

## Chapter 07.06

# Integrating Discrete Functions

*After reading this chapter, you should be able to:*

1. *integrate discrete functions by several methods,*
2. *derive the formula for trapezoidal rule with unequal segments, and*
3. *solve examples of finding integrals of discrete functions.*

### **What is integration?**

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. You can read about a few of these applications in different engineering majors in Chapters 07.00A-07.00G.

Sometimes, the function to be integrated is given at discrete data points, and the area under the curve is needed to be approximated. Here, we will discuss the integration of such discrete functions,

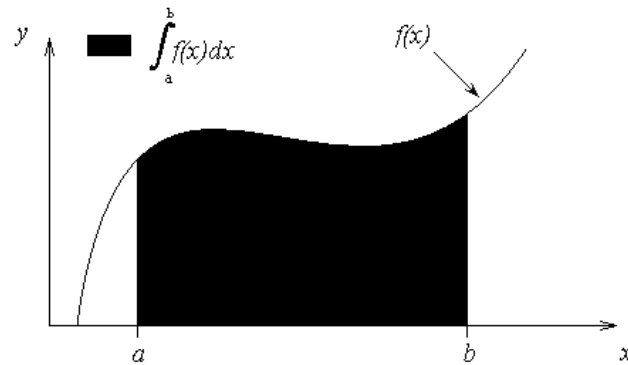
$$I = \int_a^b f(x)dx$$

where

$f(x)$  is called the integrand and is given at discrete value of  $x$ ,

$a$  = lower limit of integration

$b$  = upper limit of integration



**Figure 1** Integration of a function

### Integrating discrete functions

Multiple methods of integrating discrete functions are shown below using an example.

#### Example 1

The upward velocity of a rocket is given as a function of time in Table 1.

**Table 1** Velocity as a function of time.

$t$ (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Determine the distance,  $s$ , covered by the rocket from  $t = 11$  to  $t = 16$  using the velocity data provided and use any applicable numerical technique.

### Solution

#### Method 1: Average Velocity Method

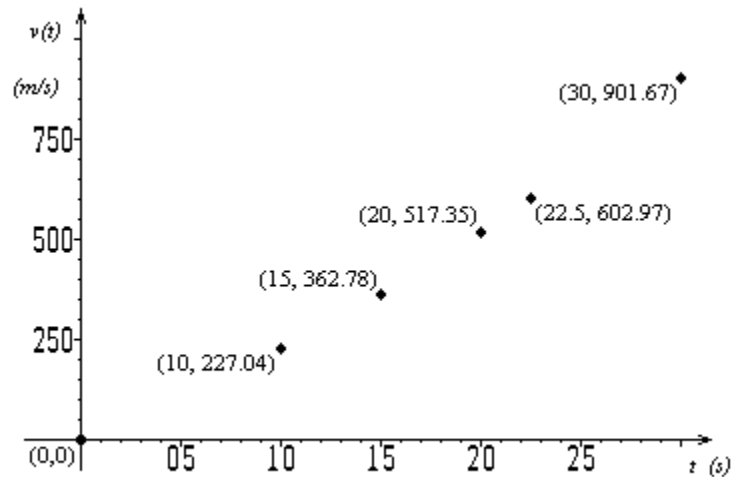
The velocity of the rocket is not provided at  $t = 11$  and  $t = 16$ , so we will have to use an interval that includes  $[11, 16]$  to find the average velocity of the rocket within that range. In this case, the interval  $[10, 20]$  will suffice.

$$v(10) = 227.04$$

$$v(15) = 362.78$$

$$v(20) = 517.35$$

$$\begin{aligned}
 \text{Average Velocity} &= \frac{v(10) + v(15) + v(20)}{3} \\
 &= \frac{227.04 + 362.78 + 517.35}{3} \\
 &= 369.06 \text{ m/s}
 \end{aligned}$$



