

Part: Simpson $\frac{3}{8}$ Rule For Integration.

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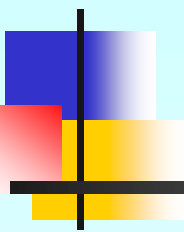
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Lecture # 1



Chapter 07.08: Simpson $\frac{3}{8}$ Rule
For Integration.

Major: All Engineering Majors

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Numerical Methods for STEM undergraduates



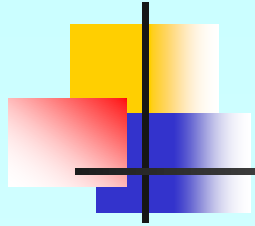
Introduction

The main objective in this chapter is to develop appropriated formulas for obtaining the integral expressed in the following form:

$$I = \int_a^b f(x)dx \quad (1)$$

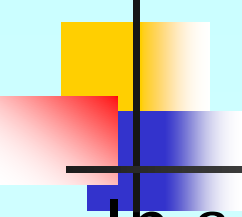
where $f(x)$ is a given function.

Most (if not all) of the developed formulas for integration is based on a simple concept of replacing a given (oftently complicated) function $f(x)$ by a simpler function (usually a polynomial function) $f_i(x)$ where i represents the order of the polynomial function.



In the previous chapter, it has been explained and illustrated that Simpsons 1/3 rule for integration can be derived by replacing the given function $f(x)$ with the 2nd -order (or quadratic) polynomial function $f_i(x) = f_2(x)$, defined as:

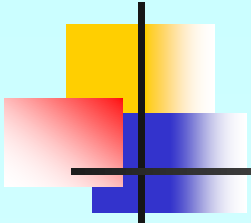
$$f_2(x) = a_0 + a_1x + a_2x^2 \quad (2)$$



In a similar fashion, Simpson $\frac{3}{8}$ rule for integration can be derived by replacing the given function $f(x)$ with the 3rd-order (or cubic) polynomial (passing through 4 known data points) function $f_i(x) = f_3(x)$ defined as

$$\begin{aligned}
 f_3(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 \\
 &= \left\{ \mathbf{1}, x, x^2, x^3 \right\} \times \left\{ \begin{array}{c} a_0 \\ a_1 \\ a_2 \\ a_3 \end{array} \right\}
 \end{aligned} \tag{3}$$

8 which can also be symbolically represented in Figure 1.



Method 1

The unknown coefficients a_0, a_1, a_2 and a_3 (in Eq. (3)) can be obtained by substituting 4 known coordinate data points $\{x_0, f(x_0), \{x_1, f(x_1)\}, \{x_2, f(x_2)\}$ and $\{x_3, f(x_3)\}\}$ into Eq. (3), as following

$$\left. \begin{aligned} f(x_0) &= a_0 + a_1x_0 + a_2x_0^2 + a_3x_0^2 \\ f(x_1) &= a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^2 \\ f(x_2) &= a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^2 \\ f(x_3) &= a_0 + a_1x_3 + a_2x_3^2 + a_3x_3^2 \end{aligned} \right\} \quad (4)$$



Eq. (4) can be expressed in matrix notation as

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \end{Bmatrix} \quad (5)$$

The above Eq. (5) can be symbolically represented as

$$[A]_{4 \times 4} \vec{a}_{4 \times 1} = \vec{f}_{4 \times 1} \quad (6)$$



Thus,

$$\vec{a} = \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = [A]^{-1} \times \vec{f} \quad (7)$$

Substituting Eq. (7) into Eq. (3), one gets

$$f_3(x) = \{1, x, x^2, x^3\} \times [A]^{-1} \times \vec{f} \quad (8)$$



Remarks

As indicated in Figure 1, one has

$$\left. \begin{aligned} x_0 &= a \\ x_1 &= a + h = a + \frac{b-a}{3} = \frac{2a+b}{3} \\ x_2 &= a + 2h = a + \frac{2b-2a}{3} = \frac{a+2b}{3} \\ x_3 &= a + 3h = a + \frac{3b-3a}{3} = b \end{aligned} \right\} \quad (9)$$

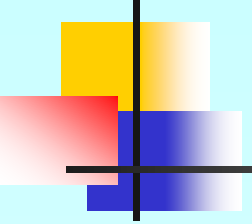
With the help from MATLAB [2], the unknown vector \vec{a} (shown in Eq. 7) can be solved.



