

Shooting Method

Major: All Engineering Majors

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Transforming Numerical Methods Education for STEM
Undergraduates

Shooting Method

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Shooting Method

The shooting method uses the methods used in solving initial value problems. This is done by assuming initial values that would have been given if the ordinary differential equation were a initial value problem. The boundary value obtained is compared with the actual boundary value. Using trial and error or some scientific approach, one tries to get as close to the boundary value as possible.

Example

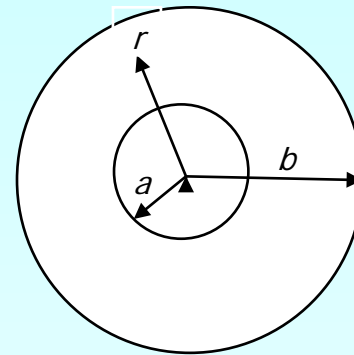
$$\frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} = 0,$$
$$u(5) = 0.0038731,$$
$$u(8) = 0.0030770$$

Let

$$\frac{du}{dr} = w$$

Then

$$\frac{dw}{dr} + \frac{1}{r} w - \frac{u}{r^2} = 0$$



Where $a = 5$
and $b = 8$

Solution

Two first order differential equations are given as

$$\frac{du}{dr} = w, \quad u(5) = 0.0038371$$

$$\frac{dw}{dr} = -\frac{w}{r} + \frac{u}{r^2}, \quad w(5) = \text{not known}$$

Let us assume

$$w(5) = \frac{du}{dr}(5) \approx \frac{u(8) - u(5)}{8 - 5} = -0.00026538$$

To set up initial value problem

$$\frac{du}{dr} = w = f_1(r, u, w), \quad u(5) = 0.0038371$$

$$\frac{dw}{dr} = -\frac{w}{r} + \frac{u}{r^2} = f_2(r, u, w), \quad w(5) = -0.00026538$$

Solution Cont

Using Euler's method,

$$u_{i+1} = u_i + f_1(r_i, u_i, w_i)h$$

$$w_{i+1} = w_i + f_2(r_i, u_i, w_i)h$$

Let us consider 4 segments between the two boundaries, $r = 5$ and $r = 8$ then,

$$h = \frac{8-5}{4} = 0.75$$

Solution Cont

For $i = 0, r_0 = 5, u_0 = 0.0038371, w_0 = -0.00026538$

$$\begin{aligned}u_1 &= u_0 + f_1(r_0, u_0, w_0)h \\&= 0.0038371 + f_1(5, 0.0038371, -0.00026538)(0.75) \\&= 0.0038371 + (-0.00026538)(0.75) \\&= 0.0036741\end{aligned}$$

$$\begin{aligned}w_1 &= w_0 + f_2(r_0, u_0, w_0)h \\&= -0.00026538 + f_2(5, 0.0038371, -0.00026538)(0.75) \\&= -0.00026538 + \left(-\frac{-0.00026538}{5} + \frac{0.0038371}{5^2} \right)(0.75) \\&= -0.00010938\end{aligned}$$

Solution Cont

For $i = 1, r_1 = r_0 + h = 5 + 0.75 = 5.75, u_1 = 0.0036741, w_1 = -0.00010940$

$$\begin{aligned}u_2 &= u_1 + f_1(r_1, u_1, w_1)h \\&= 0.0036741 + f_1(5.75, 0.0036741, -0.00010938)(0.75) \\&= 0.0036741 + (-0.00010938)(0.75) \\&= 0.0035920\end{aligned}$$

$$\begin{aligned}w_2 &= w_1 + f_2(r_1, u_1, w_1)h \\&= -0.00010938 + f_2(5.75, 0.0036741, -0.00010938)(0.75) \\&= -0.00010938 + (0.00013015)(0.75) \\&= -0.000011769\end{aligned}$$

