Elliptic Partial Differential Equations

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Transforming Numerical Methods Education for STEM Undergraduates

Defining Elliptic PDE's

- The general form for a second order linear PDE with two independent variables (x, y) and one dependent variable (u) is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

- Recall the criteria for an equation of this type to be considered elliptic

$$B^2 - 4AC < 0$$

- For example, examine the Laplace equation given by

$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$, where $A = 1$, $B = 0$, $C = 1$ and $D = 0$

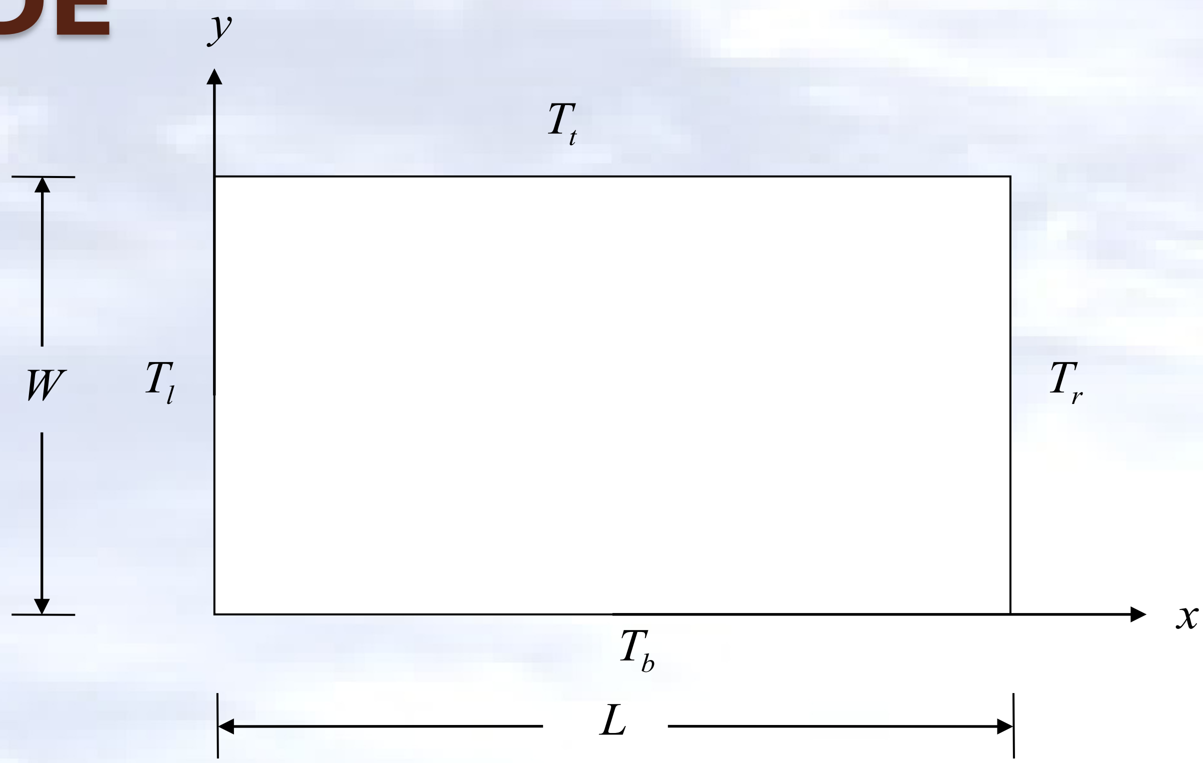
then

$$B^2 - 4AC = 0 - 4(1)(1)$$

$$= -4 < 0$$

thus allowing us to classify this equation as elliptic.

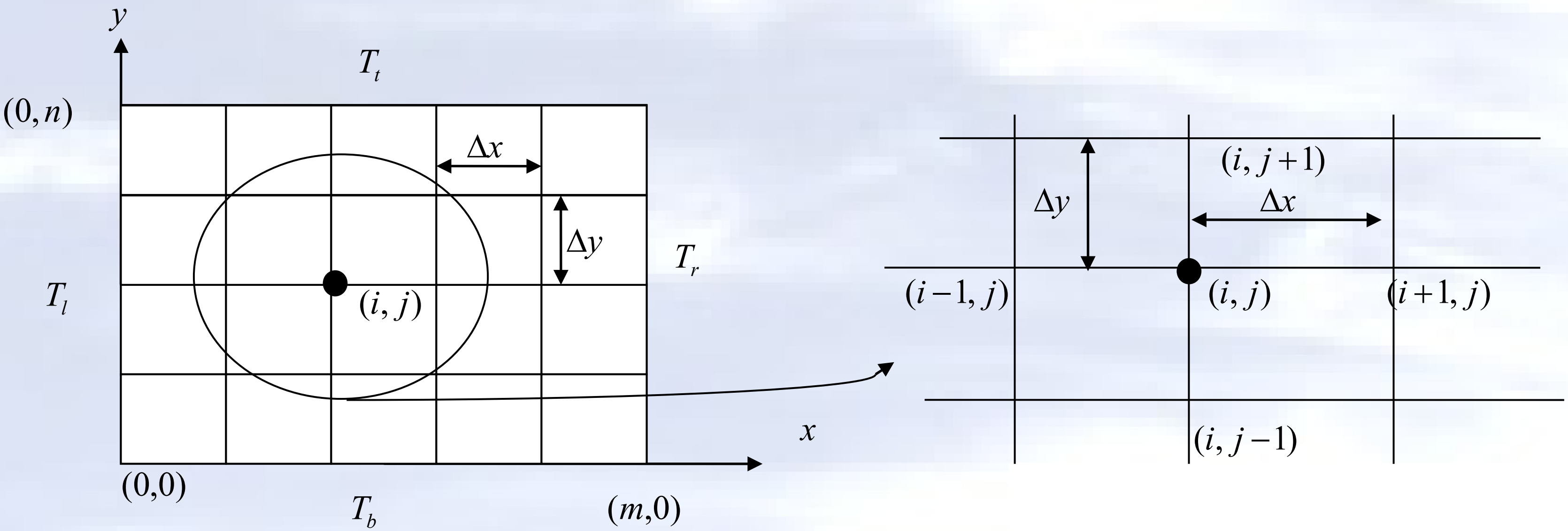
Physical Example of an Elliptic PDE



Schematic diagram of a plate with specified temperature boundary conditions

The Laplace equation governs the temperature: $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$

Discretizing the Elliptic PDE



If we define $\Delta x = \dfrac{L}{m}$ **and** $\Delta y = \dfrac{W}{n}$ **,we can then write the finite difference approximation of the partial derivatives at a general interior node** (i, j) **as**

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{i,j} \cong \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} \quad \textbf{and} \quad \left.\frac{\partial^2 T}{\partial y^2}\right|_{i,j} \cong \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}$$

Discretizing the Elliptic PDE

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Substituting these approximations into the Laplace equation yields:

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0$$

if,

$$\Delta x = \Delta y$$

the Laplace equation can be rewritten as

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

Discretizing the Elliptic PDE

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

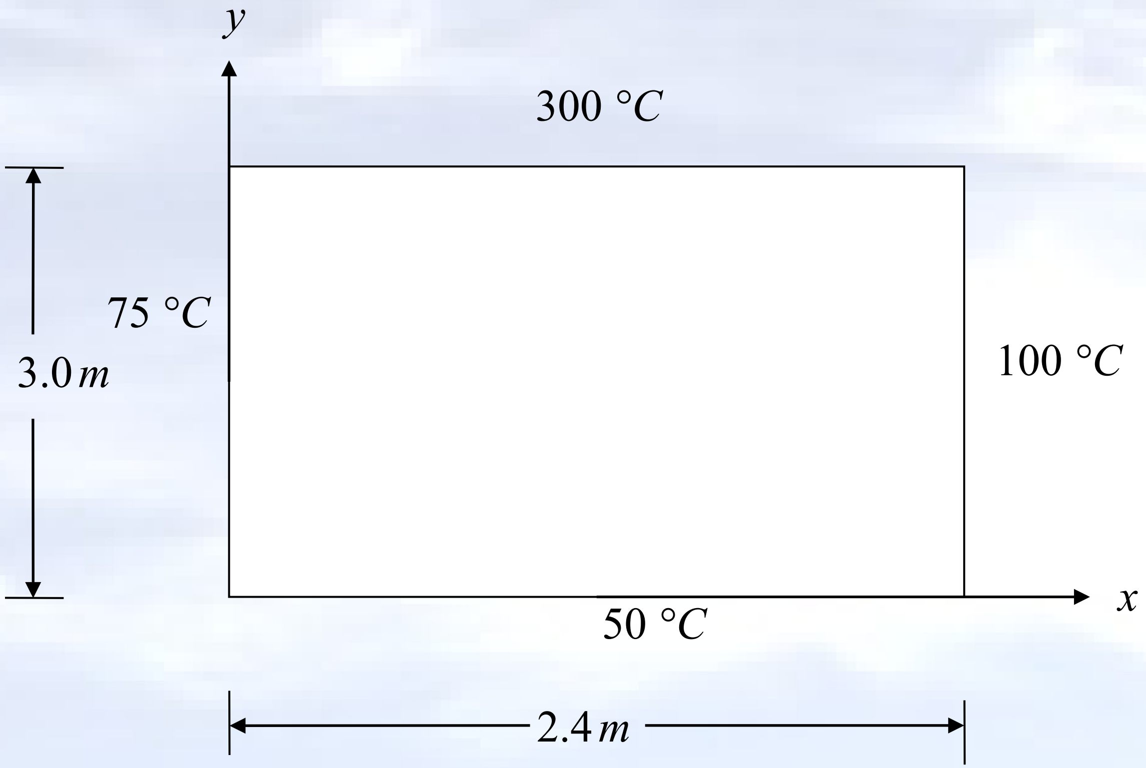
Once the governing equation has been discretized there are several numerical methods that can be used to solve the problem.