Method 2

Using Lagrange interpolation, the cubic polynomial function $f_{i=3}(x)$ that passes through 4 data points (see Figure 1) can be explicitly given as

$$\begin{aligned} f_3(x) = & \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times f(x_1) \\ & + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times f(x_3) \end{aligned} \quad (10)$$



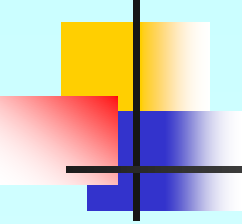
Simpsons $\frac{3}{8}$ Rule For Integration

Thus, Eq. (1) can be calculated as (See Eqs. 8, 10 for Method 1 and Method 2, respectively):

$$I = \int_a^b f(x)dx \approx \int_a^b f_3(x)dx$$

Integrating the right-hand-side of the above equations, one obtains

$$I = (b - a) \times \frac{\{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\}}{8} \quad (11)$$



Since $h = \frac{b-a}{3}$ hence $b-a = 3h$, and the above equation becomes:

$$I \approx \frac{3h}{8} \times \{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\} \quad (12)$$

The error introduced by the Simpson 3/8 rule can be derived as [Ref. 1]:

$$E_t = -\frac{(b-a)^5}{6480} \times f''''(\zeta), \text{ where } a \leq \zeta \leq b \quad (13)$$



Example 1 (Single Simpson $\frac{3}{8}$ rule)

Compute

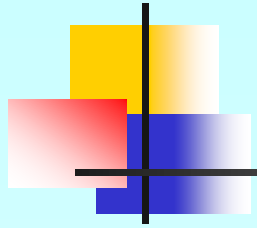
$$I = \int_{a=8}^{b=30} \left\{ 2000 \ln \left(\frac{140,000}{140,000 - 2100x} \right) - 9.8x \right\} dx,$$

by using a single segment Simpson $\frac{3}{8}$ rule

Solution

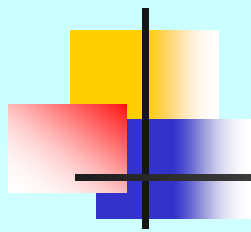
In this example:

$$h = \frac{b-a}{3} = \frac{30-8}{3} = 7.3333$$



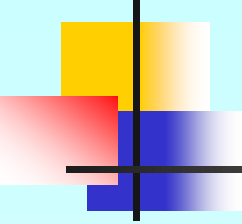
$$x_0 = 8 \Rightarrow f(x_0) = 2000 \ln\left(\frac{140000}{140000 - 2100 \times 8}\right) - 9.8 \times 8 = 177.2667$$

$$\begin{cases} x_1 = x_0 + h = 8 + 7.3333 = 15.3333 \\ f(x_1) = 2000 \ln\left(\frac{140000}{140000 - 2100 \times 15.3333}\right) - 9.8 \times 15.3333 = 372.4629 \end{cases}$$



$$\begin{cases} x_2 = x_0 + 2h = 8 + 2(7.3333) = 22.6666 \\ f(x_2) = 2000 \ln\left(\frac{140000}{140000 - 2100 \times 22.6666}\right) - 9.8 \times 22.6666 = 608.8976 \end{cases}$$

$$\begin{cases} x_3 = x_0 + 3h = 8 + 3(7.3333) = 30 \\ f(x_3) = 2000 \ln\left(\frac{140000}{140000 - 2100 \times 30}\right) - 9.8 \times 30 = 901.6740 \end{cases}$$



Applying Eq. (12), one has:

$$I = \frac{3}{8} \times 7.3333 \times \{177.2667 + 3 \times 372.4629 + 3 \times 608.8976 + 901.6740\}$$

$$I = 11063.3104$$

The “exact” answer can be computed as

$$I_{exact} = 11061.34$$



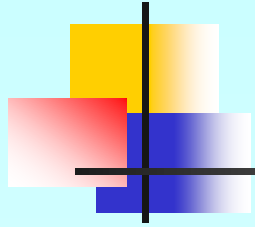
3. Multiple Segments for Simpson $\frac{3}{8}$ Rule