Solution Cont

For $i = 2, r_2 = r_1 + h = 5.75 + 0.75 = 6.5$ $u_2 = 0.0035920, w_2 = -0.000011785$

$$\begin{aligned}u_3 &= u_2 + f_1(r_2, u_2, w_2)h \\&= 0.0035920 + f_1(6.5, 0.0035920, -0.000011769)(0.75) \\&= 0.0035920 + (-0.000011769)(0.75) \\&= 0.0035832\end{aligned}$$

$$\begin{aligned}w_3 &= w_2 + f_2(r_2, u_2, w_2)h \\&= -0.000011769 + f_2(6.5, 0.0035920, -0.000011769)(0.75) \\&= -0.000011769 + (0.000086829)(0.75) \\&= 0.000053352\end{aligned}$$

Solution Cont

For $i = 3, r_3 = r_2 + h = 6.50 + 0.75 = 7.25$ $u_3 = 0.0035832, w_3 = 0.000053332$

$$\begin{aligned}u_4 &= u_3 + f_1(r_3, u_3, w_3)h \\&= 0.0035832 + f_1(7.25, 0.0035832, 0.000053352)(0.75) \\&= 0.0035832 + (0.000053352)(0.75) \\&= 0.0036232\end{aligned}$$

$$\begin{aligned}w_4 &= w_3 + f_2(r_3, u_3, w_3)h \\&= -0.000011785 + f_2(5.75, 0.0035832, -0.000053352)(0.75) \\&= 0.000053352 + (0.000060811)(0.75) \\&= 0.000098961\end{aligned}$$

So at $r = r_4 = r_3 + h = 7.25 + 0.75 = 8$

$$u(8) \approx u_4 = 0.0036232$$

Solution Cont

Let us assume a new value for $\frac{du}{dr}(5)$

$$w(5) = 2 \frac{du}{dr}(5) \approx 2 \frac{u(8) - u(5)}{8 - 5} = 2(-0.00026538) = -0.00053076$$

Using $h = 0.75$ and Euler's method, we get

$$u(8) \approx u_4 = 0.0029665''$$

While the given value of this boundary condition is

$$u(8) \approx u_4 = 0.0030770$$

Solution Cont

Using linear interpolation on the obtained data for the two assumed values of

$\frac{du}{dr}(5)$ we get

$$u(8) = 0.00030770$$

$$\begin{aligned}\frac{du}{dr}(5) &\approx \frac{-0.00053076 - (-0.00026538)}{0.0029645 - 0.0036232} (0.0030770 - 0.0036232) + (-0.00026538) \\ &= -0.00048611\end{aligned}$$

