Part: Simpson $\frac{3}{8}$ Rule For Integration.

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Lecture # 1

Chapter 07.08: Simpson $^3\!/_8$ Rule For Integration.

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Numerical Methods for STEM undergraduates

Introduction

The main objective in this chapter is to develop appropriated formulas for obtaining the integral expressed in the following form:

$$I = \int_a^b f(x)dx \qquad (1)$$

where $f(x)$ is a given function.

Most (if not all) of the developed formulas for integration is based on a simple concept of replacing a given (oftently complicated) function $f(x)$ by a simpler function (usually a polynomial function) $f_i(x)$ where i represents the order of the polynomial function.

In the previous chapter, it has been explained and illustrated that Simpsons 1/3 rule for integration can be derived by replacing the given function $f(x)$ with the 2nd –order (or quadratic) polynomial function $f_i(x) = f_2(x)$, defined as:

$$f_2(x) = a_0 + a_1 x + a_2 x^2 \qquad (2)$$

In a similar fashion, Simpson $^3\!/_8$ rule for integration can be derived by replacing the given function $f(x)$ with the 3rd-order (or cubic) polynomial (passing through 4 known data points) function $f_i(x) = f_3(x)$ defined as

$$\left.\begin{aligned} f_3(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\ &= \left\{1, x, x^2, x^3\right\} \times \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} \end{aligned}\right\} \qquad (3)$$

which can also be symbolically represented in Figure 1.

Method 1

The unknown coefficients $a_0, a_1, a_2 \ and \ a_3$ (in Eq. (3)) can be obtained by substituting 4 known coordinate data points $\{x_0, f(x_0)\}, \{x_1, f(x_1)\}, \{x_2, f(x_2)\} \ and \ \{x_3, f(x_3)\}$ into Eq. (3), as following

$$\left.\begin{aligned} f(x_0) &= a_0 + a_1x_0 + a_2x_0^2 + a_3x_0^2 \\ f(x_1) &= a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^2 \\ f(x_2) &= a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^2 \\ f(x_3) &= a_0 + a_1x_3 + a_2x_3^2 + a_3x_3^2 \end{aligned}\right\} \qquad (4)$$

Eq. (4) can be expressed in matrix notation as

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \end{Bmatrix} \qquad (5)$$

The above Eq. (5) can be symbolically represented as

$$[A]_{4\times 4}\vec{a}_{4\times 1} = \vec{f}_{4\times 1} \qquad (6)$$

Thus,

$$\vec{a} = \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = [A]^{-1} \times \vec{f} \qquad (7)$$

Substituting Eq. (7) into Eq. (3), one gets

$$f_3(x) = \left\{1, x, x^2, x^3\right\} \times [A]^{-1} \times \vec{f} \qquad (8)$$

Remarks

As indicated in Figure 1, one has

$$\left.\begin{aligned} x_0 &= a \\ x_1 &= a + h = a + \frac{b-a}{3} = \frac{2a+b}{3} \\ x_2 &= a + 2h = a + \frac{2b-2a}{3} = \frac{a+2b}{3} \\ x_3 &= a + 3h = a + \frac{3b-3a}{3} = b \end{aligned}\right\} \quad (9)$$

With the help from MATLAB [2], the unknown vector $\vec{a}$ (shown in Eq. 7) can be solved.

Method 2

Using Lagrange interpolation, the cubic polynomial function $f_{i=3}(x)$ that passes through 4 data points (see Figure 1) can be explicitly given as

$$f_3(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\times f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\times f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\times f(x_3)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\times f(x_3) \tag{10}$$

Simpsons $\frac{3}{8}$ Rule For Integration

Thus, Eq. (1) can be calculated as (See Eqs. 8, 10 for Method 1 and Method 2, respectively):

$$I = \int_a^b f(x)dx \approx \int_a^b f_3(x)dx$$

Integrating the right-hand-side of the above equations, one obtains

$$I = (b-a) \times \frac{\{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\}}{8} \qquad (11)$$

Since $h=\frac{b-a}{3}$ hence $b-a=3h$, and the above equation becomes:

$$I \approx \frac{3h}{8}\times\left\{f\left(x_0\right)+3f\left(x_1\right)+3f\left(x_2\right)+f\left(x_3\right)\right\} \qquad (12)$$