Using " n " = number of equal (small) segments, the width " h " can be defined as

$$h = \frac{b - a}{3} \quad (14)$$

Notes:

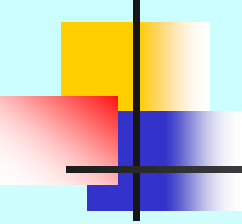
n = multiple of 3 = number of small " h " segments



The integral, shown in Eq. (1), can be expressed as

$$I = \int_a^b f(x)dx \approx \int_a^b f_3(x)dx$$

$$I \approx \int_{x_0=a}^{x_3} f_3(x)dx + \int_{x_3}^{x_6} f_3(x)dx + \dots + \int_{x_{n-3}}^{x_n=b} f_3(x)dx \quad (15)$$



Substituting Simpson $\frac{3}{8}$ rule (See Eq. 12) into Eq. (15), one gets

$$I = \frac{3h}{8} \left\{ f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) + f(x_3) + 3f(x_4) + 3f(x_5) + f(x_6) \right\} \\ \left\{ + \dots + f(x_{n-3}) + 3f(x_{n-2}) + 3f(x_{n-1}) + f(x_n) \right\} \quad (16)$$

$$I = \frac{3h}{8} \left\{ f(x_0) + 3 \sum_{i=1,4,7,\dots}^{n-2} f(x_i) + 3 \sum_{i=2,5,8,\dots}^{n-1} f(x_i) + 2 \sum_{i=3,6,9,\dots}^{n-3} f(x_i) + f(x_n) \right\} \quad (17)$$



Example 2 (Multiple segments Simpson $\frac{3}{8}$ rule)

Compute $I = \int_{a=8}^{b=30} \left\{ 2000 \ln \left(\frac{140,000}{140,000 - 2100x} \right) - 9.8x \right\} dx,$

using Simple $\frac{3}{8}$ multiple segments rule, with number (of "h") segments = $n = 6$ (which corresponds to 2 "big" segments).



Solution

In this example, one has (see Eq. 14):

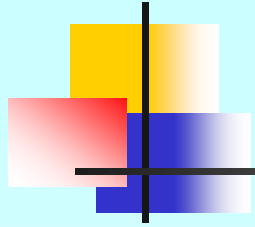
$$h = \frac{30 - 8}{6} = 3.6666$$

$$\{x_0, f(x_0)\} = \{8, 177.2667\}$$

$$\{x_1, f(x_1)\} = \{11.6666, 270.4104\}; \text{ where } x_1 = x_0 + h = 8 + 3.6666 \\ = 11.6666$$

$$\{x_2, f(x_2)\} = \{15.3333, 372.4629\}; \text{ where } x_2 = x_0 + 2h = 15.3333$$

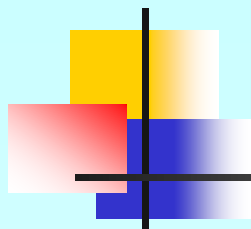
$$\{x_3, f(x_3)\} = \{19, 484.7455\}; \text{ where } x_3 = x_0 + 3h = 19$$



$$\{x_4, f(x_4)\} = \{22.6666, 608.8976\}; \text{ where } x_4 = x_0 + 4h = 22.6666$$

$$\{x_5, f(x_5)\} = \{26.3333, 746.9870\}; \text{ where } x_5 = x_0 + 5h = 26.3333$$

$$\{x_6, f(x_6)\} = \{30, 901.6740\}; \text{ where } x_6 = x_0 + 6h = 30$$

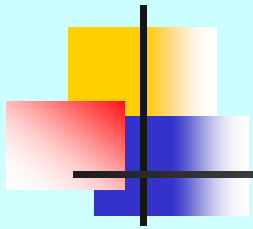


Applying Eq. (17), one obtains:

$$I = \frac{3}{8}(3.6666) \left\{ 177.2667 + 3 \sum_{i=1,4,\dots}^{n-2=4} f(x_i) + 3 \sum_{i=2,5,\dots}^{n-1=5} f(x_i) + 2 \sum_{i=3,6,\dots}^{n-3=3} f(x_i) + 901.6740 \right\}$$

$$I = (1.3750) \left\{ 177.2667 + 3(270.4104 + 608.8976) + 3(372.4629 + 746.9870) \right\} \\ + 2(484.7455) + 901.6740$$

$$I = 11,601.4696$$



Example 3 (Mixed, multiple segments Simpson $\frac{1}{3}$ and $\frac{3}{8}$ rules)