Using $h = 0.75$ and repeating the Euler's method with $w(5) = -0.00048611$

$$u(8) \approx u_4 = 0.0030769$$

Solution Cont

Using linear interpolation to refine the value of u_4

till one gets close to the actual value of $u(8)$ which gives you,

$$u_1 = u(5) = 0.0038731$$

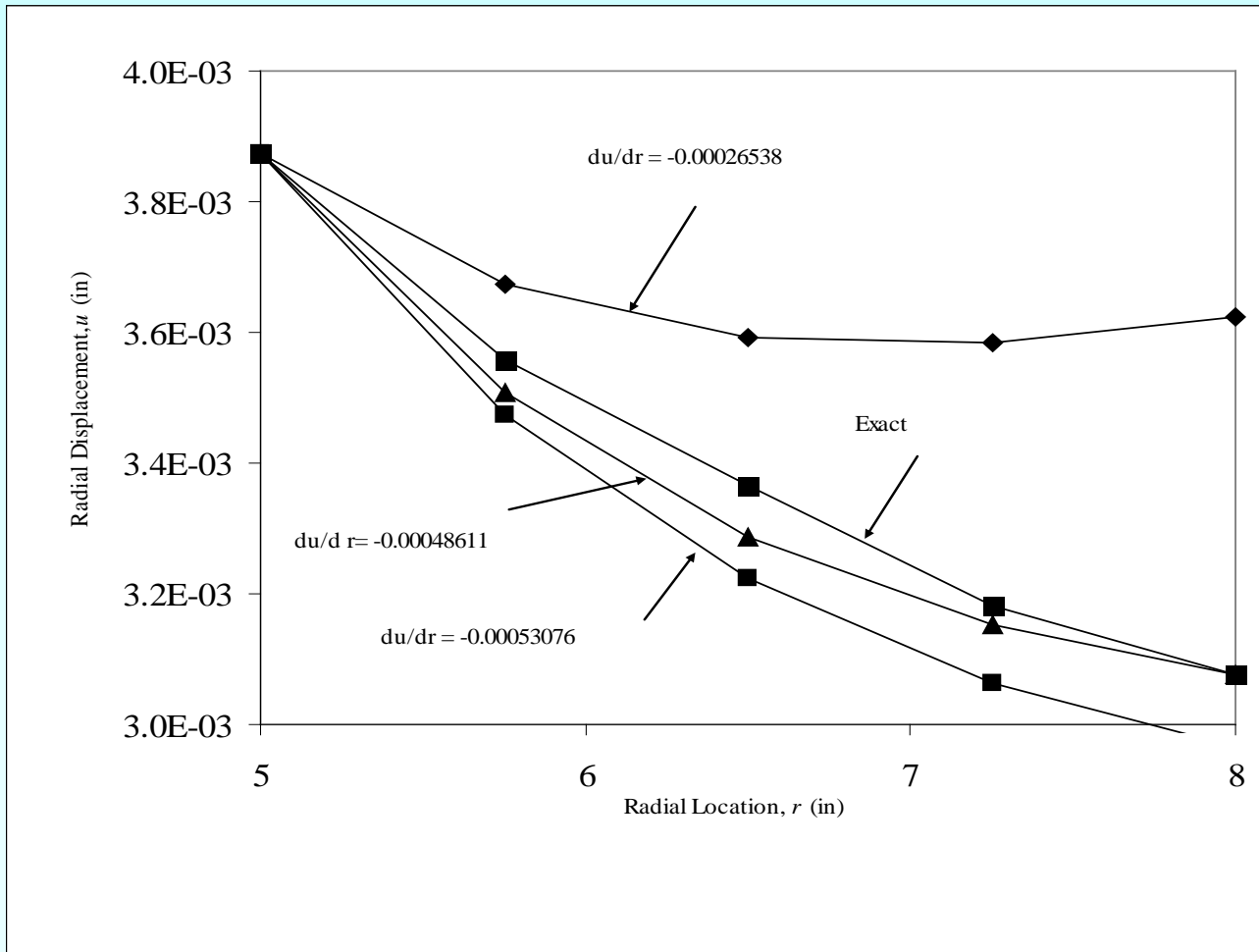
$$u(5.75) \approx u_2 = 0.0035085$$

$$u(6.50) \approx u_3 = 0.0032858$$

$$u(7.25) \approx u_4 = 0.0031518$$

$$u(8.00) \approx u_5 = 0.0030770$$

Comparisons of different initial guesses



Comparison of Euler and Runge-Kutta Results with exact results

Table 1 Comparison of Euler and Runge-Kutta results with exact results.

r (in)	Exact (in)	Euler (in)	$ \epsilon_t \%$	Runge-Kutta (in)	$ \epsilon_t \%$
5	3.8731×10^{-3}	3.8731×10^{-3}	0.0000	3.8731×10^{-3}	0.0000
5.75	3.5567×10^{-3}	3.5085×10^{-3}	1.3731	3.5554×10^{-3}	3.5824×10^{-2}
6.5	3.3366×10^{-3}	3.2858×10^{-3}	1.5482	3.3341×10^{-3}	7.4037×10^{-2}
7.25	3.1829×10^{-3}	3.1518×10^{-3}	9.8967×10^{-1}	3.1792×10^{-3}	1.1612×10^{-1}
8	3.0770×10^{-3}	3.0770×10^{-3}	1.9500×10^{-3}	3.0723×10^{-3}	1.5168×10^{-1}

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/shooting_method.html

THE END

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