We will examine the:

- **Direct Method**
- **Gauss-Seidel Method**
- **Lieberman Method**

Example 1: Direct Method

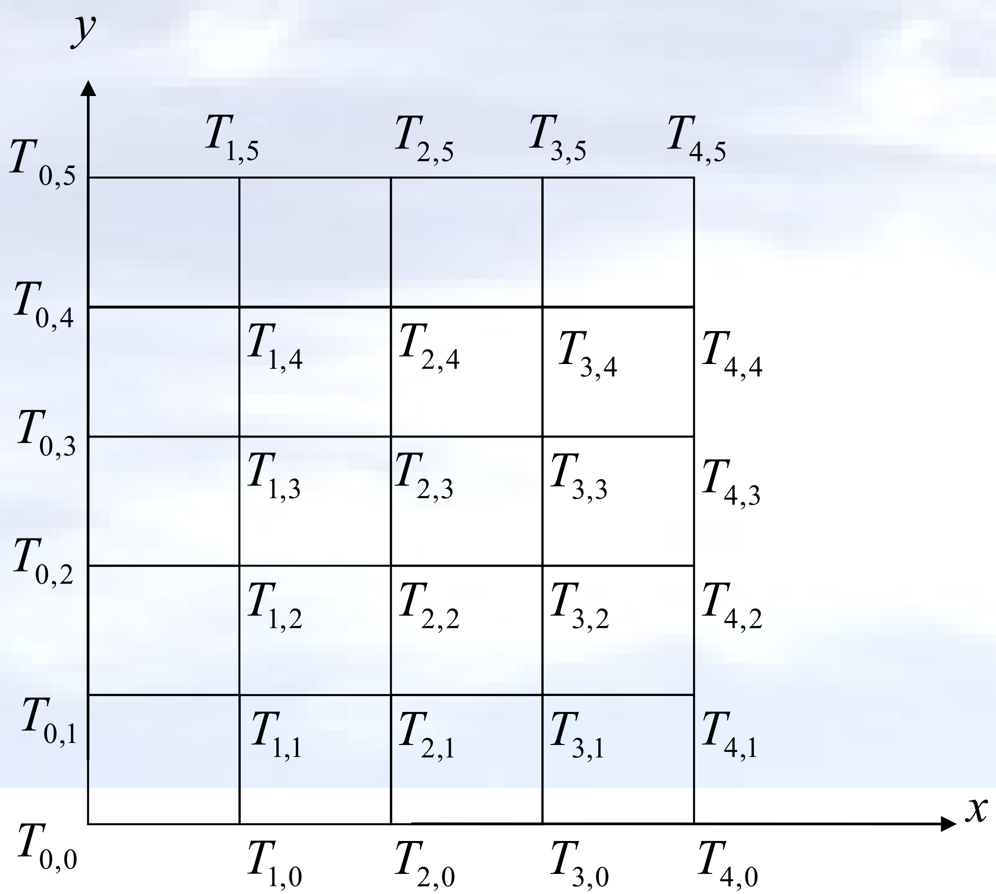
Consider a plate $2.4\,m \times 3.0\,m$ that is subjected to the boundary conditions shown below. Find the temperature at the interior nodes using a square grid with a length of $0.6\,m$ by using the direct method.



Example 1: Direct Method

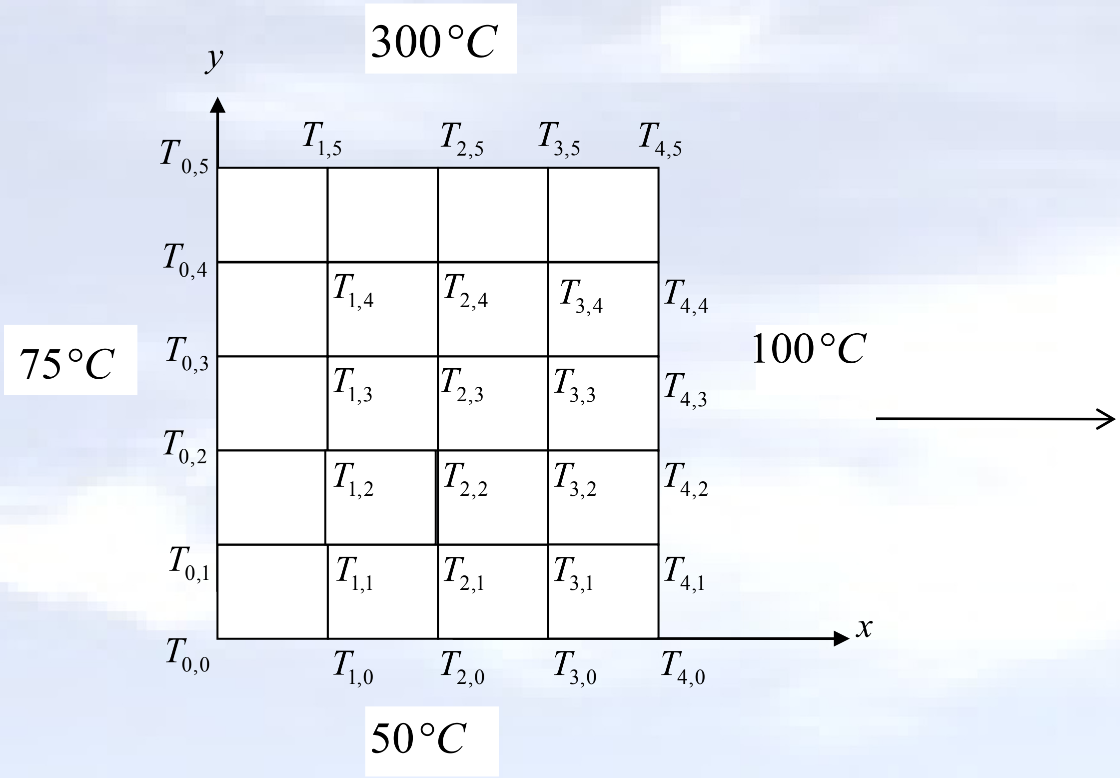
We can discretize the plate by taking,

$$\Delta x = \Delta y = 0.6m$$



Example 1: Direct Method

The nodal temperatures at the boundary nodes are given by:



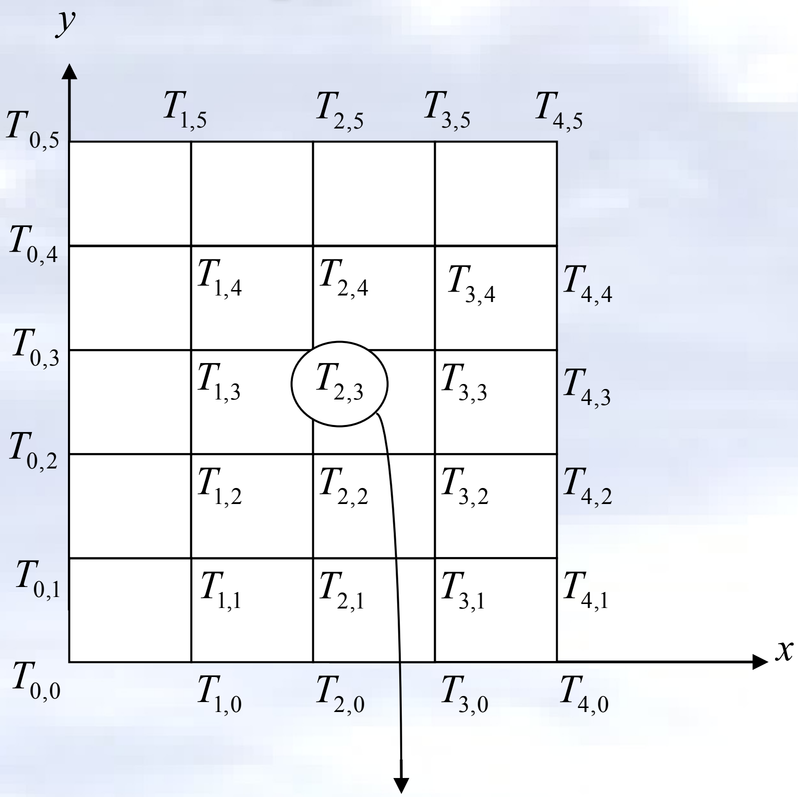
$$T_{0,j} = 75,\ j = 1,2,3,4$$

$$T_{4,j} = 100,\ j = 1,2,3,4$$

$$T_{i,0} = 50,\ i = 1,2,3$$

$$T_{i,5} = 300,\ i = 1,2,3$$

Example 1: Direct Method



Here we develop the equation for the temperature at the node (2,3)

***i*=2 and *j*=3**

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

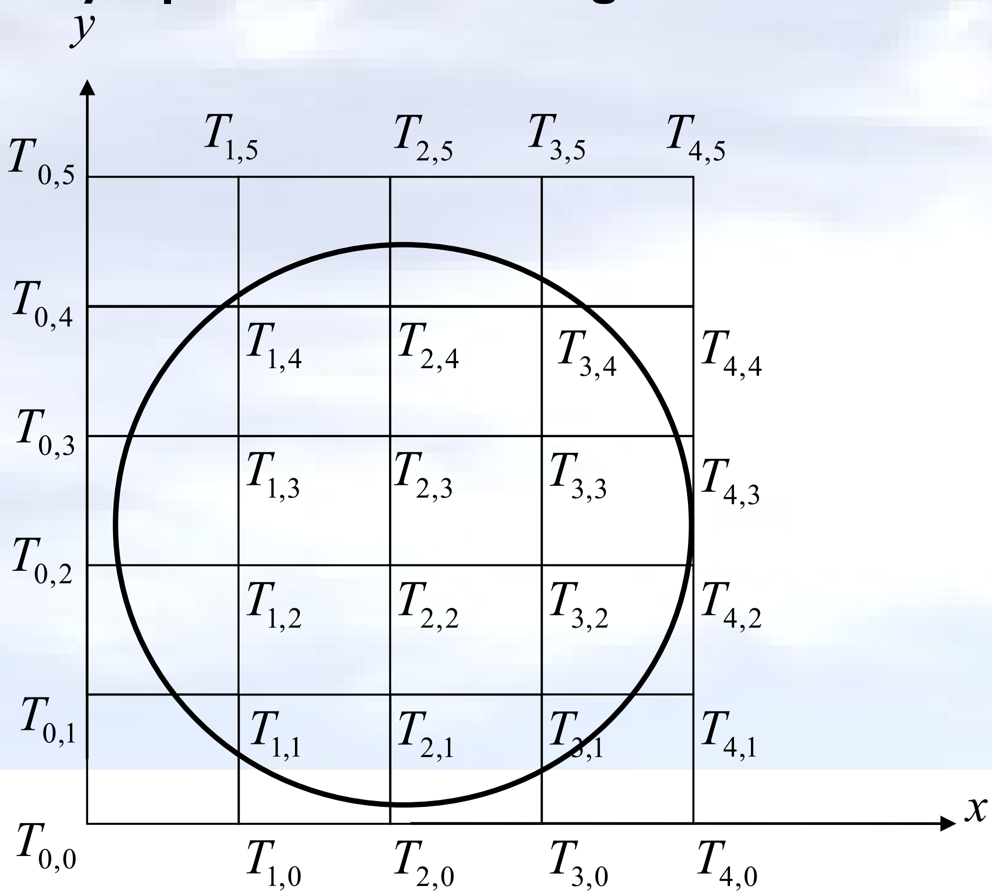
$$T_{3,3} + T_{1,3} + T_{2,4} + T_{2,2} - 4T_{2,3} = 0$$

$$T_{1,3} + T_{2,2} - 4T_{2,3} + T_{2,4} + T_{3,3} = 0$$

Example 1: Direct Method

We can develop similar equations for every interior node leaving us with an equal number of equations and unknowns.