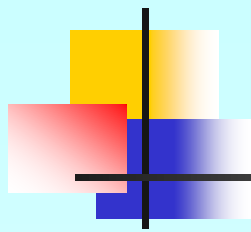
Compute
$$I = \int_{a=8}^{b=30} \left\{ 2000 \ln \left(\frac{140,000}{140,000 - 2100x} \right) - 9.8x \right\} dx,$$

using Simpson $\frac{1}{3}$ rule (with $n_1 = 4$ small segments), and Simpson $\frac{3}{8}$ rule (with $n_2 = 3$ small segments).

Solution:

In this example, one has:

$$h = \frac{b-a}{n} = \frac{b-a}{n_1 + n_2} = \frac{30-8}{(4+3)} = 3.1429$$



$$x_0 = a = 8$$

$$x_1 = x_0 + 1h = 8 + 3.1429 = 11.1429$$

$$x_2 = x_0 + 2h = 8 + 2(3.1429) = 14.2857$$

$$x_3 = x_0 + 3h = 8 + 3(3.1429) = 17.4286$$

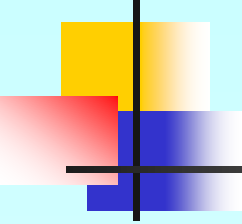
$$x_4 = x_0 + 4h = 8 + 4(3.1429) = 20.5714$$

$$x_5 = x_0 + 5h = 8 + 5(3.1429) = 23.7143$$

$$x_6 = x_0 + 6h = 8 + 6(3.1429) = 26.8571$$

$$x_7 = x_0 + 7h = 8 + 7(3.1429) = 30$$

} *Simpson $\frac{1}{3}$ rule*



$$f(x_0 = 8) = 2000 \ln\left(\frac{140,000}{140,000 - 2100 \times 8}\right) - 9.8 \times 8 = 177.2667$$

Similarly:

$$f(x_1 = 11.1429) = 256.5863$$

$$f(x_2) = 342.3241$$

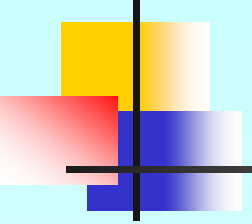
$$f(x_3) = 435.2749$$

$$f(x_4) = 536.3909$$

$$f(x_5) = 646.8260$$

$$f(x_6) = 767.9978$$

$$f(x_7) = 901.6740$$

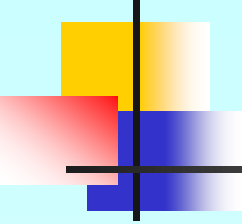


For multiple segments ($n_1 = \text{first 4 segments}$)
using Simpson $\frac{1}{3}$ rule, one obtains (See Eq. 19):

$$I_1 = \left(\frac{h}{3}\right) \left\{ f(x_0) + 4 \sum_{i=1,3,\dots}^{n_1-1=3} f(x_i) + 2 \sum_{i=2,\dots}^{n_1-2=2} f(x_i) + f(x_{n_1}) \right\}$$

$$I_1 = \left(\frac{3.1429}{3}\right) \{177.2667 + 4(256.5863 + 435.2749) + 2(342.3241) + 536.3909\}$$

$$I_1 = 4364.1197$$



For multiple segments ($n_2 = \text{last 3 segments}$)
using Simpson 3/8 rule, one obtains (See Eq. 17):

$$I_2 = \left(\frac{3h}{8}\right) \left\{ f(x_0) + 3 \sum_{i=1,3,\dots}^{n_2-2=1} f(x_i) + 3 \sum_{i=2,\dots}^{n_2-1=2} f(x_i) + 2 \sum_{i=3,6,\dots}^{n_2-3=0} f(x_i) + f(x_{n_1}) \right\}$$

$$I_2 = \left(\frac{3}{8} \times 3.1429\right) \{177.2667 + 3(256.5863) + 3(342.3241) + (\text{skip!}) + 435.2749\}$$

$$I_2 = 6697.2748$$

The mixed (combined) Simpson 1/3 and 3/8 rules give:

$$I = I_1 + I_2 = 4364.1197 + 6697.2748$$

$$I = 11,061.3946$$

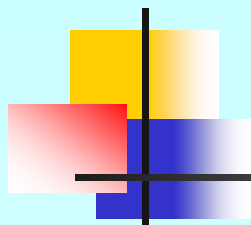


Remarks:

(a) Comparing the truncated error of Simpson 1/3 rule

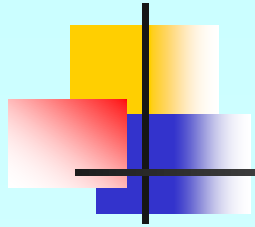
$$E_t = -\frac{(b-a)^5}{2880} \times f''''(\zeta) \quad (18)$$

With Simple 3/8 rule (See Eq. 13), the latter seems to offer slightly more accurate answer than the former. However, the cost associated with Simpson 3/8 rule (using 3rd order polynomial function) is significant higher than the one associated with Simpson 1/3 rule (using 2nd order polynomial function).



(b) The number of multiple segments that can be used in the conjunction with Simpson 1/3 rule is 2,4,6,8,.. (any even numbers).

$$I_1 = \left(\frac{h}{3}\right) \{f(x_0) + 4f(x_1) + f(x_2) + f(x_2) + 4f(x_3) + f(x_4) + \dots + f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\}$$
$$I_2 = \left(\frac{h}{3}\right) \left\{ f(x_0) + 4 \sum_{i=1,3,\dots}^{n-1} f(x_i) + 2 \sum_{i=2,4,6\dots}^{n-2} f(x_i) + f(x_n) \right\} \tag{19}$$



However, Simpson 3/8 rule can be used with the number of segments equal to 3,6,9,12,.. (can be either certain odd or even numbers).

(c) If the user wishes to use, say 7 segments, then the mixed Simpson 1/3 rule (for the first 4 segments), and Simpson 3/8 rule (for the last 3 segments).

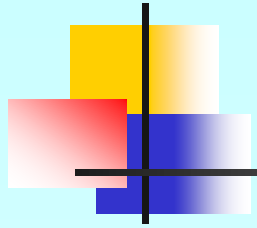
4. Computer Algorithm For Mixed Simpson 1/3 and 3/8 rule For Integration

Based on the earlier discussions on (Single and Multiple segments) Simpson 1/3 and 3/8 rules, the following "pseudo" step-by-step mixed Simpson rules can be given as

Step 1 User's input information, such as

Given function $f(x)$, integral limits " a, b ",

n_1 = number of small, "h" segments, in conjunction with Simpson 1/3 rule.



n_2 = number of small, “h” segments, in conjunction with Simpson 3/8 rule.

Notes:

n_1 = a multiple of 2 (any even numbers)

n_2 = a multiple of 3 (can be certain odd, or even numbers)



Step 2

Compute $n = n_1 + n_2$

$$h = \frac{b - a}{n}$$

$$x_0 = a$$

$$x_1 = a + 1h$$

$$x_2 = a + 2h$$

•

•

$$x_i = a + ih$$

•

•

$$x_n = a + nh = b$$



Step 3

Compute “multiple segments” Simpson 1/3 rule (See Eq. 19)

$$I_1 = \left(\frac{h}{3}\right) \left\{ f(x_0) + 4 \sum_{i=1,3,\dots}^{n_1-1} f(x_i) + 2 \sum_{i=2,4,6\dots}^{n_1-2} f(x_i) + f(x_n) \right\}$$

(19, repeated)



Step 4

Compute "multiple segments" Simpson 3/8 rule (See Eq. 17)

$$I_2 = \left(\frac{3h}{8}\right) \left\{ f(x_0) + 3 \sum_{i=1,4,7\dots}^{n_2-2} f(x_i) + 3 \sum_{i=2,5,8\dots}^{n_2-1} f(x_i) + 2 \sum_{i=3,6,9,\dots}^{n_2-3} f(x_i) + f(x_{n_2}) \right\}$$

(17, repeated)

Step 5

$$I = I_1 + I_2 \quad (20)$$

and print out the final approximated answer for I .

THE END

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Acknowledgement

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