

Chapter 09.02

Newton's Method

After reading this chapter, you should be able to:

1. *Understand how Newton's method is different from the Golden Section Search method*
2. *Understand how Newton's method works*
3. *Solve one-dimensional optimization problems using Newton's method*

How is the Newton's method different from the Golden Section Search method?

The Golden Section Search method requires explicitly indicating lower and upper boundaries for the search region in which the optimal solution lies. Such methods where the boundaries need to be specified are known as bracketing approaches in the sense that the optimal solution is bracketed by these boundaries.

Newton's method is an open (instead of bracketing) approach, where the optimum of the one-dimensional function $f(x)$ is found using an initial guess of the optimal value without the need for specifying lower and upper boundary values for the search region.

Unlike the bracketing approaches, open approaches are not guaranteed to converge. However, if they do converge, they do so much faster than bracketed approaches. Therefore, open approaches are more useful if there is reasonable evidence that the initial guess is close to the optimal value. Otherwise, if there is doubt about the quality of the initial guess, it is advisable to use bracketing approaches to bring the guess closer to the optimal value and then use an open approach benefiting from the advantages presented by both techniques.

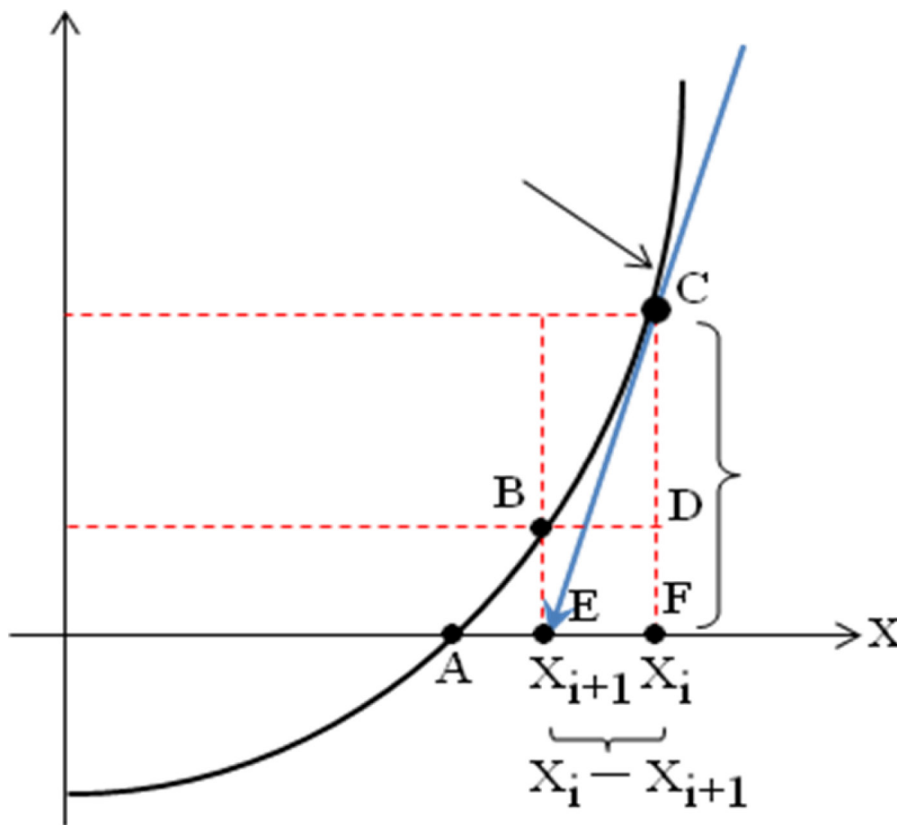
What is the Newton's method and how does it work?

Newton's method is an open approach to find the minimum or the maximum of a function $f(x)$. It is very similar to the Newton-Raphson method http://numericalmethods.eng.usf.edu/topics/newton_raphson.html to find the roots of a function such that $f(x)=0$. Since the derivative of the function $f(x)$, $f'(x)=0$ at the functions maximum and minimum, the minima and the maxima can be found by applying the Newton-Raphson method to the derivative, essentially obtaining

$$x_{i+1} = x_i - \frac{f'(x_i)}{f''(x_i)} \quad (1)$$

We caution that before using Newton's method to determine the minimum or the maximum of a function, one should have a reasonably good estimate of the solution to ensure convergence, and that the function should be easily twice differentiable.

Derivation of the Newton-Raphson Equation



Slope at point

$$C \approx \frac{F(X_i) - F(X_{i+1})}{X_i - X_{i+1}}$$

We "wish" that in the next iteration X_{i+1} will be the root, or $F(X_{i+1}) = 0$.

Thus:

Slope at point

$$C = \frac{F(X_i) - 0}{X_i - X_{i+1}}$$

or

$$F'(X_i) = \frac{F(X_i)}{X_i - X_{i+1}}$$

Hence :

$$X_{i+1} = X_i - \frac{F(X_i)}{F'(X_i)}$$

Remarks:

1. If $F(X) \equiv f'(X)$, then $X_{i+1} = X_i - \frac{f'(X_i)}{f''(X_i)}$
2. For Multi-variable case, then NR method becomes

$$\vec{X}_{i+1} = \vec{X}_i - [f''(X_i)]^{-1} \times \nabla f(X_i)$$

Step by step use of Newton's method

The following algorithm implements Newton's method to determine the maximum or minimum of a function $f(x)$.