Question: How many equations would this generate?



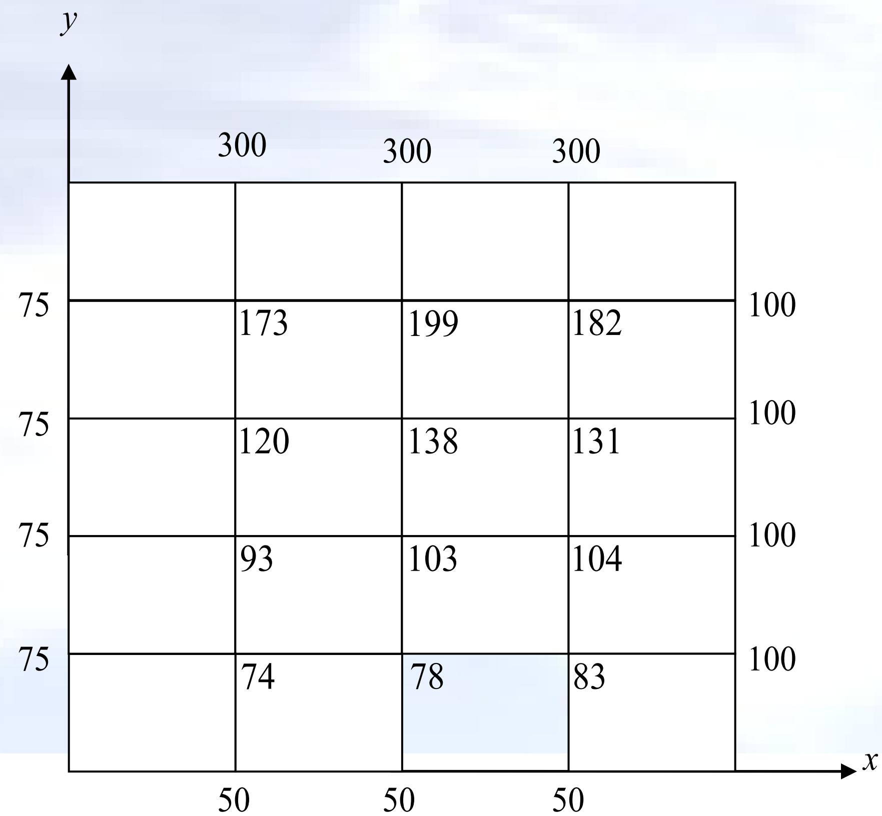
Example 1: Direct Method

We can develop similar equations for every interior node leaving us with an equal number of equations and unknowns.

Question: How many equations would this generate? **Answer: 12**

Solving yields:

$$\begin{bmatrix} T_{1,1} \\ T_{1,2} \\ T_{1,3} \\ T_{1,4} \\ T_{2,1} \\ T_{2,2} \\ T_{2,3} \\ T_{2,4} \\ T_{3,1} \\ T_{3,2} \\ T_{3,3} \\ T_{3,4} \end{bmatrix} = \begin{bmatrix} 73.8924 \\ 93.0252 \\ 119.907 \\ 173.355 \\ 77.5443 \\ 103.302 \\ 138.248 \\ 198.512 \\ 82.9833 \\ 104.389 \\ 131.271 \\ 182.446 \end{bmatrix} {}^{\circ}C \longrightarrow$$



The Gauss-Seidel Method

- Recall the discretized equation

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

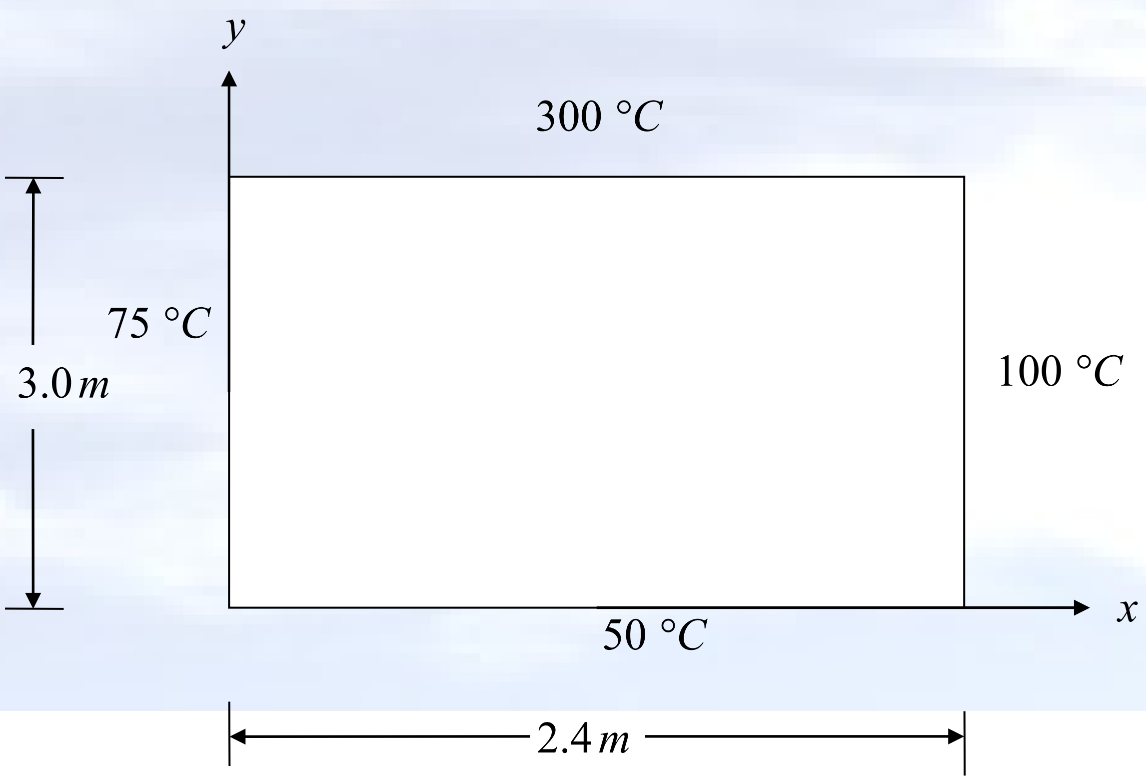
- This can be rewritten as

$$T_{i,j} = \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}}{4}$$

- For the Gauss-Seidel Method, this equation is solved iteratively for all interior nodes until a pre-specified tolerance is met.

Example 2: Gauss-Seidel Method

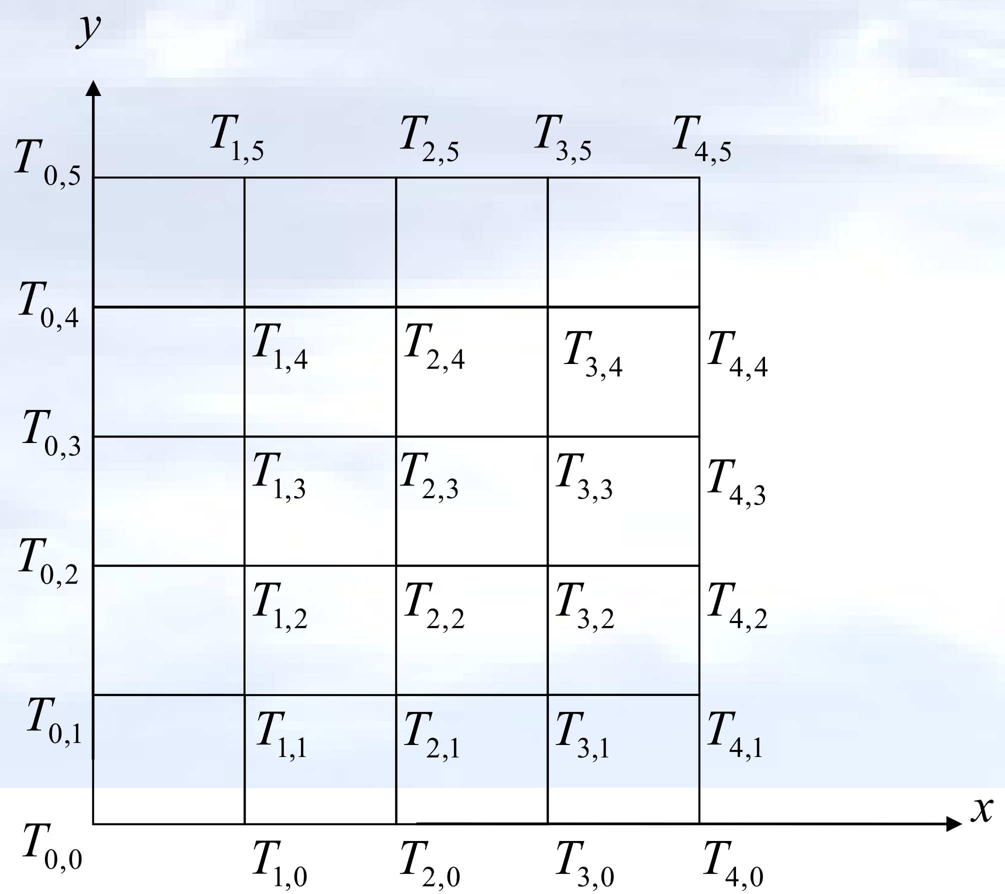
Consider a plate $2.4\,m \times 3.0\,m$ that is subjected to the boundary conditions shown below. Find the temperature at the interior nodes using a square grid with a length of $0.6\,m$ using the Gauss-Siedel method. Assume the initial temperature at all interior nodes to be $0\,°C$.