**Figure 1** Velocity vs. time data for the rocket example

Using

$$s = \bar{v}\Delta t,$$

we get

$$s = (369.06)(16 - 11) = 1845.3 \text{ m}$$

### Method 2: Trapezoidal Rule

If we were finding the distance traveled between times in the data table, we would simply find the area of the trapezoids with the corner points given as the velocity and time data points. For example

$$\int_{10}^{20} v(t) dt = \int_{10}^{15} v(t) dt + \int_{15}^{20} v(t) dt$$

and applying the trapezoidal rule over each of the above integrals gives

$$\int_{10}^{20} v(t) dt \approx \frac{15-10}{2} [v(10) + v(15)] + \frac{20-15}{2} [v(15) + v(20)]$$

The values of  $v(10)$ ,  $v(15)$  and  $v(20)$  are given in Table 1.

However, we are interested in finding

$$\int_{11}^{16} v(t) dt = \int_{11}^{15} v(t) dt + \int_{15}^{16} v(t) dt$$

and applying the trapezoidal rule over each of the above integrals gives

$$\begin{aligned}\int_{11}^{16} v(t) dt &\approx \frac{15-11}{2} [v(11) + v(15)] + \frac{16-15}{2} [v(15) + v(16)] \\ &= \frac{15-11}{2} (v(11) + 362.78) + \frac{16-15}{2} (362.78 + v(16))\end{aligned}$$

How do we find  $v(11)$  and  $v(16)$ ? We use linear interpolation. To find  $v(11)$ ,

$$\begin{aligned}v(t) &= 227.04 + 27.148(t - 10), \quad 10 \leq t \leq 15 \\ v(11) &= 227.04 + 27.148(11 - 10) \\ &= 254.19 \text{ m/s}\end{aligned}$$

and to find  $v(16)$

$$\begin{aligned}v(t) &= 362.78 + 30.913(t - 15), \quad 15 \leq t \leq 20 \\ v(16) &= 362.78 + 30.913(16 - 15) \\ &= 393.69 \text{ m/s}\end{aligned}$$

Then

$$\begin{aligned}\int_{11}^{16} v(t) dt &\approx \frac{15-11}{2} (v(11) + 362.78) + \frac{16-15}{2} (362.78 + v(16)) \\ &= \frac{15-11}{2} (254.19 + 362.78) + \frac{16-15}{2} (362.78 + 393.69) \\ &= 1612.2 \text{ m}\end{aligned}$$

### Method 3: Polynomial interpolation to find the velocity profile

Because we are finding the area under the curve from  $[10, 20]$ , we must use three points,  $t = 10$ ,  $t = 15$ , and  $t = 20$ , to fit a quadratic polynomial through the data. Using polynomial interpolation, our resulting velocity function is (refer to notes on direct method of interpolation)

$$v(t) = 12.05 + 17.733t + 0.3766t^2, \quad 10 \leq t \leq 20.$$

Now, we simply take the integral of the quadratic within our limits, giving us

$$\begin{aligned}s &\approx \int_{11}^{16} (12.05 + 17.733t + 0.3766t^2) dt \\ &= \left[ 12.05t + \frac{17.733t^2}{2} + \frac{0.3766t^3}{3} \right]_{11}^{16} \\ &= 12.05(16 - 11) + \frac{17.733}{2} (16^2 - 11^2) + \frac{0.3766}{3} (16^3 - 11^3) \\ &= 1604.3 \text{ m}\end{aligned}$$

### Method 4: Spline interpolation to find the velocity profile

Fitting quadratic splines (refer to notes on spline method of interpolation) through the data results in the following set of quadratics

$$\begin{aligned}v(t) &= 22.704t, & 0 \leq t \leq 10 \\ &= 0.8888t^2 + 4.928t + 88.88, & 10 \leq t \leq 15\end{aligned}$$

$$\begin{aligned}
 &= -0.1356t^2 + 35.66t - 141.61, & 15 \leq t \leq 20 \\
 &= 1.6048t^2 - 33.956t + 554.55, & 20 \leq t \leq 22.5 \\
 &= 0.20889t^2 + 28.86t - 152.13, & 22.5 \leq t \leq 30
 \end{aligned}$$

