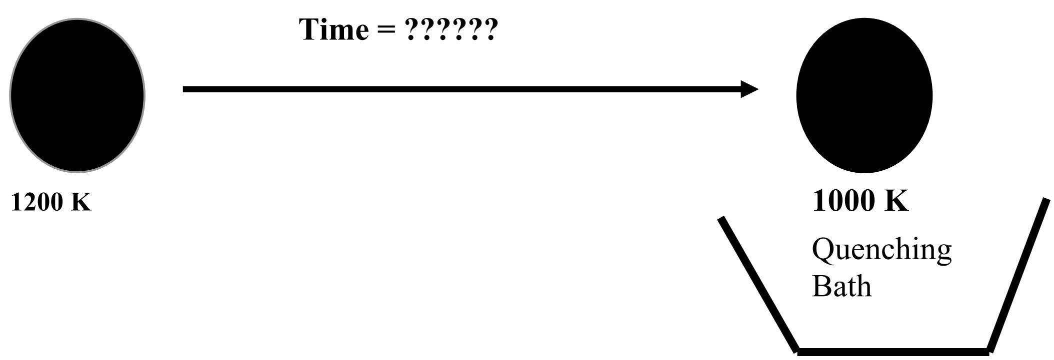
Chapter 08.00A

Physical Problem for Ordinary Differential Equations
General Engineering



Problem
You are working for a ball bearing company - Ralph's Bearings. For long lasting life of some spherical bearings made in this company, they need to be quenched for 30 seconds in water that is maintained at a room temperature of 300K after heating them to a high temperature of 1000K. However, it takes time to take the ball from the furnace to the quenching bath and its temperature falls. If the temperature of the furnace is 1200K and it takes 10 seconds to take the ball to the quenching bath. Is less or more time needed to get to a temperature of 1000K?

Mathematical model
To keep the mathematical simple, one can assume the spherical bearing to be a lumped mass system. What does a lumped system mean? It implies that the internal conduction in the sphere is large enough that the temperature throughout the ball is uniform. This allows us to make the assumption that the temperature is only a function of time and not of the location in the spherical ball. This means that if a differential equation governs this physical problem, it would be an ordinary differential equation for a lumped system and a partial differential equation for a non-lumped system. In your heat transfer course, you will learn when a system can be considered lumped or nonlumped. In simplistic terms, this distinction is based on the material, geometry and heat exchange factors of the ball with its surroundings.

Now assuming a lumped-mass system, let us develop the mathematical model for the above problem. When the ball is taken out of the furnace at the initial temperature of θ_0 and is cooled by radiation to its surroundings at the temperature of θ_a, the rate at which heat is lost to radiation is

Rate of heat lost due to radiation $= A \in \sigma\left(\theta^4 - {\theta_a}^4\right)$

where

A = surface area of ball, m^2

$\in$ = emittance[1]

σ = Stefan-Boltzmann[2] constant, $5.67\times10^{-8}\ \dfrac{\text{J}}{\text{s.m}^2.\text{K}^4}$

θ = temperature of the ball at a given time, K

The energy stored in the mass is given by

Energy stored by mass $= mC\theta$

where

m = mass of ball, kg

C = specific heat of the ball, $\text{J}/(\text{kg}-\text{K})$

From an energy balance,

Rate at which heat is gained - Rate at which heat is lost =Rate at which heat is stored

gives

$$-A \in \sigma\left(\theta^4 - \theta_a^4\right) = mC\frac{d\theta}{dt}$$

Given the

Radius of ball, $r = 2.0$ cm

Density of ball, $\rho = 7800\ \text{kg/m}^3$

Specific heat, C = 420 J/(kg.K)

Emittance, $\in = 0.85$

Stefan-Boltzmann Constant, $\sigma = 5.67\times10^{-8}\ \text{J}/(\text{s.m}^2.\text{K}^4)$

Initial temperature of the ball, $\theta(0) = 1200\,\text{K}$,

Ambient temperature, $\theta_a = 300\,\text{K}$,

we have

Surface area of the ball

$$A = 4\pi r^2$$
$$= 4\pi(0.02)^2$$

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[1] Emittance is defined as the total radiation emitted divided by total radiation that would be emitted by a blackbody at the same temperature. The emittance is always between 0 and 1. A black body is a body that emits and absorbs at any temperature the maximum possible amount of radiation at any given wavelength.