Initialization

Determine a reasonably good estimate x_0 for the maxima or the minima of the function $f(x)$.

Step 1

Determine $f'(x)$ and $f''(x)$.

Step 2

Substitute x_{i+1} , the initial estimate x_0 for the first iteration, $f'(x)$ and $f''(x)$ into Eqn. 1 to determine x_i and the function value in iteration i .

Step 3

If the value of the first derivative of the function is zero, then you have reached the optimum (maxima or minima), otherwise repeat Step 2 with the new value of x_i until the absolute relative approximate error is less than the pre-specified tolerance.

Example 1

Consider Figure 1 below. The cross-sectional area A of a gutter with equal base and edge length of 2 is given by

$$A = 4 \sin \theta (1 + \cos \theta)$$

Find the angle θ which maximizes the cross-sectional area of the gutter.

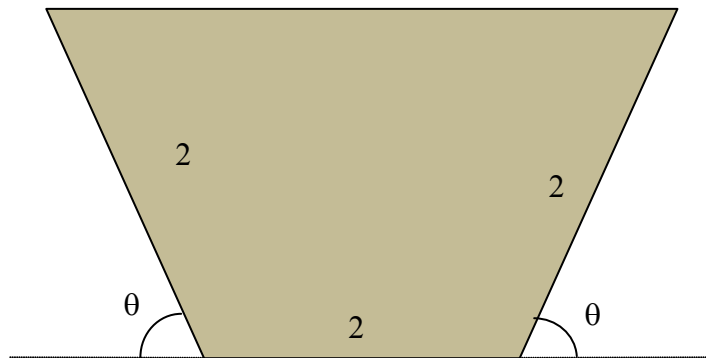


Figure 1: Cross section of the gutter

Solution

The function to be maximized is $f(\theta) = 4 \sin \theta(1 + \cos \theta)$. The first and second derivative of the function is shown below.

$$f'(\theta) = 4(\cos \theta + \cos^2 \theta - \sin^2 \theta)$$

$$f''(\theta) = -4 \sin \theta(1 + 4 \cos \theta)$$

Let us use $\theta_0 = \pi/4$ as the initial estimate of θ . Using Eqn. (1), we can calculate the first iteration follows:

$i = 0$

$$\begin{aligned} \theta_1 &= \theta_0 - \frac{f'(\theta_0)}{f''(\theta_0)} \\ &= \frac{\pi}{4} - \frac{f'\left(\frac{\pi}{4}\right)}{f''\left(\frac{\pi}{4}\right)} \\ &= \frac{\pi}{4} - \frac{4\left(\cos \frac{\pi}{4} + \cos^2 \frac{\pi}{4} - \sin^2 \frac{\pi}{4}\right)}{-4 \sin \frac{\pi}{4} (1 + 4 \cos \frac{\pi}{4})} \\ &= 1.0466 \end{aligned}$$

The function is evaluated at the first estimate as $f(1.0466) = 5.1962$. The next iteration uses $\theta_1 = 1.0466$ as the best estimate of θ . Using Eqn(1) again, the second iteration is calculated as follows:

$i = 1$

$$\begin{aligned}
 \theta_2 &= \theta_1 - \frac{f'(\theta_1)}{f''(\theta_1)} \\
 &= 1.0466 - \frac{f'(1.0466)}{f''(1.0466)} \\
 &= 1.0466 - \frac{4(\cos 1.0466 + \cos^2 1.0466 - \sin^2 1.0466)}{-4 \sin 1.0466(1 + 4 \cos 1.0466)} \\
 &= 1.0472
 \end{aligned}$$

The iterations will continue until the solution converges to a single optimal solution. Summary results of all the iterations are shown in Table 1.

Several important observations regarding the 5th iteration can be made. At each iteration, the magnitude of the first derivative gets smaller and approaches zero. A value of zero of the first derivative tells us that we have reached the optimal and we can stop. Also note that the sign of the second derivative is negative which tells us that we are at a maximum. This value would have been positive if we had reached a minimum. The solution tells us that the optimal angle is 1.0472. Remember that the actual solution to the problem is at 60 degrees or 1.0472 radians. See Example 2 in Golden Search Method http://numericalmethods.eng.usf.edu/topics/opt_goldensearch.html.

Table 1. Summary of iterations for Example 1

Iteration	θ_i	$f'(\theta_i)$	$f''(\theta_i)$	θ_{i+1}	$f(\theta_{i+1})$
1	0.78540	2.8284	-10.828	1.0466	5.1962
2	1.0466	0.0061898	-10.396	1.0472	5.1962
3	1.0472	1.0613E-06	-10.392	1.0472	5.1962
4	1.0472	3.0642E-14	-10.392	1.0472	5.1962
5	1.0472	1.3323E-15	-10.392	1.0472	5.1962

OPTIMIZATION	
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