The error introduced by the Simpson 3/8 rule can be derived as [Ref. 1]:

$$E_t=-\frac{(b-a)^5}{6480}\times f''''(\zeta)$$, where $a \le \zeta \le b$ (13)

Example 1 (Single Simpson $^3\!/_8$ rule)

Compute

$$I = \int_{a=8}^{b=30} \left\{ 2000 \ln\left(\frac{140{,}000}{140{,}000 - 2100x} \right) - 9.8x \right\} dx,$$

by using a single segment Simpson $^3\!/_8$ rule

Solution

In this example:

$$h = \frac{b-a}{3} = \frac{30-8}{3} = 7.3333$$

$$x_0 = 8 \Rightarrow f(x_0) = 2000\ln\left(\frac{140000}{140000 - 2100 \times 8}\right) - 9.8 \times 8 = 177.2667$$

$$\begin{cases} x_1 = x_0 + h = 8 + 7.3333 = 15.3333 \\ f(x_1) = 2000\ln\left(\frac{140000}{140000 - 2100 \times 15.3333}\right) - 9.8 \times 15.3333 = 372.4629 \end{cases}$$

$$\begin{cases} x_2 = x_0 + 2h = 8 + 2(7.3333) = 22.6666 \\ f\left(x_2\right) = 2000\ln\left(\dfrac{140000}{140000 - 2100 \times 22.6666}\right) - 9.8 \times 22.6666 = 608.8976 \end{cases}$$

$$\begin{cases} x_3 = x_0 + 3h = 8 + 3(7.3333) = 30 \\ f\left(x_3\right) = 2000\ln\left(\dfrac{140000}{140000 - 2100 \times 30}\right) - 9.8 \times 30 = 901.6740 \end{cases}$$

Applying Eq. (12), one has:

$$I = \frac{3}{8} \times 7.3333 \times \{177.2667 + 3 \times 372.4629 + 3 \times 608.8976 + 901.6740\}$$

$$I = 11063.3104$$

The “exact” answer can be computed as

$$I_{exact} = 11061.34$$

3. Multiple Segments for Simpson $\frac{3}{8}$ Rule

Using $"n"$ = number of equal (small) segments, the width $"h"$ can be defined as

$$h = \frac{b-a}{3} \qquad (14)$$

<u>Notes:</u>

n = multiple of 3 = number of small $"h"$ segments

The integral, shown in Eq. (1), can be expressed as

$$I = \int_{a}^{b} f(x)dx \approx \int_{a}^{b} f_3(x)dx$$

$$I \approx \int_{x_0=a}^{x_3} f_3(x)dx + \int_{x_3}^{x_6} f_3(x)dx + ... + \int_{x_{n-3}}^{x_n=b} f_3(x)dx \quad (15)$$

Substituting Simpson $\frac{3}{8}$ rule (See Eq. 12) into Eq. (15), one gets

$$I=\frac{3h}{8}\begin{Bmatrix} f(x_0)+3f(x_1)+3f(x_2)+f(x_3)+f(x_3)+3f(x_4)+3f(x_5)+f(x_6) \\ +.....+f(x_{n-3})+3f(x_{n-2})+3f(x_{n-1})+f(x_n) \end{Bmatrix}$$

(16)

$$I=\frac{3h}{8}\left\{f(x_0)+3\sum_{i=1,4,7,..}^{n-2}f(x_i)+3\sum_{i=2,5,8,..}^{n-1}f(x_i)+2\sum_{i=3,6,9,..}^{n-3}f(x_i)+f(x_n)\right\}$$

(17)

Example 2 (Multiple segments Simpson $3/8$ rule)

Compute $$I = \int_{a=8}^{b=30} \left\{ 2000 \ln\left(\frac{140{,}000}{140{,}000 - 2100x} \right) - 9.8x \right\} dx,$$

using Simple $3/8$ multiple segments rule, with number (of "h") segments = n = 6 (which corresponds to 2 "big" segments).