[2] Stefan-Botzmann constant was discovered by two Austrian scientists – J. Stefan and L. Boltzmann. Stefan found it experimentally in 1879 and Boltzmann derived it theoretically in 1884.

$$= 5.02654 \times 10^{-3}\,\text{m}^2$$

Mass of the ball

$$M = pV$$

$$= \rho\left[\frac{4}{3}\pi r^3\right]$$

$$= 7800 \times \left(\frac{4}{3}\right)\pi(0.02)^3$$

$$= 0.261380 kg$$

Hence

$$-A \in \sigma(\theta^4 - \theta_a{}^4) = mC\frac{d\theta}{dt}$$

reduces to

$$-(5.02654 \times 10^{-3})\,(0.85)\,(5.67 \times 10^{-8})\,(\theta^4 - 300^4) = (0.261380)\,(420)\,\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = -2.20673 \times 10^{-12}\,(\theta^4 - 81 \times 10^8)$$

An improved mathematical model

The heat from the ball can also be lost due to convection. The rate of heat lost due to convection is

Rate of heat lost due to convection $= hA(\theta - \theta_a)$,

where

h = the convective cooling coefficient $\left[\text{W}/\left(\text{m}^2 - \text{K}\right)\right]$.

Hence the heat is lost is due to both, convection and radiation and is given by

Rate of heat lost due to convection and radiation $= A \in \sigma\left(\theta^4 - \theta_a^4\right) + hA(\theta - \theta_a)$

Energy stored by mass $= mC\theta$

From an energy balance,

Rate at which heat is gained -Rate at which heat is lost
=Rate at which heat is stored

gives

$$-A \in \sigma\left(\theta^4 - \theta_a^4\right) - hA(\theta - \theta_a) = mC\frac{d\theta}{dt}$$

Given $h = 350\dfrac{\text{J}}{\text{s} - \text{m}^2 - \text{K}}$ and considering both convection and radiation, and substituting the values of the constants given before

$$-(5.02654 \times 10^{-3})\,(0.85)\,(5.67 \times 10^{-8})\,(\theta^4 - 300^4)$$
$$-(350)\,(5.02654 \times 10^{-3})\,(\theta - 300) = (0.261380)\,(420)\,\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = -2.20673 \times 10^{-13}(\theta^4 - 81 \times 10^8) - 1.60256 \times 10^{-2}\,(\theta - 300)$$

The solution to the above ordinary differential equation with the initial condition of $\theta(0) = 1200\text{K}$ would give us the temperature of the ball as a function of time. We can then find at what time the ball temperature drops to 1000K.

Questions

1. Note that the above ordinary differential equation is non-linear. Is there is an exact solution to the problem?
2. You are asked to solve the inverse problem, that is, when is the dependent variable temperature 1000K. How would you go about solving the inverse problem using different numerical methods such as Euler's and Runge-Kutta methods
3. Can you find if convection or radiation can be neglected? How would you quantify the effect of neglecting one or the other?
4. Find the following at $t = 5\ \text{s}$
   a) Rate of change of temperature,
   b) Rate at which heat is lost due to convection,
   c) Rate at which heat is lost due to radiation,
   d) Rate at which heat is stored in the ball.

ORDINARY DIFFERENTIAL EQUATIONS	
Topic	Ordinary differential equations
Summary	To find the temperature of a heated ball as a function of time, a first-order ordinary differential equation must be solved.
Major	General Engineering
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