Chapter 07.03

Simpson's 1/3 Rule of Integration

After reading this chapter, you should be able to

1. *derive the formula for Simpson's 1/3 rule of integration,*
2. *use Simpson's 1/3 rule it to solve integrals,*
3. *develop the formula for multiple-segment Simpson's 1/3 rule of integration,*
4. *use multiple-segment Simpson's 1/3 rule of integration to solve integrals, and*
5. *derive the true error formula for multiple-segment Simpson's 1/3 rule.*

What is integration?

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. You can read about some of these applications in Chapters 07.00A-07.00G.

Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral. Here, we will discuss Simpson's 1/3 rule of integral approximation, which improves upon the accuracy of the trapezoidal rule.

Here, we will discuss the Simpson's 1/3 rule of approximating integrals of the form

$$I = \int_a^b f(x)dx$$

where

$f(x)$ is called the integrand,

$a =$ lower limit of integration

$b =$ upper limit of integration

Simpson's 1/3 Rule

The trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial over interval of integration. Simpson's 1/3 rule is an extension of Trapezoidal rule where the integrand is approximated by a second order polynomial.

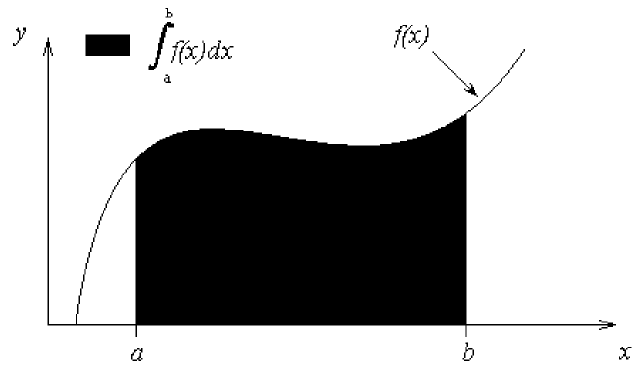

Figure 1 Integration of a function

Method 1:

Hence

$$I = \int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

where $f_2(x)$ is a second order polynomial given by

$$f_2(x) = a_0 + a_1 x + a_2 x^2 .$$

Choose

$$(a, f(a)), \left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right), \text{ and } (b, f(b))$$

as the three points of the function to evaluate a_0, a_1 and a_2.

$$f(a) = f_2(a) = a_0 + a_1 a + a_2 a^2$$

$$f\left(\frac{a+b}{2}\right) = f_2\left(\frac{a+b}{2}\right) = a_0 + a_1\left(\frac{a+b}{2}\right) + a_2\left(\frac{a+b}{2}\right)^2$$

$$f(b) = f_2(b) = a_0 + a_1 b + a_2 b^2$$

Solving the above three equations for unknowns, a_0, a_1 and a_2 give

$$a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left(\frac{a+b}{2}\right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2}$$

$$a_1 = -\frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}$$

$$a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2}$$

Then

$$I \approx \int_a^b f_2(x)dx$$

$$= \int_a^b \left(a_0 + a_1 x + a_2 x^2\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3}\right]_a^b$$

$$= a_0(b-a) + a_1 \frac{b^2 - a^2}{2} + a_2 \frac{b^3 - a^3}{3}$$

Substituting values of a_0, a_1 and a_2 give

$$\int_a^b f_2(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Since for Simpson 1/3 rule, the interval $[a,b]$ is broken into 2 segments, the segment width

$$h = \frac{b-a}{2}$$

Hence the Simpson's 1/3 rule is given by

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Since the above form has 1/3 in its formula, it is called Simpson's 1/3 rule.

<u>Method 2:</u>

Simpson's 1/3 rule can also be derived by approximating $f(x)$ by a second order polynomial using Newton's divided difference polynomial as

$$f_2(x) = b_0 + b_1(x-a) + b_2(x-a)\left(x - \frac{a+b}{2}\right)$$

where

$$b_0 = f(a)$$

$$b_1 = \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a}$$

$$b_2 = \frac{\dfrac{f(b) - f\left(\dfrac{a+b}{2}\right)}{b - \dfrac{a+b}{2}} - \dfrac{f\left(\dfrac{a+b}{2}\right) - f(a)}{\dfrac{a+b}{2} - a}}{b-a}$$

Integrating Newton's divided difference polynomial gives us

$$\int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

$$= \int_a^b \left[b_0 + b_1(x-a) + b_2(x-a)\left(x - \frac{a+b}{2}\right)\right]dx$$

$$= \left[b_0 x + b_1\left(\frac{x^2}{2} - ax\right) + b_2\left(\frac{x^3}{3} - \frac{(3a+b)x^2}{4} + \frac{a(a+b)x}{2}\right)\right]_a^b$$

$$= b_0(b-a) + b_1\left(\frac{b^2 - a^2}{2} - a(b-a)\right)$$

$$+ b_2\left(\frac{b^3 - a^3}{3} - \frac{(3a+b)(b^2 - a^2)}{4} + \frac{a(a+b)(b-a)}{2}\right)$$

