frac{\pi}{2} - \frac{\sin(2\pi)}{4} \right) - \left(\frac{\pi}{4} - \frac{\sin 2\left(\frac{\pi}{2} \right)}{4} \right) \right]$$

$$= \left[\left(\frac{\pi}{2} - 0 \right) - \left(\frac{\pi}{4} - 0 \right) \right]$$

$$= \frac{\pi}{4}$$

Example 14

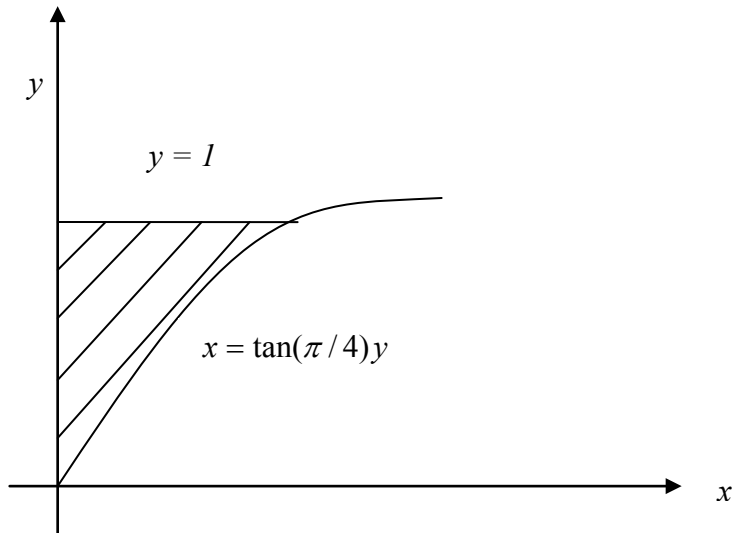Find the volume of the solid generated by revolving the shaded region in Figure 9 about the y-axis.

Figure 9   Volume generated by revolving shaded region.

Solution

We use the formula $V = \int\limits_{a}^{b} \pi \, (radius)^2 dy$

Let

$$u = \frac{\pi}{4}y, \, du = \frac{\pi}{4}dy \, .$$

Therefore, at $y = 0, \, u = 0$

$$y = 1, \, u = \frac{\pi}{4}$$

$$V = \int\limits_{0}^{1} \pi [R(y)]^2 \, dy$$

$$= \pi \int\limits_{0}^{1} \left[\tan\left(\frac{\pi}{4}y \right) \right]^2 dy$$

$$= \pi \times \frac{4}{\pi} \int\limits_{0}^{1} \left[\tan\left(\frac{\pi}{4}y \right) \right]^2 \frac{\pi}{4} dy$$

$$= 4 \int_{0}^{\pi/4} (\tan u)^2 \, du \quad (\text{Choosing } u = \frac{\pi}{4} y)$$

$$= 4 \int_{0}^{\pi/4} \left(-1 + \sec^2 u\right) du$$

$$= 4 \left[-u + \tan u\right]_{0}^{\pi/4}$$

$$= 4 \left[\left(-\frac{\pi}{4} + \tan \frac{\pi}{4}\right) - (0 + \tan 0)\right]$$

$$= 4 \left[\left(-\frac{\pi}{4} + 1\right) - (0 + 0)\right]$$

$$= 0.8584$$

INTEGRATION	
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Major	General Engineering
Authors	Autar Kaw, Loubna Guennoun
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