The value of the integral would then simply be

$$\begin{aligned}
 s &= \int_{11}^{15} v(t) dt + \int_{15}^{16} v(t) dt \\
 &\approx \int_{11}^{15} (0.8888t^2 + 4.928t + 88.88) dt + \int_{15}^{16} (-0.1356t^2 + 35.66t - 141.61) dt \\
 &= \left[ \frac{0.8888t^3}{3} + \frac{4.928t^2}{2} + 88.88t \right]_{11}^{15} + \left[ \frac{-0.1356t^3}{3} + \frac{35.66t^2}{2} - 141.61t \right]_{15}^{16} \\
 &= \frac{0.8888}{3} (15^3 - 11^3) + \frac{4.928}{2} (15^2 - 11^2) + 88.88(15 - 11) \\
 &\quad + \frac{-0.1356}{3} (16^3 - 15^3) + \frac{35.66}{2} (16^2 - 15^2) - 141.61(16 - 15) \\
 &= 1595.9 \text{ m}
 \end{aligned}$$

### Example 2

What is the absolute relative true error for each of the four methods used in Example 1 if the data in Table 1 was actually obtained from the velocity profile of

$$v(t) = \left( 2000 \ln \left[ \frac{140000}{140000 - 2100t} \right] - 9.8t \right),$$

where  $v$  is given in m/s and  $t$  in s.

#### Solution

The distance covered between  $t = 11$  and  $t = 16$  is

$$\begin{aligned}
 s &= \int_{11}^{16} \left( 2000 \ln \left[ \frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt \\
 &= 1604.9 \text{ m}
 \end{aligned}$$

#### Method 1

The approximate value obtained using average velocity method was 1845.3m. Hence, the absolute relative true error,  $|\epsilon_t|$ , is

$$\begin{aligned}
 |\epsilon_t| &= \left| \frac{1604.9 - 1845.3}{1604.9} \right| \times 100\% \\
 &= 14.976\%
 \end{aligned}$$

#### Method 2:

The approximate value obtained using the trapezoidal rule was 1612.2 m. Hence, the absolute relative true error,  $|\epsilon_t|$ , is

$$|\epsilon_t| = \left| \frac{1604.9 - 1612.2}{1604.9} \right| \times 100\% \\ = 0.451\%$$

**Method 3:**

The approximate value obtained using the direct polynomial was 1604.3 m. Hence, the absolute relative true error,  $|\epsilon_t|$ , is

$$|\epsilon_t| = \left| \frac{1604.9 - 1604.3}{1604.9} \right| \times 100\% \\ = 0.037\%$$

**Method 4:**

The approximate value obtained using the spline interpolation was 1595.9 m, hence, the absolute relative true error,  $|\epsilon_t|$ , is

$$|\epsilon_t| = \left| \frac{1604.9 - 1595.9}{1604.9} \right| \times 100\% \\ = 0.564\%$$

**Table 2** Comparison of discrete function methods of numerical integration

Method	Approximate Value	$ \epsilon_t $
Average Velocity	1845.3	14.976%
Trapezoidal Rule	1612.2	0.451%
Polynomial Interpolation	1604.3	0.037%
Spline Interpolation	1595.9	0.564%