Substituting values of b_0, b_1, and b_2 into this equation yields the same result as before

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Method 3:

One could even use the Lagrange polynomial to derive Simpson's formula. Notice any method of three-point quadratic interpolation can be used to accomplish this task. In this case, the interpolating function becomes

$$f_2(x) = \frac{\left(x - \dfrac{a+b}{2}\right)(x-b)}{\left(a - \dfrac{a+b}{2}\right)(a-b)} f(a) + \frac{(x-a)(x-b)}{\left(\dfrac{a+b}{2} - a\right)\left(\dfrac{a+b}{2} - b\right)} f\left(\frac{a+b}{2}\right) + \frac{(x-a)\left(x - \dfrac{a+b}{2}\right)}{(b-a)\left(b - \dfrac{a+b}{2}\right)} f(b)$$

Integrating this function gets

$$\int_a^b f_2(x)dx = \left[\frac{\frac{x^3}{3}-\frac{(a+3b)x^2}{4}+\frac{b(a+b)x}{2}}{\left(a-\frac{a+b}{2}\right)(a-b)}f(a)+\frac{\frac{x^3}{3}-\frac{(a+b)x^2}{2}+abx}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}f\left(\frac{a+b}{2}\right) + \frac{\frac{x^3}{3}-\frac{(3a+b)x^2}{4}+\frac{a(a+b)x}{2}}{(b-a)\left(b-\frac{a+b}{2}\right)}f(b)\right]_a^b$$

$$= \frac{\frac{b^3-a^3}{3}-\frac{(a+3b)(b^2-a^2)}{4}+\frac{b(a+b)(b-a)}{2}}{\left(a-\frac{a+b}{2}\right)(a-b)}f(a)$$

$$+\frac{\frac{b^3-a^3}{3}-\frac{(a+b)(b^2-a^2)}{2}+ab(b-a)}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}f\left(\frac{a+b}{2}\right)$$

$$+\frac{\frac{b^3-a^3}{3}-\frac{(3a+b)(b^2-a^2)}{4}+\frac{a(a+b)(b-a)}{2}}{(b-a)\left(b-\frac{a+b}{2}\right)}f(b)$$

Believe it or not, simplifying and factoring this large expression yields you the same result as before

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

$$= \frac{h}{3}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].$$

<u>Method 4:</u>
Simpson's 1/3 rule can also be derived by the method of coefficients. Assume

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f\left(\frac{a+b}{2}\right) + c_3 f(b)$$

Let the right-hand side be an exact expression for the integrals $\int_a^b 1dx$, $\int_a^b xdx$, and $\int_a^b x^2dx$. This implies that the right hand side will be exact expressions for integrals of any linear combination of the three integrals for a general second order polynomial. Now

$$\int_a^b 1dx = b-a = c_1 + c_2 + c_3$$

$$\int_a^b x dx = \frac{b^2 - a^2}{2} = c_1 a + c_2 \frac{a+b}{2} + c_3 b$$

$$\int_a^b x^2 dx = \frac{b^3 - a^3}{3} = c_1 a^2 + c_2 \left(\frac{a+b}{2}\right)^2 + c_3 b^2$$

Solving the above three equations for c_0, c_1 and c_2 give

$$c_1 = \frac{b-a}{6}$$

$$c_2 = \frac{2(b-a)}{3}$$

$$c_3 = \frac{b-a}{6}$$

This gives

$$\begin{aligned}\int_a^b f(x)dx &\approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b) \\ &= \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \\ &= \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\end{aligned}$$

The integral from the first method

$$\int_a^b f(x)dx \approx \int_a^b (a_0 + a_1 x + a_2 x^2) dx$$

can be viewed as the area under the second order polynomial, while the equation from Method 4

$$\int_a^b f(x)dx \approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b)$$

can be viewed as the sum of the areas of three rectangles.

Example 1

The distance covered by a rocket in meters from $t = 8s$ to $t = 30s$ is given by

$$x = \int_8^{30}\left(2000\ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t\right)dt$$

a) Use Simpson's 1/3 rule to find the approximate value of x.
b) Find the true error, E_t.
c) Find the absolute relative true error, $|\in_t|$.

