Romberg Rule of Integration

Major: All Engineering Majors

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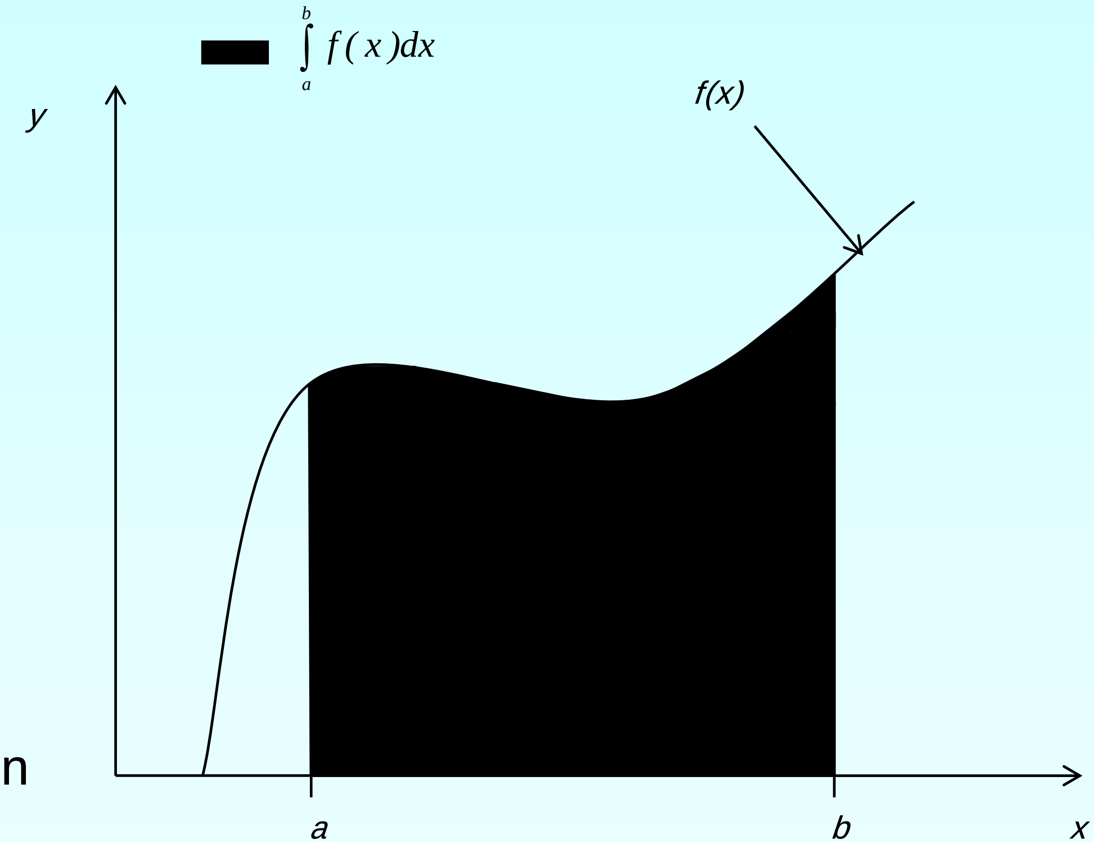
Basis of Romberg Rule

Integration

The process of measuring the area under a curve.

$$I = \int_a^b f(x)dx$$



Where:

f(x) is the integrand

a= lower limit of integration

b= upper limit of integration

What is The Romberg Rule?

Romberg Integration is an extrapolation formula of the Trapezoidal Rule for integration. It provides a better approximation of the integral by reducing the True Error.

Error in Multiple Segment Trapezoidal Rule

The true error in a multiple segment Trapezoidal Rule with n segments for an integral

$$I = \int_a^b f(x)dx$$

Is given by

$$E_t = \frac{(b-a)^3}{12n^2} \frac{\sum_{i=1}^{n} f''(\xi_i)}{n}$$

where for each *i*, ξ_i is a point somewhere in the domain , $[a+(i-1)h, a+ih]$.

Error in Multiple Segment Trapezoidal Rule

The term $\dfrac{\sum_{i=1}^{n} f''(\xi_i)}{n}$ can be viewed as an approximate average value of $f''(x)$ in $[a,b]$.

This leads us to say that the true error, E_t previously defined can be approximated as

$$E_t \cong \alpha \frac{1}{n^2}$$

Error in Multiple Segment Trapezoidal Rule

Table 1 shows the results obtained for the integral using multiple segment Trapezoidal rule for

$$x = \int_{8}^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

n	Value	E_t	$\left|\in_t\right|\%$	$\left|\in_a\right|\%$
1	11868	807	7.296	---
2	11266	205	1.854	5.343
3	11153	91.4	0.8265	1.019
4	11113	51.5	0.4655	0.3594
5	11094	33.0	0.2981	0.1669
6	11084	22.9	0.2070	0.09082
7	11078	16.8	0.1521	0.05482
8	11074	12.9	0.1165	0.03560

Table 1: Multiple Segment Trapezoidal Rule Values

Error in Multiple Segment Trapezoidal Rule

The true error gets approximately quartered as the number of segments is doubled. This information is used to get a better approximation of the integral, and is the basis of Richardson's extrapolation.

Richardson's Extrapolation for Trapezoidal Rule

The true error, E_t in the *n*-segment Trapezoidal rule is estimated as

$$E_t \approx \frac{C}{n^2}$$

where *C* is an *approximate constant* of proportionality. Since

$$E_t = TV - I_n$$

Where TV = true value and I_n = approx. value

Richardson's Extrapolation for Trapezoidal Rule

From the previous development, it can be shown that

$$\frac{C}{(2n)^2} \approx TV - I_{2n}$$

when the segment size is doubled and that

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

which is Richardson's Extrapolation.

