

# Parabolic Partial Differential Equations

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Transforming Numerical Methods Education for STEM Undergraduates

# Defining Parabolic PDE's

- The general form for a second order linear PDE with two independent variables and one dependent variable is

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D = 0$$

- Recall the criteria for an equation of this type to be considered parabolic
- For example, examine the heat-conduction equation given by

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}, \text{ where } A = \alpha, B = 0, C = 0, D = -1$$

Then

$$\begin{aligned} B^2 - 4AC &= 0 - 4(\alpha)(0) \\ &= 0 \end{aligned}$$

thus allowing us to classify this equation as parabolic.

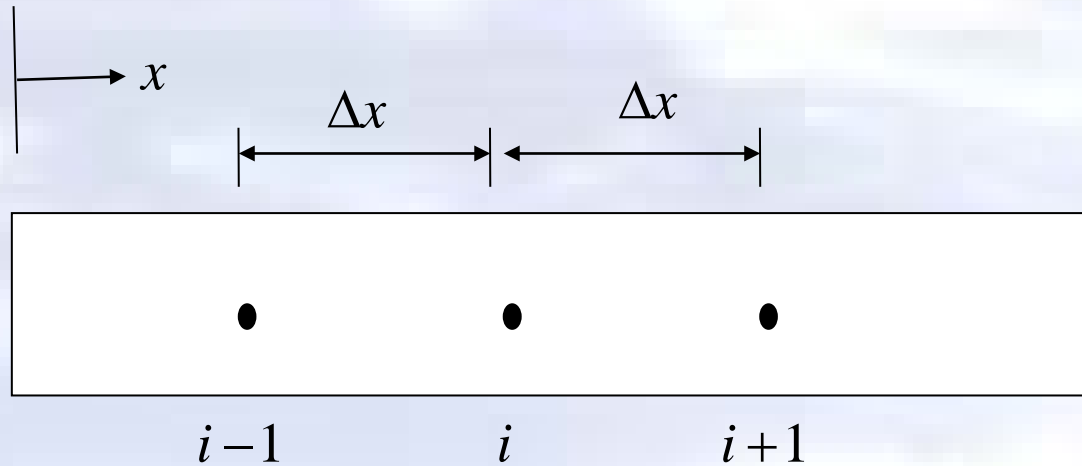
# Physical Example of an Elliptic PDE



The internal temperature of a metal rod exposed to two different temperatures at each end can be found using the heat conduction equation.

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

# Discretizing the Parabolic PDE



Schematic diagram showing interior nodes

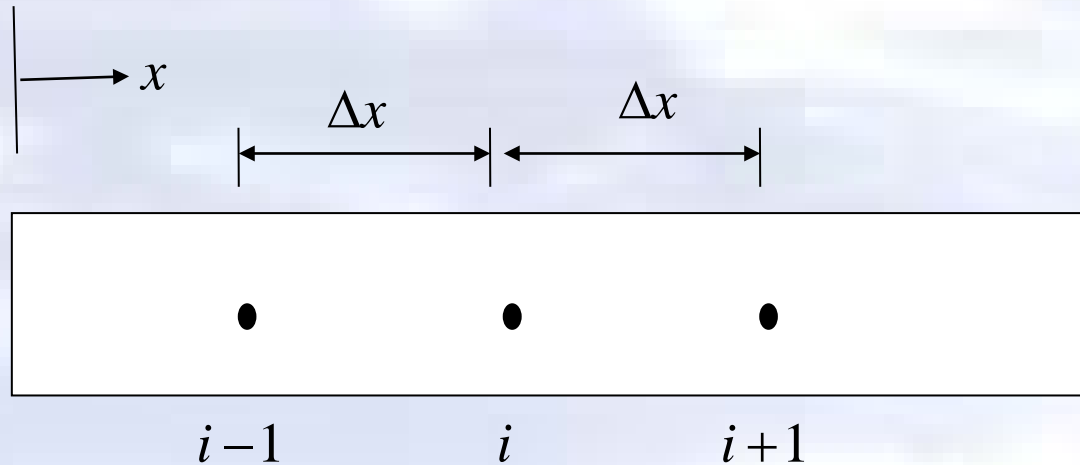
For a rod of length  $L$  divided into  $n + 1$  nodes  $\Delta x = \frac{L}{n}$