Solution

a) $$x \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

$$a = 8$$
$$b = 30$$
$$\frac{a+b}{2} = 19$$

$$f(t) = 2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t$$

$$f(8) = 2000\ln\left[\frac{140000}{140000-2100(8)}\right]-9.8(8) = 177.27 m/s$$

$$f(30) = 2000\ln\left[\frac{140000}{140000-2100(30)}\right]-9.8(30) = 901.67 m/s$$

$$f(19) = 2000\ln\left(\frac{140000}{140000-2100(19)}\right)-9.8(19) = 484.75 m/s$$

$$\begin{aligned} x &\approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \\ &= \left(\frac{30-8}{6}\right)[f(8)+4f(19)+f(30)] \\ &= \frac{22}{6}[177.27+4\times 484.75+901.67] \\ &= 11065.72 \text{ m} \end{aligned}$$

b) The exact value of the above integral is

$$x = \int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$
$$= 11061.34 \text{ m}$$

So the true error is

$$\begin{aligned} E_t &= True\ Value - Approximate\ Value \\ &= 11061.34 - 11065.72 \\ &= -4.38\ m \end{aligned}$$

c) The absolute relative true error is

$$\begin{aligned} |\in_t| &= \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100 \\ &= \left|\frac{-4.38}{11061.34}\right| \times 100 \end{aligned}$$

$= 0.0396\%$

Multiple-segment Simpson's 1/3 Rule

Just like in multiple-segment trapezoidal rule, one can subdivide the interval $[a,b]$ into n segments and apply Simpson's 1/3 rule repeatedly over every two segments. Note that n needs to be even. Divide interval $[a,b]$ into n equal segments, so that the segment width is given by

$$h = \frac{b-a}{n}.$$

Now

$$\int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx$$

where

$$x_0 = a$$

$$x_n = b$$

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + ... + \int_{x_{n-4}}^{x_{n-2}} f(x)dx + \int_{x_{n-2}}^{x_n} f(x)dx$$

Apply Simpson's 1/3rd Rule over each interval,

$$\int_a^b f(x)dx \cong (x_2 - x_0)\left[\frac{f(x_0)+4f(x_1)+f(x_2)}{6}\right] + (x_4 - x_2)\left[\frac{f(x_2)+4f(x_3)+f(x_4)}{6}\right] + ...$$

$$+ (x_{n-2} - x_{n-4})\left[\frac{f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})}{6}\right] + (x_n - x_{n-2})\left[\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right]$$

Since

$$x_i - x_{i-2} = 2h$$

$$i = 2, 4, ..., n$$

then

$$\int_a^b f(x)dx \cong 2h\left[\frac{f(x_0)+4f(x_1)+f(x_2)}{6}\right] + 2h\left[\frac{f(x_2)+4f(x_3)+f(x_4)}{6}\right] + ...$$

$$+ 2h\left[\frac{f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})}{6}\right] + 2h\left[\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4\{f(x_1) + f(x_3) + ... + f(x_{n-1})\} + 2\{f(x_2) + f(x_4) + ... + f(x_{n-2})\} + f(x_n)\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(x_i) + f(x_n) \right]$$

$$\boxed{\int_a^b f(x)dx \cong \frac{b-a}{3n}\left[f(x_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(x_i) + f(x_n) \right]}$$

Example 2

Use 4-segment Simpson's 1/3 rule to approximate the distance covered by a rocket in meters from $t = 8$ s to $t = 30$ s as given by

$$x = \int_8^{30} \left(2000\ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t \right) dt$$

a) Use four segment Simpson's 1/3rd Rule to estimate *x*.
b) Find the true error, E_t for part (a).
c) Find the absolute relative true error, $|\in_t|$ for part (a).

Solution:

a) Using n segment Simpson's 1/3 rule,

$$x \approx \frac{b-a}{3n}\left[f(t_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(t_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(t_i) + f(t_n) \right]$$

$$n = 4$$
$$a = 8$$
$$b = 30$$
$$h = \frac{b-a}{n}$$
$$= \frac{30-8}{4}$$
$$= 5.5$$

$$f(t) = 2000\ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t$$

So

$$f(t_0) = f(8)$$

$$f(8) = 2000\ln\left[\frac{140000}{140000 - 2100(8)}\right] - 9.8(8) = 177.27 m/s$$

$$f(t_1) = f(8 + 5.5) = f(13.5)$$

$$f(13.5) = 2000\ln\left[\frac{140000}{140000-2100(13.5)}\right] - 9.8(13.5) = 320.25 m/s$$

$$f(t_2) = f(13.5+5.5) = f(19)$$

$$f(19) = 2000\ln\left(\frac{140000}{140000-2100(19)}\right) - 9.8(19) = 484.75 m/s$$

