Chapter 10.00A

Physical Problem for Partial Differential Equations General Engineering

Problem Statement

Large amount of heat is generated inside the Internal Combustion (IC) engines. This heat is to be removed to prevent the over heating of the engine components. IC engines are used in motorcycles and motorcycle engine heads are equipped with array of rectangular fins to dissipate heat generated inside the engine (Figure 1).

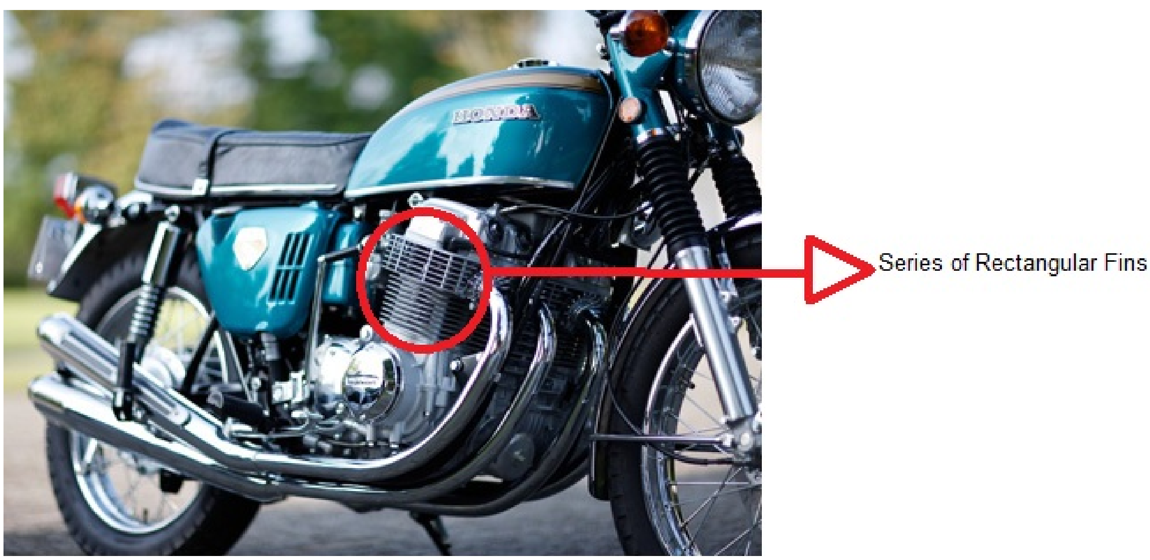

Figure 1: Motorcycle engine head equipped with array of rectangular fins

In order to design the cooling system for the motorcycle engine, one has to calculate the heat transfer from one rectangular fin. The temperature inside the engine should not exceed a maximum threshold of T_{engine} temperature in order for the engine to perform with out failure. In order to make the problem simple, lets assume the surfaces of the fin which are exposed to air (ambient temperature, $T_{air} = 27°C$) are at uniform ambient temperature. This could be modeled as shown in Figure 2. During the design of the fin it is very important to know the temperature distribution along the fin.

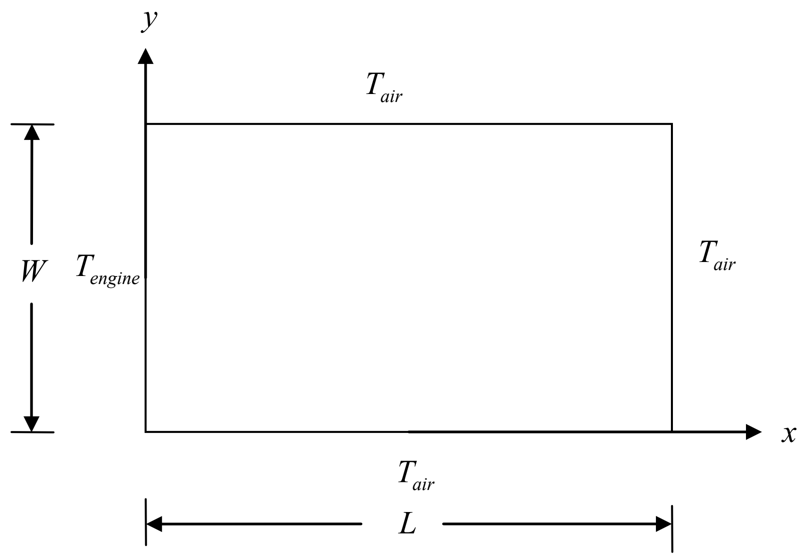

Figure 2: Model representing the single rectangular fin with boundary conditions

Note: Convective heat transfer (forced convection) occurs from the surfaces of the fin which are expose to air. In order to keep the problem simple boundary conditions are simplified.