**Trapezoidal Rule for Discrete Functions with Unequal Segments**

For a general case of a function given at  $n$  data points  $(x_1, f(x_1))$ ,  $(x_2, f(x_2))$ ,  $(x_3, f(x_3))$ , ...,  $(x_n, f(x_n))$ , where,  $x_1, x_2, \dots, x_n$  are in an ascending order, the approximate value of the

integral  $\int_{x_1}^{x_n} f(x) dx$  is given by

$$\int_{x_1}^{x_n} f(x) dx = \int_{x_1}^{x_2} f(x) dx + \int_{x_2}^{x_3} f(x) dx + \dots + \int_{x_{n-1}}^{x_n} f(x) dx \\ \approx (x_2 - x_1) \frac{f(x_1) + f(x_2)}{2} + (x_3 - x_2) \frac{f(x_2) + f(x_3)}{2} + \dots \\ \dots + (x_n - x_{n-1}) \frac{f(x_{n-1}) + f(x_n)}{2}$$

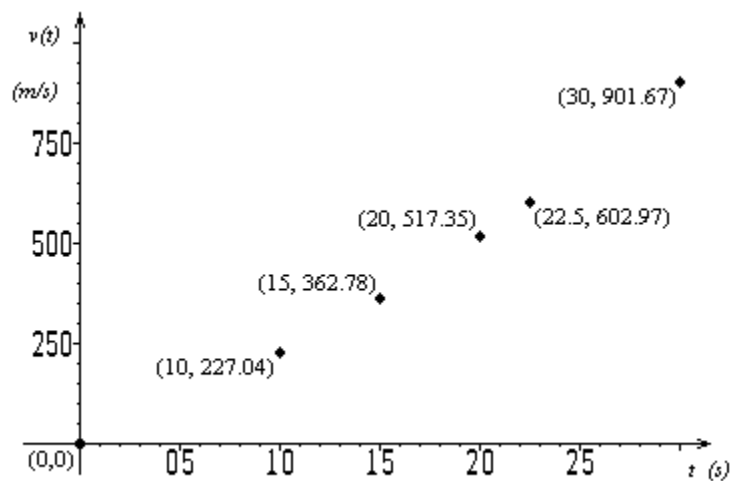
This approach uses the trapezoidal rule in the intervals  $[x_1, x_2]$ ,  $[x_2, x_3]$ , ...,  $[x_{n-1}, x_n]$  and then adds the obtained values.

**Example 3**

The upward velocity of a rocket is given as a function of time in Table 3.

**Table 3.** Velocity as a function of time.

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



**Figure 2** Velocity vs. time data for the rocket example

Determine the distance,  $s$ , covered by the rocket from  $t = 0$  to  $t = 30$  using the velocity data provided and the trapezoidal rule for discrete data with unequal segments.

**Solution**

$$\begin{aligned}
 \int_0^{30} v(t) dt &= \int_0^{10} v(t) dt + \int_{10}^{15} v(t) dt + \int_{15}^{20} v(t) dt + \int_{20}^{22.5} v(t) dt + \int_{22.5}^{30} v(t) dt \\
 &= (10-0) \frac{v(0)+v(10)}{2} + (15-10) \frac{v(10)+v(15)}{2} \\
 &\quad + (20-15) \frac{v(15)+v(20)}{2} + (22.5-20) \frac{v(20)+v(22.5)}{2}
 \end{aligned}$$

$$\begin{aligned}
& + (30 - 22.5) \frac{v(22.5) + v(30)}{2} \\
= & (10) \frac{0 + 227.04}{2} + (5) \frac{227.04 + 362.78}{2} \\
& + (5) \frac{362.78 + 517.35}{2} + (2.5) \frac{517.35 + 602.97}{2} \\
& + (7.5) \frac{602.97 + 901.67}{2} \\
= & 1135.2 + 1474.55 + 2200.325 + 1399.9 + 5642.4 \\
= & 11852 \text{ m}
\end{aligned}$$

Can you find the value of  $\int_{10}^{20} v(t) dt$ ?

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### INTEGRATION

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Topic	Integrating discrete functions
Summary	Textbook notes on integrating discrete functions
Major	All Majors of Engineering
Authors	Autar Kaw
Last Revised	December 23, 2009
Web Site	<a href="http://numericalmethods.eng.usf.edu">http://numericalmethods.eng.usf.edu</a>

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