$$f(t_3) = f(19+5.5) = f(24.5)$$

$$f(24.5) = 2000\ln\left[\frac{140000}{140000-2100(24.5)}\right] - 9.8(24.5) = 676.05 m/s$$

$$f(t_4) = f(t_n) = f(30)$$

$$f(30) = 2000\ln\left[\frac{140000}{140000-2100(30)}\right] - 9.8(30) = 901.67 m/s$$

$$\begin{aligned}
x &= \frac{b-a}{3n}\left[f(t_0) + 4\sum_{\substack{i=1\\i=odd}}^{n-1} f(t_i) + 2\sum_{\substack{i=2\\i=even}}^{n-2} f(t_i) + f(t_n)\right]\\
&= \frac{30-8}{3(4)}\left[f(8) + 4\sum_{\substack{i=1\\i=odd}}^{3} f(t_i) + 2\sum_{\substack{i=2\\i=even}}^{2} f(t_i) + f(30)\right]\\
&= \frac{22}{12}\left[f(8) + 4f(t_1) + 4f(t_3) + 2f(t_2) + f(30)\right]\\
&= \frac{11}{6}\left[f(8) + 4f(13.5) + 4f(24.5) + 2f(19) + f(30)\right]\\
&= \frac{11}{6}\left[177.27 + 4(320.25) + 4(676.05) + 2(484.75) + 901.67\right]\\
&= 11061.64\ m
\end{aligned}$$

b) The exact value of the above integral is

$$x = \int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t\right)dt$$

=11061.34 m

So the true error is

$$E_t = True\ Value - Approximate\ Value$$

$$E_t = 11061.34 - 11061.64$$

$$= -0.30\ m$$

c) The absolute relative true error is

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100$$
$$= \left|\frac{-0.3}{11061.34}\right| \times 100$$
$$= 0.0027\%$$

Table 1 Values of Simpson's 1/3 rule for Example 2 with multiple-segments

n	Approximate Value	E_t	$\|\in_t\|$
2	11065.72	-4.38	0.0396%
4	11061.64	-0.30	0.0027%
6	11061.40	-0.06	0.0005%
8	11061.35	-0.02	0.0002%
10	11061.34	-0.01	0.0001%

Error in Multiple-segment Simpson's 1/3 rule

The true error in a single application of Simpson's 1/3rd Rule is given[1] by

$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\zeta), \quad a < \zeta < b$$

In multiple-segment Simpson's 1/3 rule, the error is the sum of the errors in each application of Simpson's 1/3 rule. The error in the n segments Simpson's 1/3rd Rule is given by

$$E_1 = -\frac{(x_2 - x_0)^5}{2880} f^{(4)}(\zeta_1), \quad x_0 < \zeta_1 < x_2$$
$$= -\frac{h^5}{90} f^{(4)}(\zeta_1)$$
$$E_2 = -\frac{(x_4 - x_2)^5}{2880} f^{(4)}(\zeta_2), \quad x_2 < \zeta_2 < x_4$$
$$= -\frac{h^5}{90} f^{(4)}(\zeta_2)$$
$$\vdots$$
$$E_i = -\frac{(x_{2i} - x_{2(i-1)})^5}{2880} f^{(4)}(\zeta_i), \quad x_{2(i-1)} < \zeta_i < x_{2i}$$
$$= -\frac{h^5}{90} f^{(4)}(\zeta_i)$$
$$\vdots$$

[1] The $f^{(4)}$ in the true error expression stands for the fourth derivative of the function $f(x)$.

$$E_{\frac{n}{2}-1} = -\frac{(x_{n-2}-x_{n-4})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right), \quad x_{n-4} < \zeta_{\frac{n}{2}-1} < x_{n-2}$$

$$= -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right)$$

$$E_{\frac{n}{2}} = -\frac{(x_n-x_{n-2})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}}\right), x_{n-2} < \zeta_{\frac{n}{2}} < x_n$$

Hence, the total error in the multiple-segment Simpson's 1/3 rule is

$$= -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}}\right)$$

$$E_t = \sum_{i=1}^{\frac{n}{2}} E_i$$

$$= -\frac{h^5}{90} \sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)$$

$$= -\frac{(b-a)^5}{90n^5} \sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)$$

$$= -\frac{(b-a)^5}{180n^4} \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{\frac{n}{2}},$$

The term $\dfrac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{\frac{n}{2}}$ is an approximate average value of $f^{(4)}(x), a < x < b$. Hence

$$E_t = -\frac{(b-a)^5}{180n^4} \overline{f}^{(4)}$$

where

$$\overline{f}^{(4)} = \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{\frac{n}{2}}$$

INTEGRATION	
Topic	Simpson's 1/3 rule
Summary	Textbook notes of Simpson's 1/3 rule
Major	General Engineering
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