Example 2: Gauss-Seidel Method

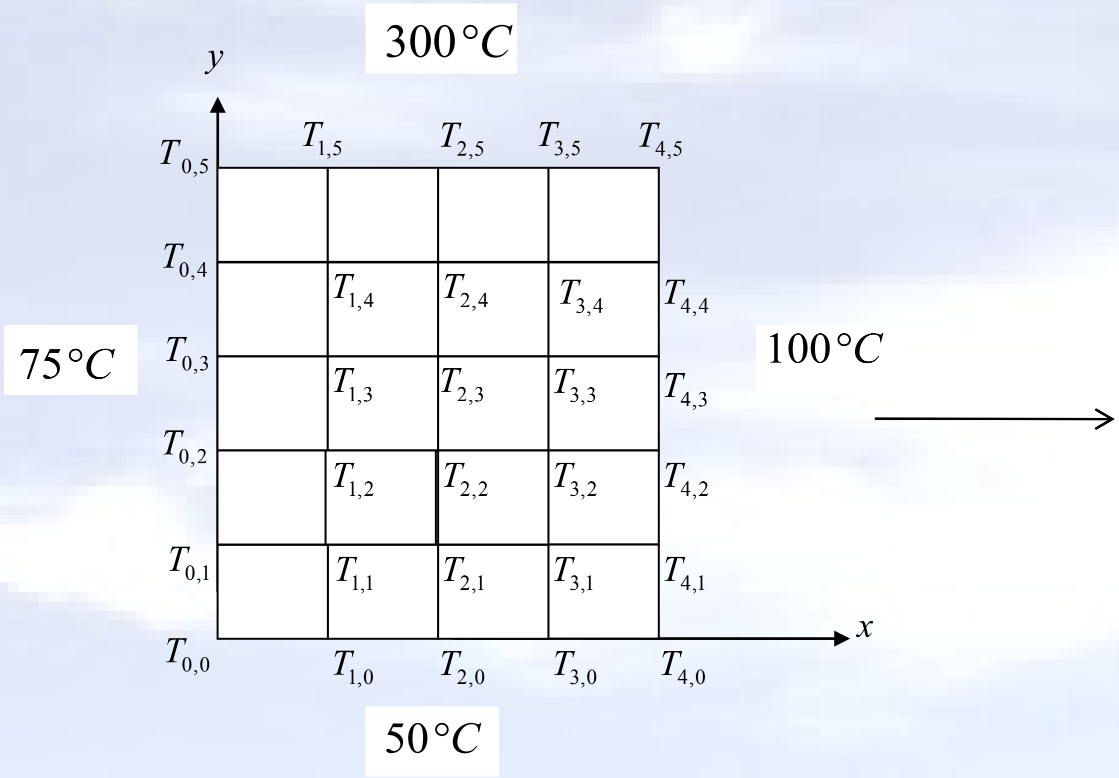
We can discretize the plate by taking

$$\Delta x = \Delta y = 0.6m$$



Example 2: Gauss-Seidel Method

The nodal temperatures at the boundary nodes are given by:



$$T_{0,j} = 75, \; j = 1,2,3,4$$

$$T_{4,j} = 100, \; j = 1,2,3,4$$

$$T_{i,0} = 50, \; i = 1,2,3$$

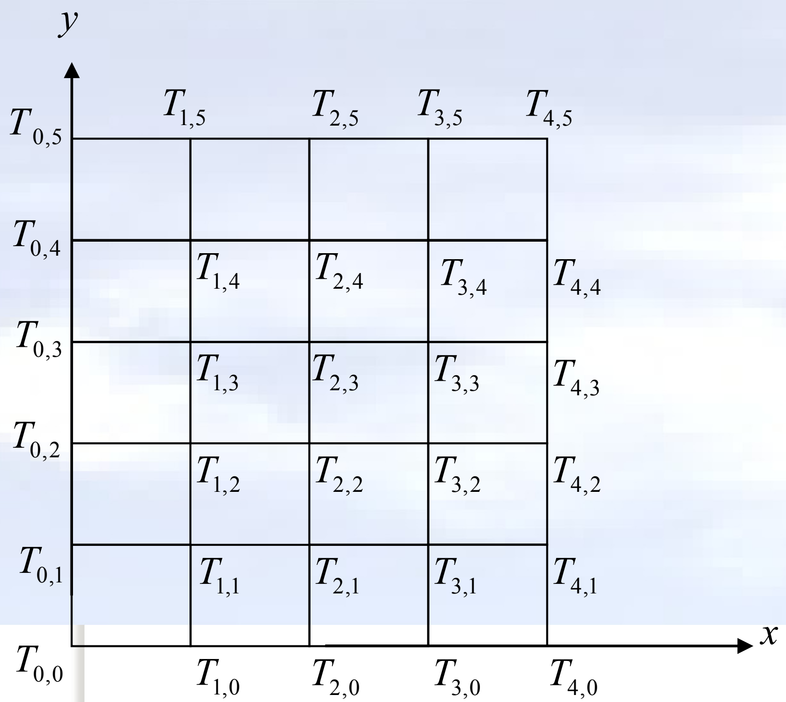
$$T_{i,5} = 300, \; i = 1,2,3$$

Example 2: Gauss-Seidel Method

- **Now we can begin to solve for the temperature at each interior node using**

$$T_{i,j} = \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}}{4}$$

- **Assume all internal nodes to have an initial temperature of <u>zero</u>.**



Iteration #1

<u>*i*=1 and *j*=1</u>

$$T_{1,1} = \frac{T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0}}{4}$$

$$= \frac{0 + 75 + 0 + 50}{4}$$

$$= 31.2500°C$$

<u>*i*=1 and *j*=2</u>

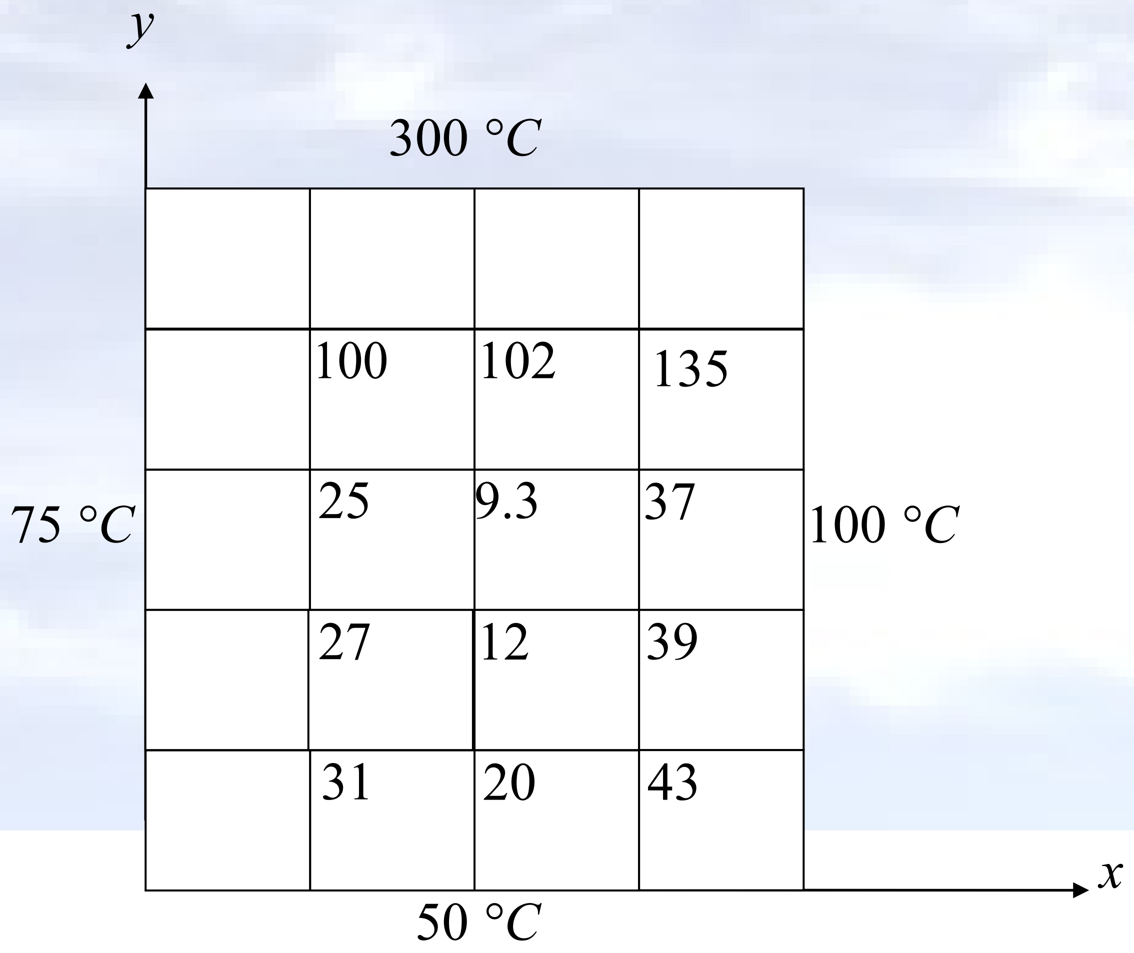
$$T_{1,2} = \frac{T_{2,2} + T_{0,2} + T_{1,3} + T_{1,1}}{4}$$

$$= \frac{0 + 75 + 0 + 31.2500}{4}$$

$$= 26.5625°C$$

Example 2: Gauss-Seidel Method

After the first iteration, the temperatures are as follows. These will now be used as the nodal temperatures for the second iteration.



Example 2: Gauss-Seidel Method

Iteration #2

***i*=1 and *j*=1**

$$T_{1,1} = \frac{T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0}}{4}$$

$$= \frac{20.3125 + 75 + 26.5625 + 50}{4}$$

$$= \underline{42.9688°C}$$

$$\left|\varepsilon_a\right|_{1,1} = \left|\frac{T_{1,1}^{present} - T_{1,1}^{previous}}{T_{1,1}^{present}}\right| \times 100$$

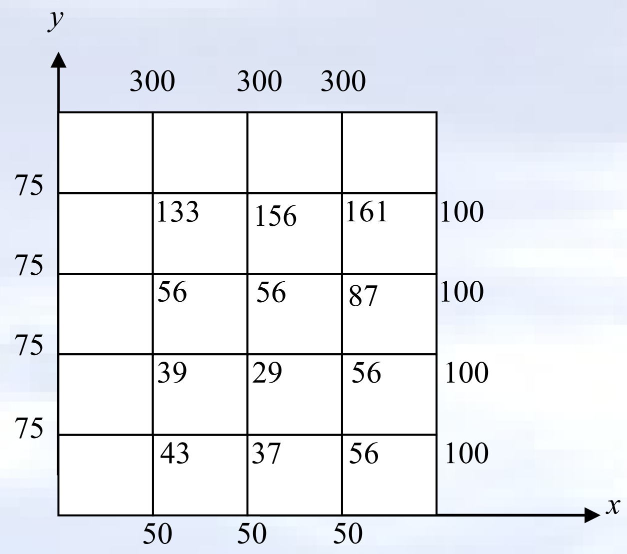
$$= \left|\frac{42.9688 - 31.2500}{42.9688}\right| \times 100$$

$$= \underline{27.27\%}$$

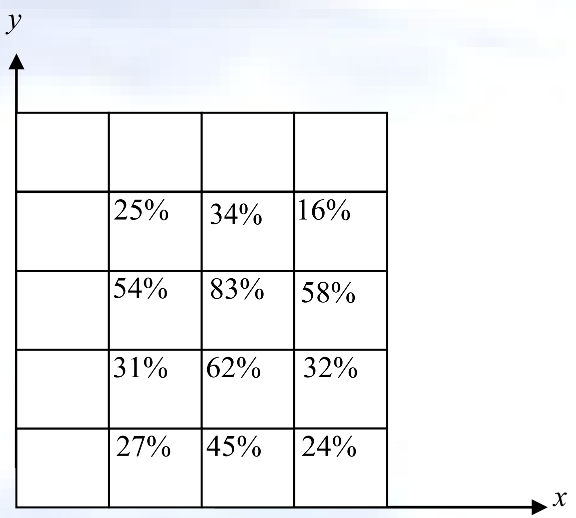
Example 2: Gauss-Seidel Method

The figures below show the temperature distribution and absolute relative error distribution in the plate after two iterations:

Temperature Distribution



Absolute Relative Approximate Error Distribution



Example 2: Gauss-Seidel Method

Node	Temperature Distribution in the Plate (°C) Number of Iterations			
	1	2	10	Exact
$T_{1,1}$	31.2500	42.9688	73.0239	
$T_{1,2}$	26.5625	38.7695	91.9585	
$T_{1,3}$	25.3906	55.7861	119.0976	
$T_{1,4}$	100.0977	133.2825	172.9755	
$T_{2,1}$	20.3125	36.8164	76.6127	
$T_{2,2}$	11.7188	30.8594	102.1577	
$T_{2,3}$	9.2773	56.4880	137.3802	
$T_{2,4}$	102.3438	156.1493	198.1055	
$T_{3,1}$	42.5781	56.3477	82.4837	
$T_{3,2}$	38.5742	56.0425	103.7757	
$T_{3,3}$	36.9629	86.8393	130.8056	
$T_{3,4}$	134.8267	160.7471	182.2278	

The Lieberman Method

- Recall the equation used in the Gauss-Siedel Method,

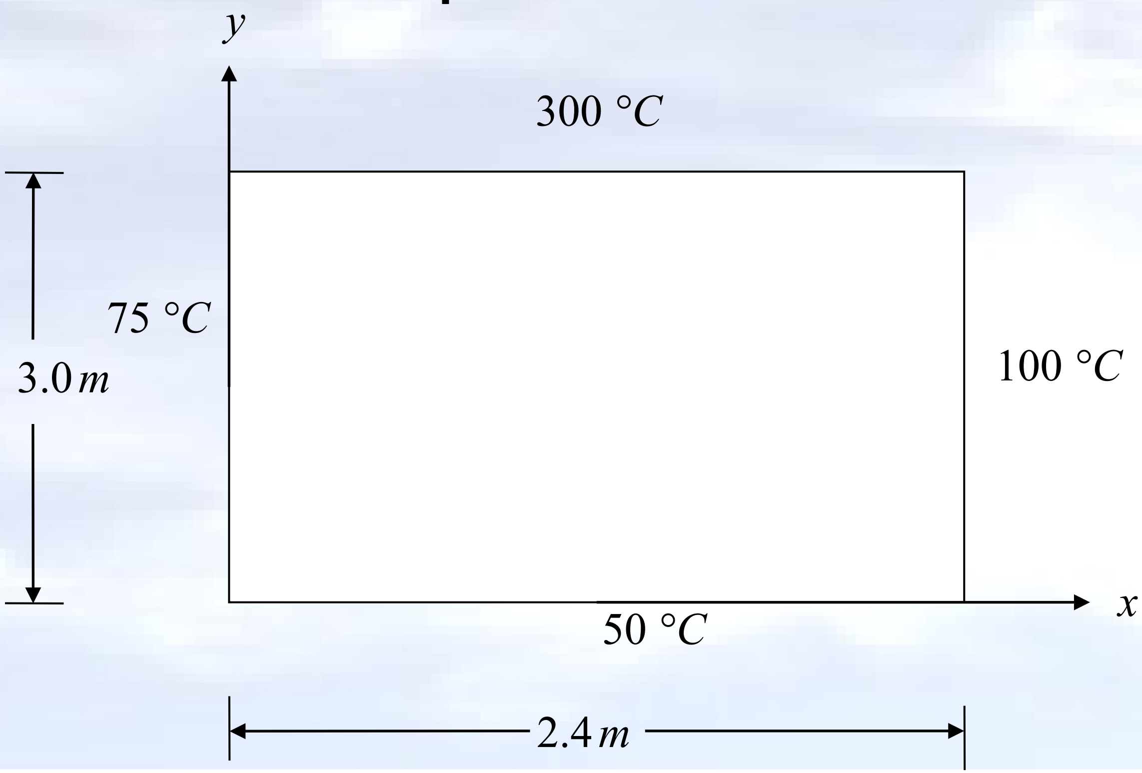
$$T_{i,j} = \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}}{4}$$

- Because the Guass-Siedel Method is guaranteed to converge, we can accelerate the process by using over- relaxation. In this case,

$$T_{i,j}^{relaxed} = \lambda T_{i,j}^{new} + (1-\lambda) T_{i,j}^{old}$$

Example 3: Lieberman Method

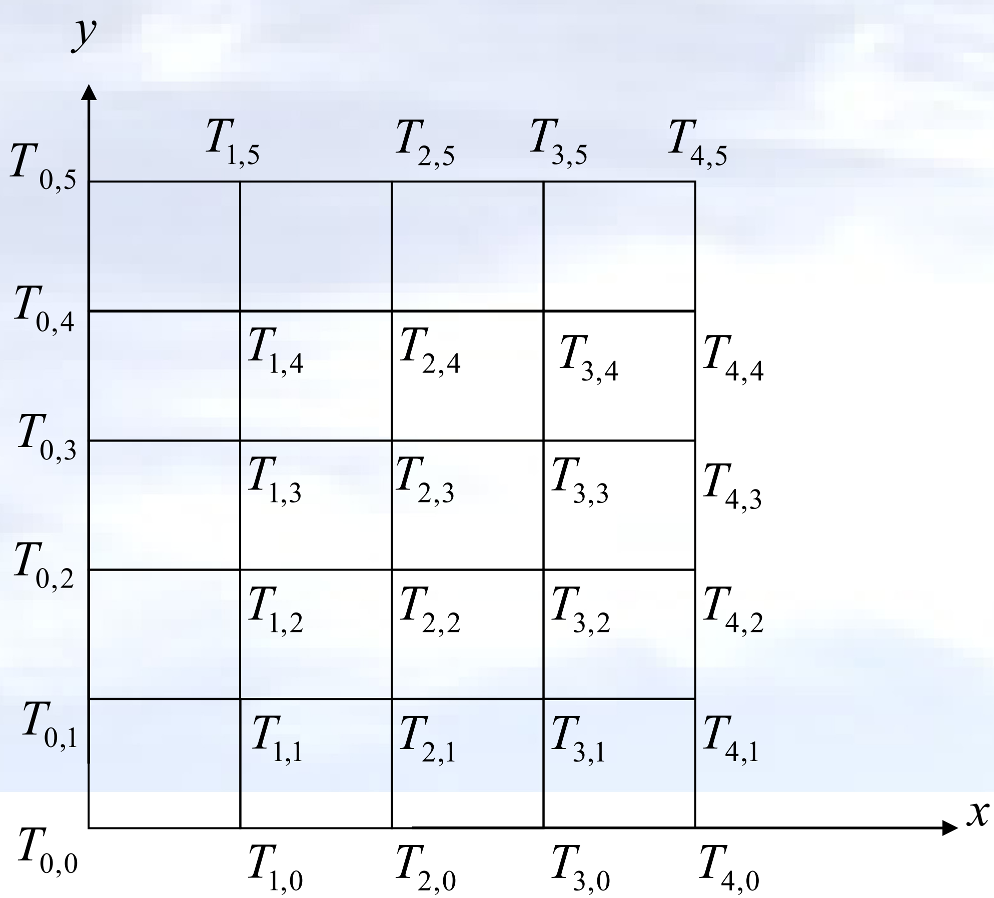
Consider a plate $2.4\,m \times 3.0\,m$ that is subjected to the boundary conditions shown below. Find the temperature at the interior nodes using a square grid with a length of $0.6\,m$. Use a weighting factor of 1.4 in the Lieberman method. Assume the initial temperature at all interior nodes to be $0\,°C$.



Example 3: Lieberman Method

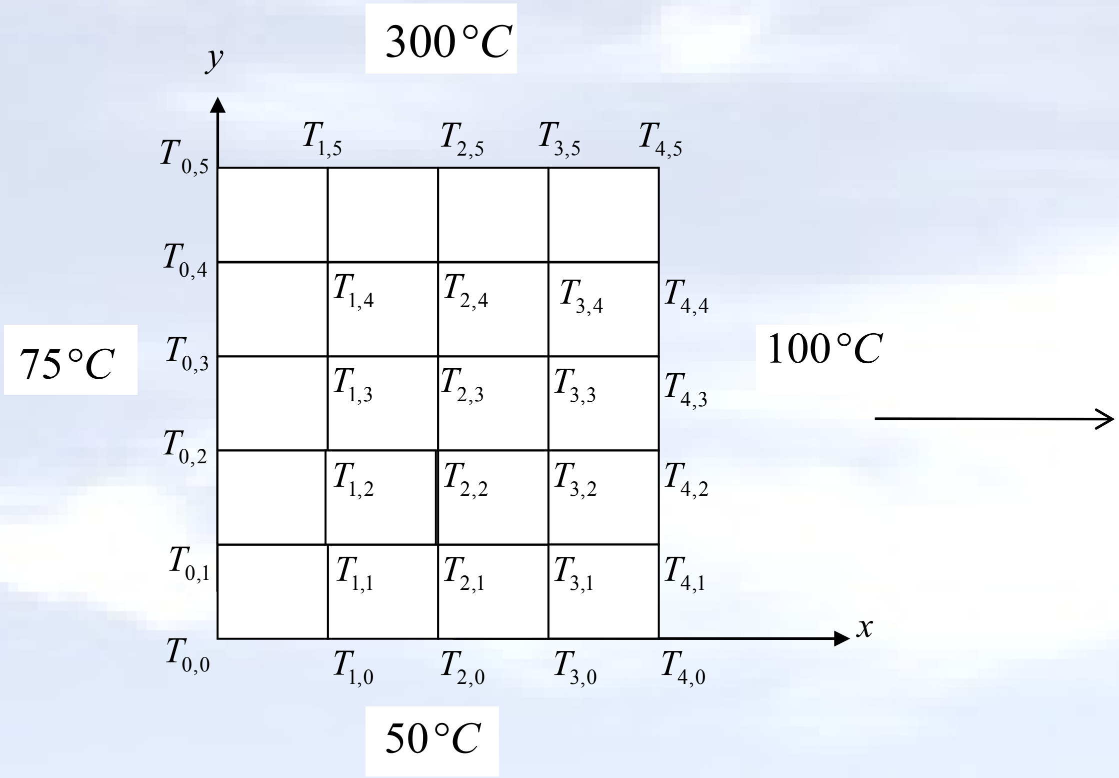
We can discretize the plate by taking

$$\Delta x = \Delta y = 0.6m$$



Example 3: Lieberman Method

We can also develop equations for the boundary conditions to define the temperature of the exterior nodes.