Example 1

The vertical distance covered by a rocket from 8 to 30 seconds is given by

$$x = \int_{8}^{30}\left(2000\,ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

a) Use Richardson's rule to find the distance covered. Use the 2-segment and 4-segment Trapezoidal rule results given in Table 1.

b) Find the true error, E_t for part (a).

c) Find the absolute relative true error, $|\in_a|$ for part (a).

Solution

a) $I_2 = 11266m$ $\qquad I_4 = 11113m$

Using Richardson's extrapolation formula for Trapezoidal rule

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

and choosing *n=2*,

$$TV \approx I_4 + \frac{I_4 - I_2}{3} \quad = 11113 + \frac{11113 - 11266}{3}$$

$$= 11062m$$

Solution (cont.)

b) The exact value of the above integral is

$$x = \int_{8}^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

$$= 11061\ m$$

Hence

$$E_t = True\ Value - Approximate\ Value$$

$$= 11061 - 11062$$

$$= -1\ m$$

Solution (cont.)

c) The absolute relative true error $\left|\in_t\right|$ would then be

$$\left|\in_t\right| = \left|\frac{11061 - 11062}{11061}\right| \times 100$$

$$= 0.00904\%$$

Table 2 shows the Richardson's extrapolation results using 1, 2, 4, 8 segments. Results are compared with those of Trapezoidal rule.

Solution (cont.)

Table 2: The values obtained using Richardson's extrapolation formula for Trapezoidal rule for

$$x = \int_{8}^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

n	Trapezoidal Rule	$\left| \in_t \right|$ for Trapezoidal Rule	Richardson's Extrapolation	$\left| \in_t \right|$ for Richardson's Extrapolation
1	11868	7.296	--	--
2	11266	1.854	11065	0.03616
4	11113	0.4655	11062	0.009041
8	11074	0.1165	11061	0.0000

Table 2: Richardson's Extrapolation Values

Romberg Integration

Romberg integration is same as Richardson's extrapolation formula as given previously. However, Romberg used a recursive algorithm for the extrapolation. Recall

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

This can alternately be written as

$$\left(I_{2n}\right)_R = I_{2n} + \frac{I_{2n} - I_n}{3} \quad = I_{2n} + \frac{I_{2n} - I_n}{4^{2-1} - 1}$$

Romberg Integration

Note that the variable TV is replaced by $\left(I_{2n}\right)_R$ as the value obtained using Richardson's extrapolation formula. Note also that the sign $\approx$ is replaced by = sign.

Hence the estimate of the true value now is

$$TV \approx \left(I_{2n}\right)_R + Ch^4$$

Where Ch^4 is an approximation of the true error.

Romberg Integration

Determine another integral value with further halving the step size (doubling the number of segments),

$$(I_{4n})_R = I_{4n} + \frac{I_{4n} - I_{2n}}{3}$$

It follows from the two previous expressions that the true value TV can be written as

$$TV \approx (I_{4n})_R + \frac{(I_{4n})_R - (I_{2n})_R}{15}$$

$$= I_{4n} + \frac{(I_{4n})_R - (I_{2n})_R}{4^{3-1} - 1}$$

Romberg Integration

A general expression for Romberg integration can be written as

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1}, k \geq 2$$

The index *k* represents the order of extrapolation. k=1 represents the values obtained from the regular Trapezoidal rule, k=2 represents values obtained using the true estimate as $O(h^2)$. The index *j* represents the more and less accurate estimate of the integral.

Example 2

The vertical distance covered by a rocket from $t = 8$ to $t = 30$ seconds is given by

$$x = \int_{8}^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

Use Romberg's rule to find the distance covered. Use the 1, 2, 4, and 8-segment Trapezoidal rule results as given in the Table 1.

Solution

From Table 1, the needed values from original Trapezoidal rule are

$$I_{1,1} = 11868 \qquad I_{1,2} = 11266$$

$$I_{1,3} = 11113 \qquad I_{1,4} = 11074$$

where the above four values correspond to using 1, 2, 4 and 8 segment Trapezoidal rule, respectively.

Solution (cont.)

To get the first order extrapolation values,

$$I_{2,1} = I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3}$$
$$= 11266 + \frac{11266 - 11868}{3}$$
$$= 11065$$

Similarly,

$$I_{2,2} = I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3}$$
$$= 11113 + \frac{11113 - 11266}{3}$$
$$= 11062$$

$$I_{2,3} = I_{1,4} + \frac{I_{1,4} - I_{1,3}}{3}$$
$$= 11074 + \frac{11074 - 11113}{3}$$
$$= 11061$$

Solution (cont.)

For the second order extrapolation values,

$$I_{3,1} = I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15}$$

$$= 11062 + \frac{11062 - 11065}{15}$$

$$= 11062$$

Similarly,

$$I_{3,2} = I_{2,3} + \frac{I_{2,3} - I_{2,2}}{15}$$

$$= 11061 + \frac{11061 - 11062}{15}$$

$$= 11061$$

Solution (cont.)

For the third order extrapolation values,

$$I_{4,1} = I_{3,2} + \frac{I_{3,2} - I_{3,1}}{63}$$

$$= 11061 + \frac{11061 - 11062}{63}$$

$$= 11061m$$

Table 3 shows these increased correct values in a tree graph.

Solution (cont.)

Table 3: Improved estimates of the integral value using Romberg Integration

		First Order	*Second Order*	*Third Order*
1-segment	11868			
		11065		
2-segment	1126		11062	
		11062		11061
4-segment	11113		11061	
		11061		
8-segment	11074			

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/romberg_method.html

THE END

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