System Model

According to fourier's law of conduction heat flow along one dimension is given by

$$q(x) = -k_x \frac{\partial T}{\partial x} \tag{1}$$

where

$q(x) =$ Heat flow per unit time per unit area

$k_x =$ Thermal conductivity of the material of the body along x - direction

$T =$ Temperature

Heat conducted in or out

$$\begin{aligned} q_x &= q(x+dx) - q(x) \\ &= \left(q(x) + \frac{dq}{dx}\right) - q(x) \\ &= \frac{dq}{dx} \\ &= \frac{d}{dx}(q) \end{aligned} \tag{2}$$

Here in one dimensional equation, heat conducted is a function of x on only, so the total differential operator and the partial differential operator are the same, i.e. $\frac{d}{dx}(q) = \frac{\partial}{\partial x}(q)$.

Substituting in equation (2) we have

$$q_x = \frac{\partial}{\partial x}(q) \qquad (3)$$

substituting equation (3) in equation (1), we have

$$q_x = \frac{\partial}{\partial x}\left(-k_x \frac{\partial T}{\partial x}\right) \qquad (4)$$

Now if we extend equation (4) for 3-dimensional heat flow due to conduction

$$\begin{aligned} q_{conduction} &= q_x + q_y + q_z \\ &= \frac{\partial}{\partial x}\left(-k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(-k_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(-k_z \frac{\partial T}{\partial z}\right) \\ &= -\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) - \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) - \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z}\right) \end{aligned} \qquad (5)$$

where

$k_x =$ Thermal conductivity of the material of the body along x - direction

$k_y =$ Thermal conductivity of the material of the body along y - direction

$k_z =$ Thermal conductivity of the material of the body along z - direction

Let heat generation rate be a function of x, y and z.

$$q_{generated} = g(x, y, z) \qquad (6)$$

Rate of heat stored in the body is given by

$$q_{stiored} = \rho C_p \frac{\partial T}{\partial t} \qquad (7)$$

where

$\rho =$ Density of the material

$C_p =$ Specific heat of the material of the body

$t =$ Time

According to the law of conservation of mass,

Rate of total heat energy generated = rate of heat stored inside body + rate of heat conducted by the body

$$q_{generated} = q_{stored} + q_{conduction} \qquad (8)$$

substituting equation (5), equation (6) and equation (7) in equation (8), we have

$$g(x, y, z) = \rho C_p \frac{\partial T}{\partial t} - \frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) - \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) - \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z}\right)$$

re-writing the above equation we have the three dimensional heat conduction equation

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z}\right) + g(x, y, z) = \rho C_p \frac{\partial T}{\partial t} \qquad (9)$$

Here we are analyzing a plate (2-dimensional body), so the differential terms along z-direction is zero, i.e. $\frac{\partial}{\partial z} = 0$. Substituting this in equation (9) gives

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + g(x, y) = \rho C_p \frac{\partial T}{\partial t} \tag{10}$$

In this problem there is no heat generation inside the plate, so $g(x, y) = 0$ and also we are solving for steady state temperature, i.e. there is no change of temperature with time, $\frac{\partial T}{\partial t} = 0$. Substituting these conditions in equation (10) gives

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) = 0 \tag{11}$$

The plate material is made of isotropic material, i.e. The properties of the material are same in all directions, $k_x = k_y = k$.

$$\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) = 0$$

$$k\left(\frac{\partial^2 T}{\partial x^2}\right) + k\left(\frac{\partial^2 T}{\partial y^2}\right) = 0$$

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = 0$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{12}$$

Equation (12) represents the mathematical model of the steady state heat conduction inside a plate made of isotropic material with no heat generation.

Boundary Conditions

$$T(0, y) = T_{engine} \tag{13}$$

$$T(L, y) = T_{air} \tag{14}$$

$$T(x,0) = T_{air} \tag{15}$$

$$T(x, W) = T_{air} \tag{16}$$

Questions

1. Solve the mathematical model represented by equation 12 with the boundary conditions to find the temperature distribution along the fin.

PARTIAL DIFFERENTIAL EQUATIONS	
Topic	Physical problem for partial differential equations for general engineering
Summary	A physical problem of ????
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