$$T_{0,j} = 75,\ j = 1,2,3,4$$

$$T_{4,j} = 100,\ j = 1,2,3,4$$

$$T_{i,0} = 50, i = 1,2,3$$

$$T_{i,5} = 300, i = 1,2,3$$

Example 3: Lieberman Method

- **Now we can begin to solve for the temperature at each interior node using the rewritten Laplace equation from the Gauss-Siedel method.**

- **Once we have the temperature value for each node we will apply the over relaxation equation of the Lieberman method**

- **Assume all internal nodes to have an initial temperature of <u>zero</u>.**

Iteration #1

<u>*i*=1 and *j*=1</u>

$$T_{1,1} = \frac{T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0}}{4} = \frac{0 + 75 + 0 + 50}{4} = 31.2500°C$$

$$T_{1,1}^{relaxed} = \lambda T_{1,1}^{new} + (1-\lambda)T_{1,1}^{old} = 1.4(31.2500) + (1-1.4)0 = 43.7500°C$$

Iteration #2

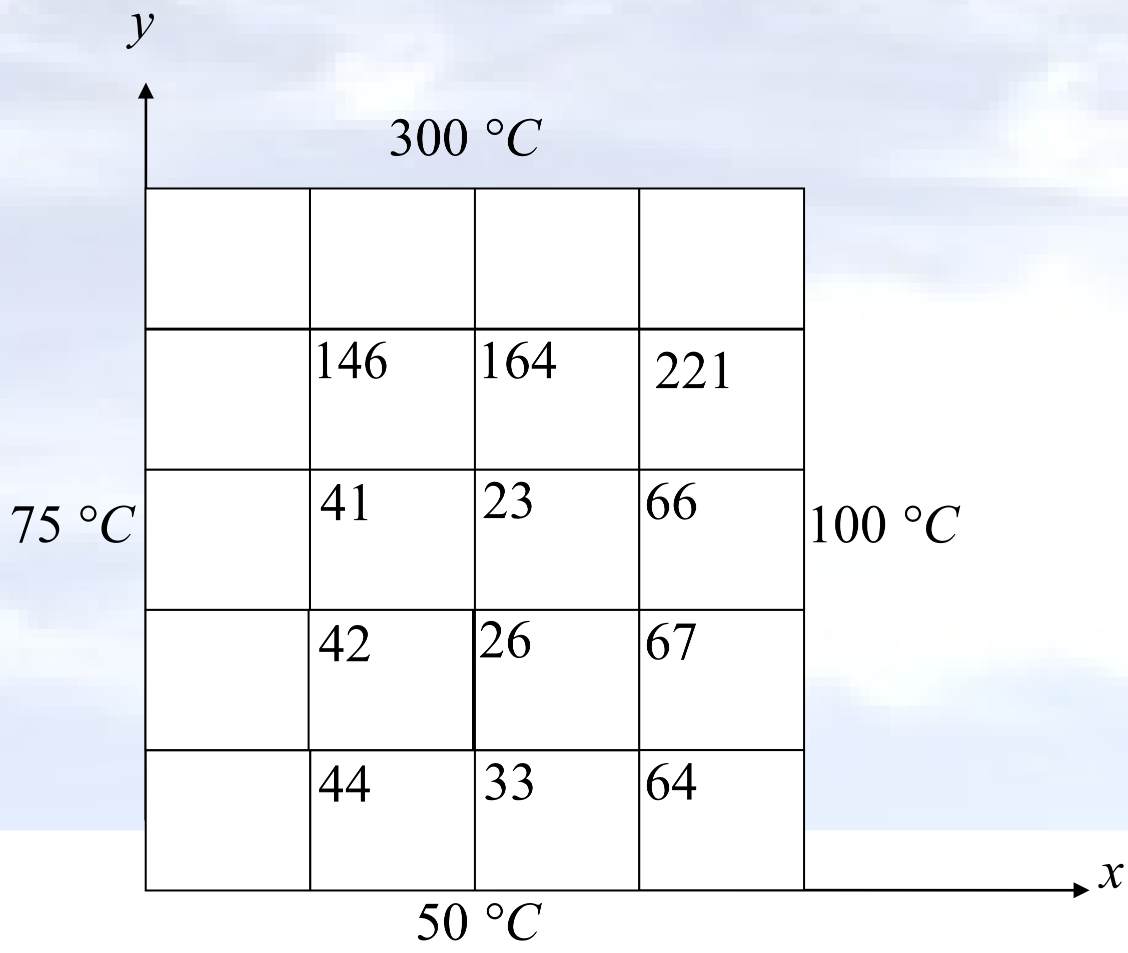
<u>*i*=1 and *j*=2</u>

$$T_{1,2} = \frac{T_{2,2} + T_{0,2} + T_{1,3} + T_{1,1}}{4} = \frac{0 + 75 + 0 + 43.75}{4} = 29.6875°C$$

$$T_{1,1}^{relaxed} = \lambda T_{1,1}^{new} + (1-\lambda)T_{1,1}^{old} = 1.4(29.6875) + (1-1.4)0 = 41.5625°C$$

Example 3: Lieberman Method

After the first iteration the temperatures are as follows. These will be used as the initial nodal temperatures during the second iteration.



Example 3: Lieberman Method

Iteration #2

<u>***i*=1 and *j*=1**</u>

$$T_{1,1} = \frac{T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0}}{4}$$

$$= \frac{32.8125 + 75 + 41.5625 + 50}{4}$$

$$= \underline{49.8438°C}$$

$$T_{1,1}^{relaxed} = \lambda T_{1,1}^{new} + (1-\lambda)T_{1,1}^{old}$$

$$= 1.4(49.8438) + (1-1.4)43.75$$

$$= 52.2813°C$$

$$\left|\varepsilon_a\right|_{1,1} = \left|\frac{T_{1,1}^{present} - T_{1,1}^{previous}}{T_{1,1}^{present}}\right| \times 100$$

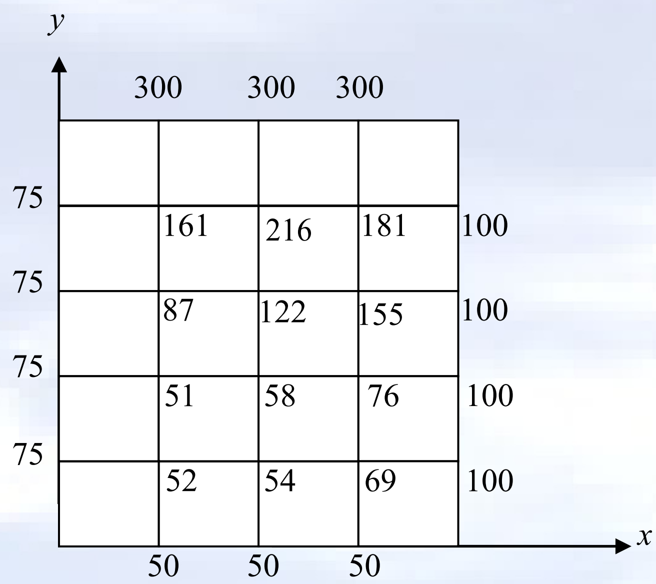
$$= \left|\frac{52.2813 - 43.7500}{52.2813}\right| \times 100$$

$$= \underline{16.32\%}$$

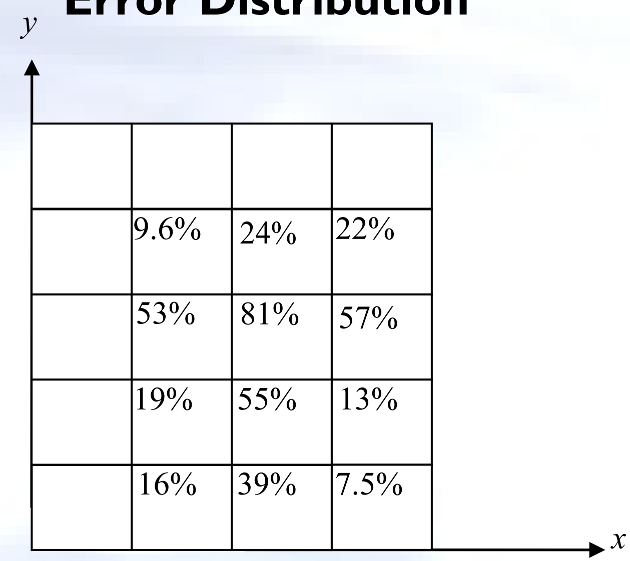
Example 3: Lieberman Method

The figures below show the temperature distribution and absolute relative error distribution in the plate after two iterations:

Temperature Distribution



Absolute Relative Approximate Error Distribution

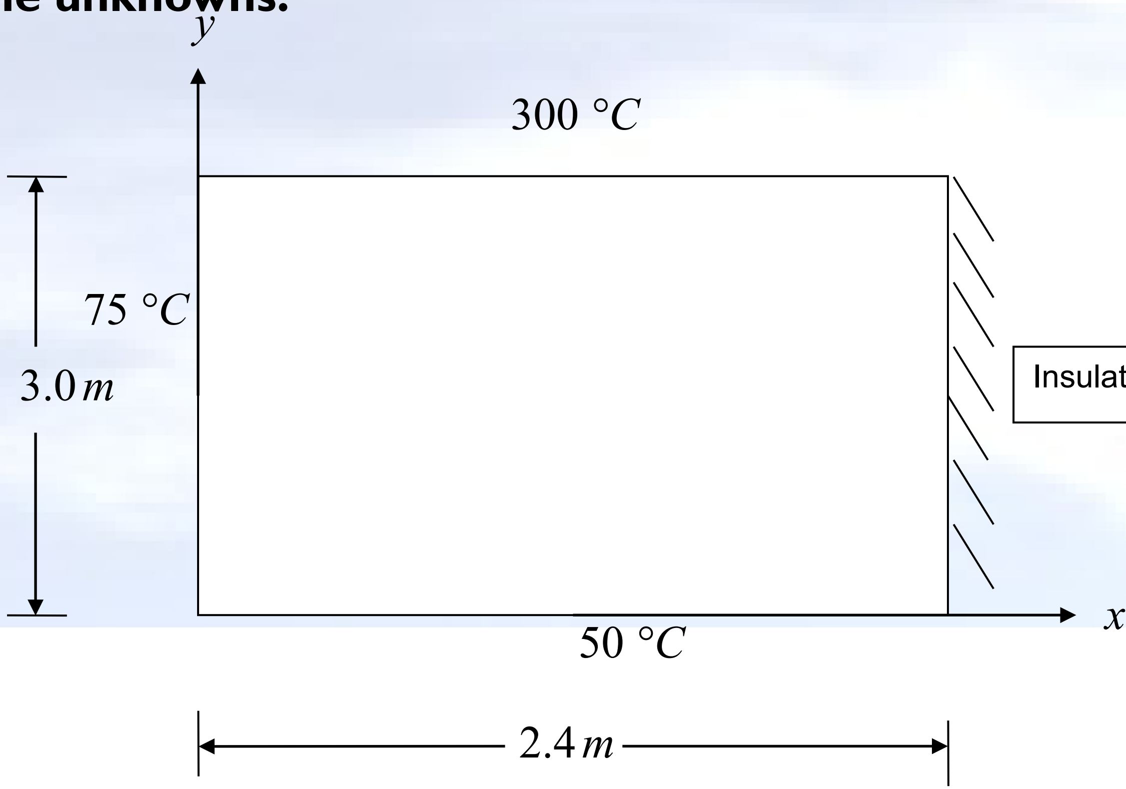


Example 3: Lieberman Method

Node	Temperature Distribution in the Plate (°C) Number of Iterations			
	1	2	9	Exact
$T_{1,1}$	43.7500	52.2813	73.7832	
$T_{1,2}$	41.5625	51.3133	92.9758	
$T_{1,3}$	40.7969	87.0125	119.9378	
$T_{1,4}$	145.5289	160.9353	173.3937	
$T_{2,1}$	32.8125	54.1789	77.5449	
$T_{2,2}$	26.0313	57.9731	103.3285	
$T_{2,3}$	23.3898	122.0937	138.3236	
$T_{2,4}$	164.1216	215.6582	198.5498	
$T_{3,1}$	63.9844	69.1458	82.9805	
$T_{3,2}$	66.5055	76.1516	104.3815	
$T_{3,3}$	66.4634	155.0472	131.2525	
$T_{3,4}$	220.7047	181.4650	182.4230	

Alternative Boundary Conditions

- **In Examples 1-3, the boundary conditions on the plate had a specified temperature on each edge. What if the conditions are different ? For example, what if one of the edges of the plate is insulated.**
- **In this case, the boundary condition would be the derivative of the temperature. Because if the right edge of the plate is insulated, then the temperatures on the right edge nodes also become unknowns.**



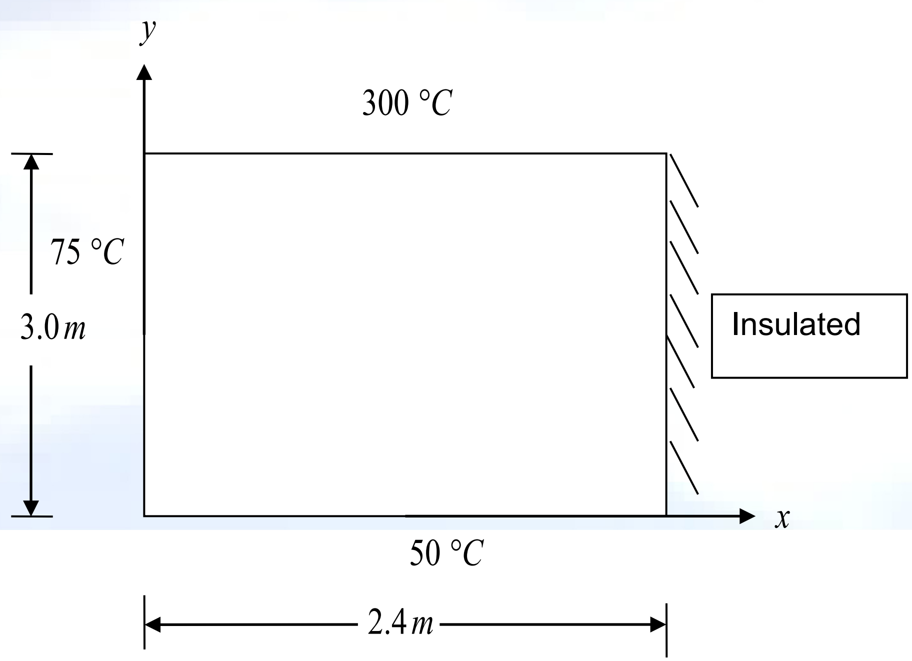
Alternative Boundary Conditions

- The finite difference equation in this case for the right edge for the nodes (m, j) for $j = 2,3,..n-1$

$$T_{m+1,j} + T_{m-1,j} + T_{m,j-1} + T_{m,j+1} - 4T_{m,j} = 0$$

- However the node $(m+1, j)$ is not inside the plate. The derivative boundary condition needs to be used to account for these additional unknown nodal temperatures on the right edge. This is done by approximating the derivative at the edge node as (m, j)

$$\left.\frac{\partial T}{\partial x}\right|_{m,j} \cong \frac{T_{m+1,j} - T_{m-1,j}}{2(\Delta x)}$$



Alternative Boundary Conditions

- **Rearranging this approximation gives us,**

$$T_{m+1,j} = T_{m-1,j} + 2(\Delta x)\left.\frac{\partial T}{\partial x}\right|_{m,j}$$

- **We can then substitute this into the original equation gives us,**

$$2T_{m-1,j} + 2(\Delta x)\left.\frac{\partial T}{\partial x}\right|_{m,j} + T_{m,j-1} + T_{m,j+1} - 4T_{m,j} = 0$$

- **Recall that is the edge is insulated then,**

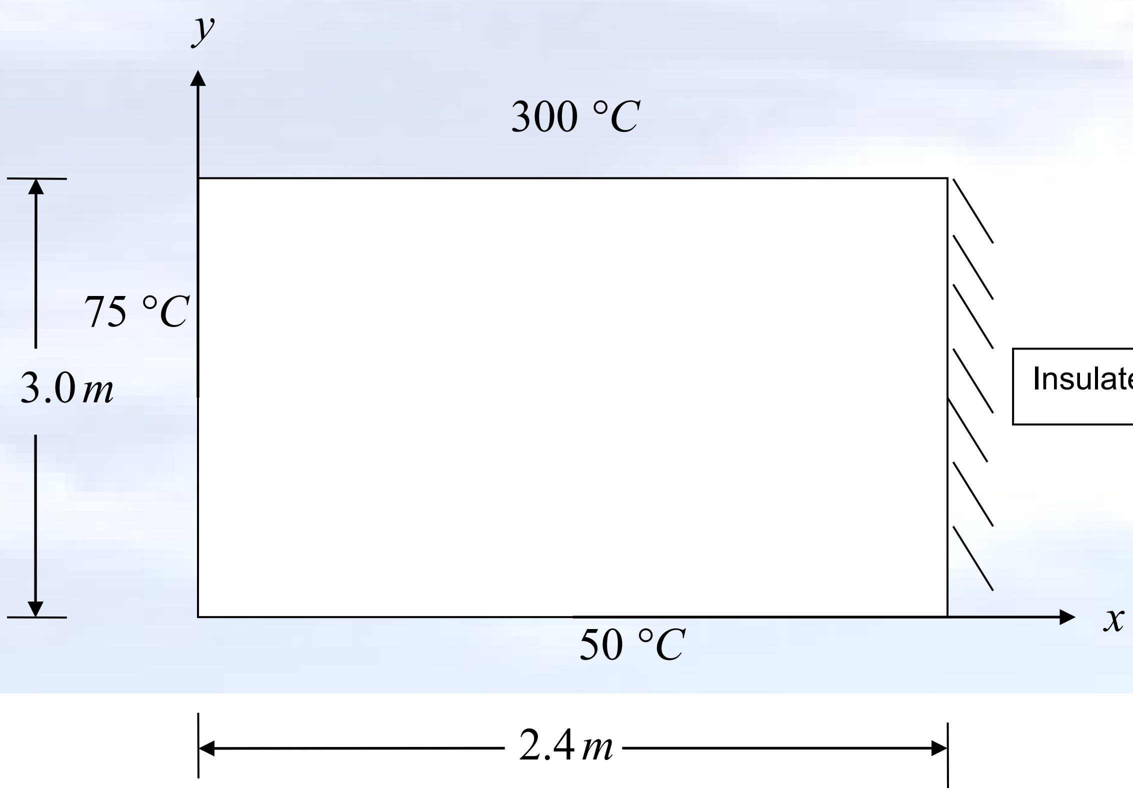
$$\left.\frac{\partial T}{\partial x}\right|_{m,j} = 0$$