Solution

In this example, one has (see Eq. 14):

$$h=\frac{30-8}{6}=3.6666$$

$$\{x_0, f(x_0)\}=\{8,177.2667\}$$

$$\{x_1, f(x_1)\}=\{11.6666,270.4104\}; \textit{where } x_1 = x_0 + h = 8 + 3.6666$$

$$=11.6666$$

$$\{x_2, f(x_2)\}=\{15.3333,372.4629\}; \textit{where } x_2 = x_0 + 2h = 15.3333$$

$$\{x_3, f(x_3)\}=\{19,484.7455\}; \textit{where } x_3 = x_0 + 3h = 19$$

$$\{x_4, f(x_4)\} = \{22.6666, 608.8976\};\ where\ x_4 = x_0 + 4h = 22.6666$$

$$\{x_5, f(x_5)\} = \{26.3333, 746.9870\};\ where\ x_5 = x_0 + 5h = 26.3333$$

$$\{x_6, f(x_6)\} = \{30, 901.6740\};\ where\ x_6 = x_0 + 6h = 30$$

Applying Eq. (17), one obtains:

$$I=\frac{3}{8}(3.6666)\left\{177.2667+3\sum_{i=1,4,..}^{n-2=4}f\left(x_i\right)+3\sum_{i=2,5,..}^{n-1=5}f\left(x_i\right)+2\sum_{i=3,6,..}^{n-3=3}f\left(x_i\right)+901.6740\right\}$$

$$I=(1.3750)\left\{\begin{array}{l}177.2667+3(270.4104+608.8976)+3(372.4629+746.9870)\\+2(484.7455)+901.6740\end{array}\right\}$$

$$I=11,601.4696$$

Example 3 (Mixed, multiple segments Simpson $\frac{1}{3}$ and $\frac{3}{8}$ rules)

Compute $I = \int_{a=8}^{b=30} \left\{ 2000 \ln\left(\frac{140{,}000}{140{,}000 - 2100x} \right) - 9.8x \right\} dx,$

using Simpson 1/3 rule (with $n_1 = 4$ small segments), and Simpson 3/8 rule (with $n_2 = 3$ small segments).

Solution:

In this example, one has:

$$h = \frac{b-a}{n} = \frac{b-a}{n_1 + n_2} = \frac{30-8}{(4+3)} = 3.1429$$

$$\left.\begin{aligned}
x_0 &= a = 8 \\
x_1 &= x_0 + 1h = 8 + 3.1429 = 11.1429 \\
x_2 &= x_0 + 2h = 8 + 2(3.1429) = 14.2857 \\
x_3 &= x_0 + 3h = 8 + 3(3.1429) = 17.4286 \\
x_4 &= x_0 + 4h = 8 + 4(3.1429) = 20.5714
\end{aligned}\right\} \mathit{Simpson}\ {}^{1}\!/_{3}\ \mathit{rule}$$

$$x_5 = x_0 + 5h = 8 + 5(3.1429) = 23.7143$$

$$x_6 = x_0 + 6h = 8 + 6(3.1429) = 26.8571$$

$$x_7 = x_0 + 7h = 8 + 7(3.1429) = 30$$

$$f\left(x_0 = 8\right) = 2000\ln\left(\frac{140{,}000}{140{,}000 - 2100\times 8}\right) - 9.8\times 8 = 177.2667$$

Similarly:

$$f\left(x_1 = 11.1429\right) = 256.5863$$
$$f\left(x_2\right) = 342.3241$$
$$f\left(x_3\right) = 435.2749$$
$$f\left(x_4\right) = 536.3909$$
$$f\left(x_5\right) = 646.8260$$
$$f\left(x_6\right) = 767.9978$$
$$f\left(x_7\right) = 901.6740$$

For multiple segments $\left(n_1 = first\ 4\ segments\right)$ using Simpson $1/3$ rule, one obtains (See Eq. 19):

$$I_1 = \left(\frac{h}{3}\right)\left\{ f\left(x_0\right) + 4\sum_{i=1,3,\ldots}^{n_1-1=3} f\left(x_i\right) + 2\sum_{i=2,\ldots}^{n_1-2=2} f\left(x_i\right) + f\left(x_{n_1}\right)\right\}$$

$$I_1 = \left(\frac{3.1429}{3}\right)\{177.2667 + 4(256.5863 + 435.2749) + 2(342.3241) + 536.3909\}$$

$$I_1 = 4364.1197$$

For multiple segments $\left(n_2 = last\ 3\ segments\right)$ using Simpson 3/8 rule, one obtains (See Eq. 17):