- **Substituting this again yields,**

$$2T_{m-1,j} + T_{m,j-1} + T_{m,j+1} - 4T_{m,j} = 0$$

Example 3: Alternative Boundary Conditions

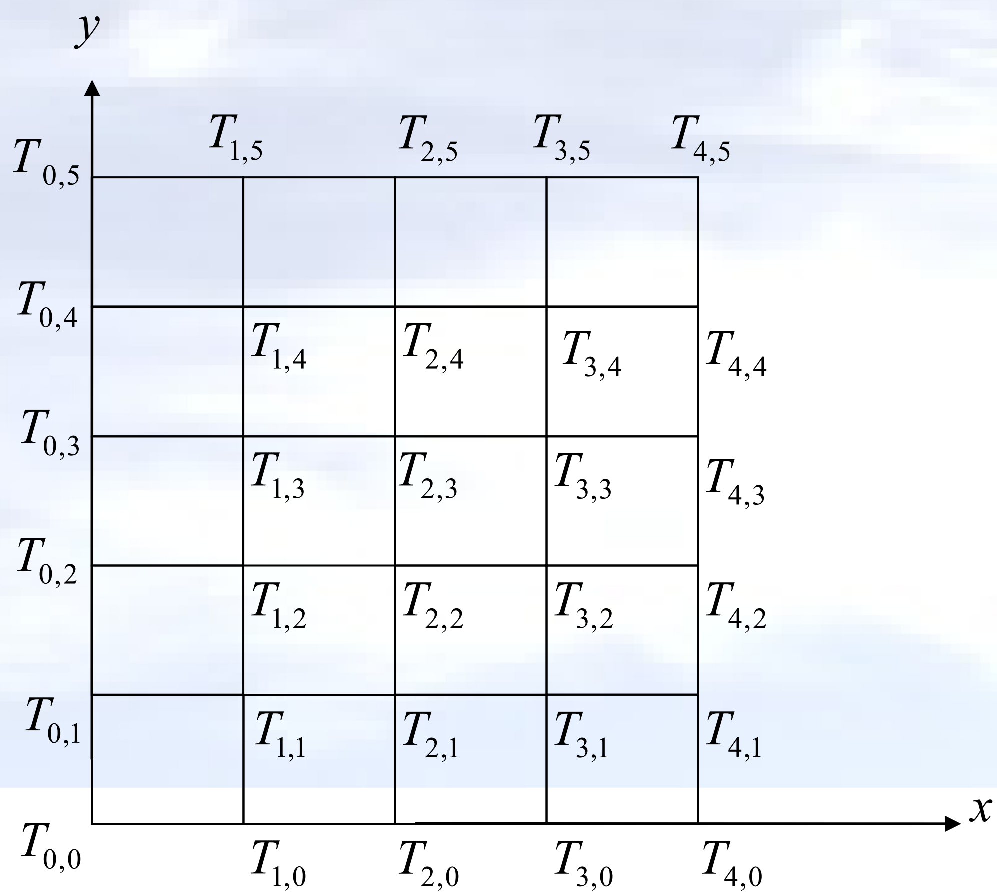
A plate $2.4\,m \times 3.0\,m$ is subjected to the temperatures and insulated boundary conditions as shown in Fig. 12. Use a square grid length of $0.6\,m$. Assume the initial temperatures at all of the interior nodes to be $0\,°C$. Find the temperatures at the interior nodes using the direct method.



Example 3: Alternative Boundary Conditions

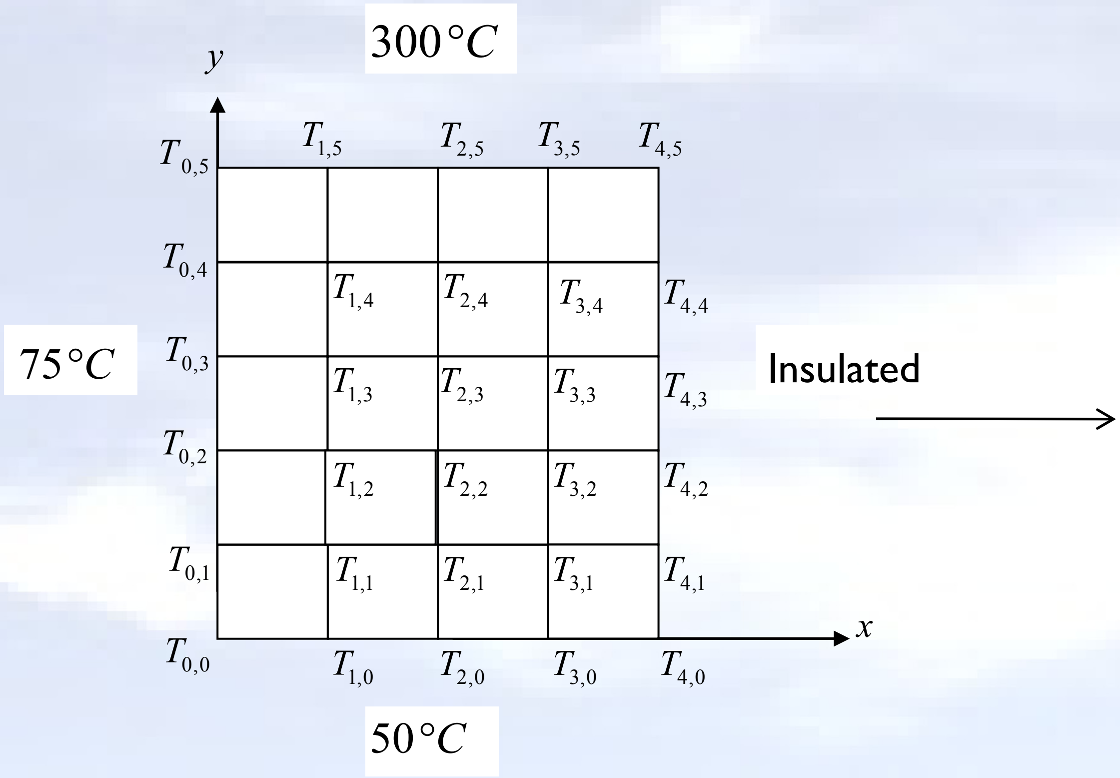
We can discretize the plate taking,

$$\Delta x = \Delta y = 0.6m$$



Example 3: Alternative Boundary Conditions

We can also develop equations for the boundary conditions to define the temperature of the exterior nodes.



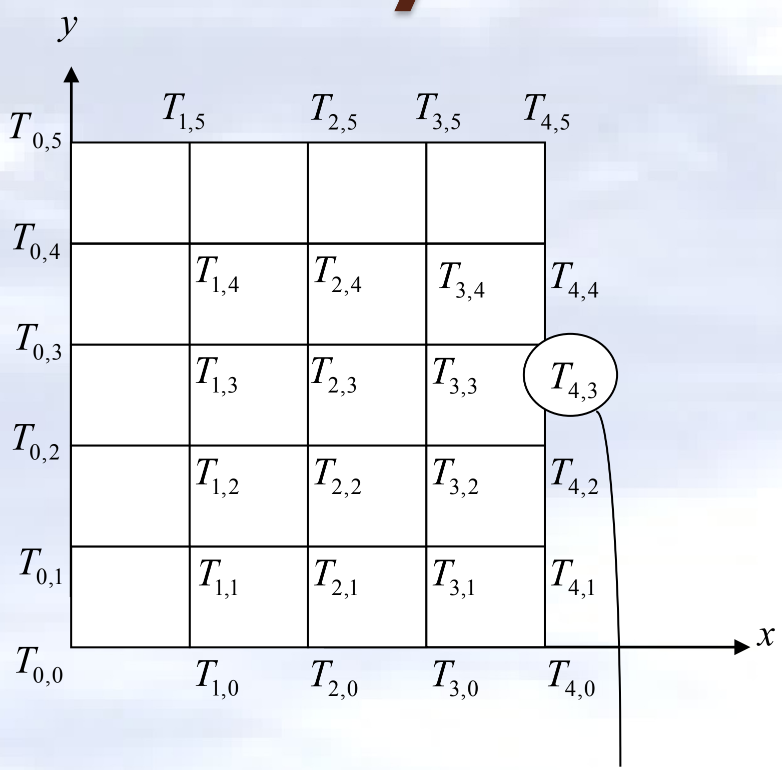
$$T_{0,j} = 75;\ j = 1,2,3,4$$

$$T_{i,0} = 50;\ i = 1,2,3,4$$

$$T_{i,5} = 300;\ i = 1,2,3,4$$

$$\left.\frac{\partial T}{\partial x}\right|_{4,j} = 0;\ j = 1,2,3,4$$

Example 3: Alternative Boundary Conditions



Here we develop the equation for the temperature at the node (4,3), to show the effects of the alternative boundary condition.

***i*=4 and *j*=3** $2T_{3,3} + T_{4,2} + T_{4,4} - 4T_{4,3} = 0$

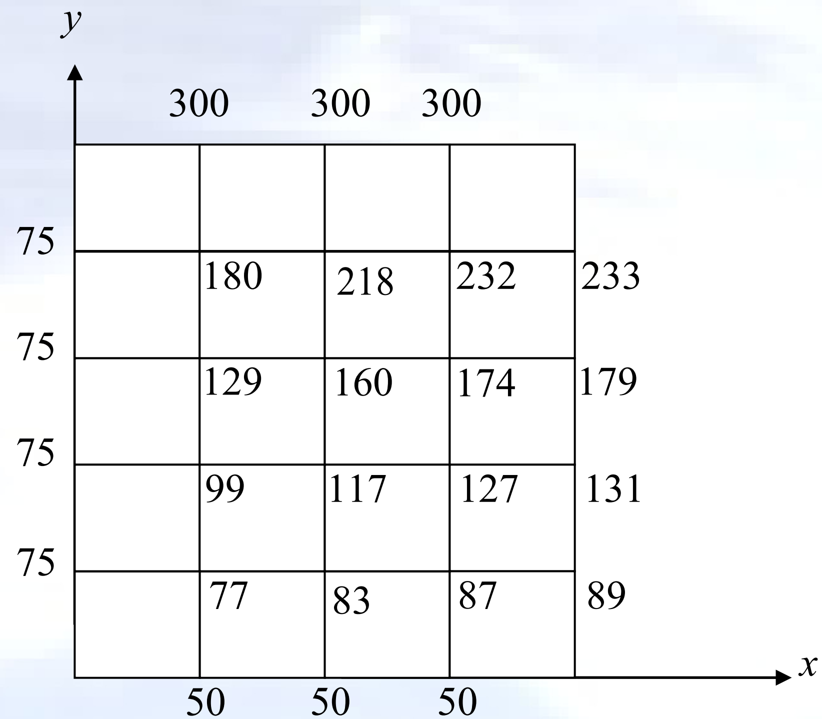
$$2T_{3,3} + T_{4,2} - 4T_{4,3} + T_{4,4} = 0$$

Example 3: Alternative Boundary Conditions

The addition of the equations for the boundary conditions gives us a system of 16 equations with 16 unknowns.

Solving yields:

$$\begin{bmatrix} T_{1,1} \\ T_{1,2} \\ T_{1,3} \\ T_{1,4} \\ T_{2,1} \\ T_{2,2} \\ T_{2,3} \\ T_{2,4} \\ T_{3,1} \\ T_{3,2} \\ T_{3,3} \\ T_{3,4} \\ T_{4,1} \\ T_{4,2} \\ T_{4,3} \\ T_{4,4} \end{bmatrix} = \begin{bmatrix} 76.8254 \\ 99.4444 \\ 128.617 \\ 180.410 \\ 82.8571 \\ 117.335 \\ 159.614 \\ 218.021 \\ 87.2678 \\ 127.426 \\ 174.483 \\ 232.060 \\ 88.7882 \\ 130.617 \\ 178.830 \\ 232.738 \end{bmatrix} {}^{\circ}C \longrightarrow$$



THE END