$$I_2 = \left(\frac{3h}{8}\right)\left\{ f\left(x_0\right) + 3\sum_{i=1,3,\ldots}^{n_2-2=1} f\left(x_i\right) + 3\sum_{i=2,\ldots}^{n_2-1=2} f\left(x_i\right) + 2\sum_{i=\cancel{3},\cancel{6},\ldots}^{n_2-3=0} f\left(x_i\right) + f\left(x_{n_1}\right) \right\}$$

$$I_2 = \left(\frac{3}{8}\times 3.1429\right)\{177.2667 + 3(256.5863) + 3(342.3241) + (skip!) + 435.2749\}$$

$$I_2 = 6697.2748$$

The mixed (combined) Simpson 1/3 and 3/8 rules give:

$$I = I_1 + I_2 = 4364.1197 + 6697.2748$$

$$I = 11{,}061.3946$$

Remarks:

(a) Comparing the truncated error of Simpson 1/3 rule

$$E_t = -\frac{(b-a)^5}{2880} \times f''''(\zeta) \qquad (18)$$

With Simple 3/8 rule (See Eq. 13), the latter seems to offer slightly more accurate answer than the former. However, the cost associated with Simpson 3/8 rule (using 3rd order polynomial function) is significant higher than the one associated with Simpson 1/3 rule (using 2nd order polynomial function).

(b) The number of multiple segments that can be used in the conjunction with Simpson 1/3 rule is 2,4,6,8,.. (any even numbers).

$$I_1 = \left(\frac{h}{3}\right)\{f(x_0) + 4f(x_1) + f(x_2) + f(x_2) + 4f(x_3) + f(x_4) + \ldots.. + f(x_{n-2}) + 4f(x_{n-1}) + f(x_n$$

$$I_2 = \left(\frac{h}{3}\right)\left\{f(x_0) + 4\sum_{i=1,3,\ldots}^{n-1} f(x_i) + 2\sum_{i=2,4,6\ldots}^{n-2} f(x_i) + f(x_n)\right\} \tag{19}$$

However, Simpson 3/8 rule can be used with the number of segments equal to 3,6,9,12,.. (can be either certain odd or even numbers).

(c) If the user wishes to use, say 7 segments, then the mixed Simpson 1/3 rule (for the first 4 segments), and Simpson 3/8 rule (for the last 3 segments).

4. Computer Algorithm For Mixed Simpson 1/3 and 3/8 rule For Integration

Based on the earlier discussions on (Single and Multiple segments) Simpson 1/3 and 3/8 rules, the following "pseudo" step-by-step mixed Simpson rules can be given as

Step 1 User's input information, such as

Given function $f(x),$ integral limits $"a, b",$

n_1= number of small, "h" segments, in conjunction with Simpson 1/3 rule.

n_2= number of small, “h” segments, in conjunction with Simpson 3/8 rule.

Notes:

n_1 = a multiple of 2 (any even numbers)

n_2 = a multiple of 3 (can be certain odd, or even numbers)

Step 2

Compute $n = n_1 + n_2$

$$h = \frac{b-a}{n}$$

$$x_0 = a$$

$$x_1 = a + 1h$$

$$x_2 = a + 2h$$

.

.

$$x_i = a + ih$$

.

.

$$x_n = a + nh = b$$

Step 3

Compute “multiple segments” Simpson 1/3 rule (See Eq. 19)

$$I_1 = \left(\frac{h}{3}\right)\left\{ f(x_0) + 4\sum_{i=1,3,...}^{n_1-1} f(x_i) + 2\sum_{i=2,4,6...}^{n_1-2} f(x_i) + f(x_n)\right\}$$

(19, repeated)

Step 4

Compute "multiple segments" Simpson 3/8 rule (See Eq. 17)

$$I_2 = \left(\frac{3h}{8}\right)\left\{ f(x_0) + 3\sum_{i=1,4,7...}^{n_2-2} f(x_i) + 3\sum_{i=2,5,8...}^{n_2-1} f(x_i) + 2\sum_{i=3,6,9,...}^{n_2-3} f(x_i) + f\left(x_{n_2}\right)\right\}$$

(17, repeated)

Step 5

$$I = I_1 + I_2 \tag{20}$$

and print out the final approximated answer for *I*.

THE END

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