The time is similarly broken into time steps of  $\Delta t$

Hence  $T_i^j$  corresponds to the temperature at node  $i$ , that is,

$$x = (i)(\Delta x) \quad \text{and time } t = (j)(\Delta t)$$

# The Explicit Method

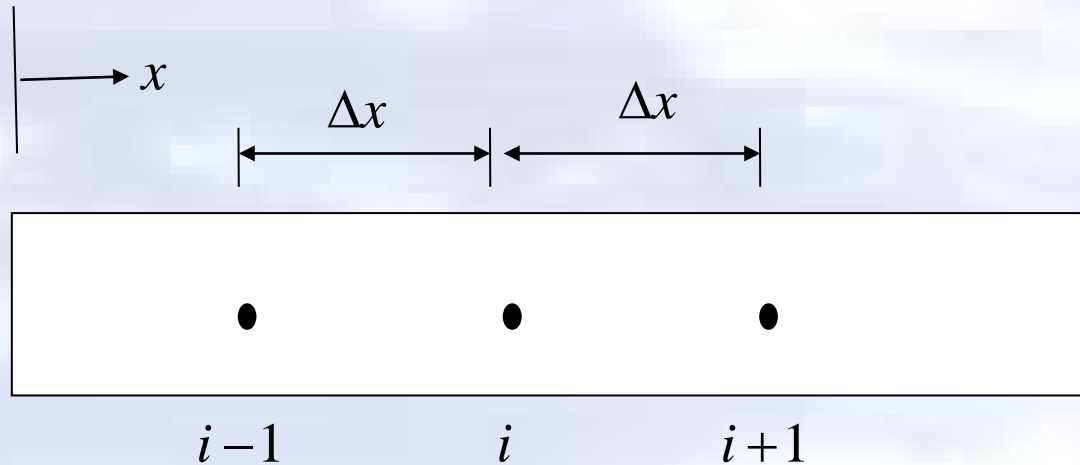


If we define  $\Delta x = \frac{L}{n}$  we can then write the finite central divided difference approximation of the left hand side at a general interior node ( $i$ ) as

$$\left. \frac{\partial^2 T}{\partial x^2} \right|_{i,j} \cong \frac{T_{i+1}^j - 2T_i^j + T_{i-1}^j}{(\Delta x)^2}$$

where ( $j$ ) is the node number along the time.

# The Explicit Method



The time derivative on the right hand side is approximated by the forward divided difference method as,

$$\left. \frac{\partial T}{\partial t} \right|_{i,j} \approx \frac{T_i^{j+1} - T_i^j}{\Delta t}$$

# The Explicit Method

Substituting these approximations into the governing equation yields

$$\alpha \frac{T_{i+1}^j - 2T_i^j + T_{i-1}^j}{(\Delta x)^2} = \frac{T_i^{j+1} - T_i^j}{\Delta t}$$

Solving for the temp at the time node  $j+1$  gives

$$T_i^{j+1} = T_i^j + \alpha \frac{\Delta t}{(\Delta x)^2} (T_{i+1}^j - 2T_i^j + T_{i-1}^j)$$

choosing,

$$\lambda = \alpha \frac{\Delta t}{(\Delta x)^2}$$

we can write the equation as,

$$T_i^{j+1} = T_i^j + \lambda (T_{i+1}^j - 2T_i^j + T_{i-1}^j).$$

# The Explicit Method

$$T_i^{j+1} = T_i^j + \lambda \left( T_{i+1}^j - 2T_i^j + T_{i-1}^j \right)$$

- This equation can be solved explicitly because it can be written for each internal location node of the rod for time node  $j+1$  in terms of the temperature at time node  $j$ .
- In other words, if we know the temperature at node  $j=0$ , and the boundary temperatures, we can find the temperature at the next time step.
- We continue the process by first finding the temperature at all nodes  $j=1$ , and using these to find the temperature at the next time node,  $j=2$ . This process continues until we reach the time at which we are interested in finding the temperature.

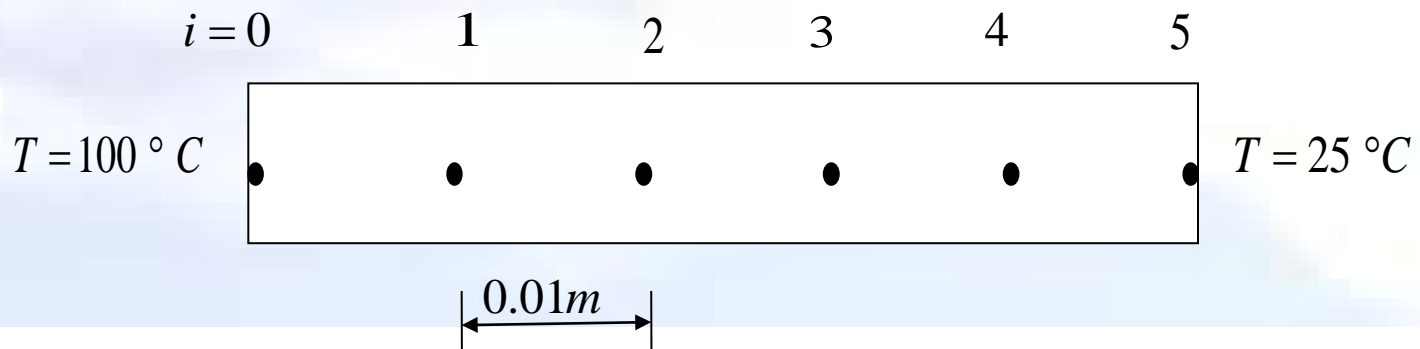


# Example I: Explicit Method

Consider a steel rod that is subjected to a temperature of  $100^{\circ}\text{C}$  on the left end and  $25^{\circ}\text{C}$  on the right end. If the rod is of length  $0.05\text{m}$ , use the explicit method to find the temperature distribution in the rod from  $t = 0$  and  $t = 9$  seconds. Use  $\Delta x = 0.01\text{m}$ ,  $\Delta t = 3\text{s}$ .

**Given:**  $k = 54 \frac{\text{W}}{\text{m} \cdot \text{K}}$ ,  $\rho = 7800 \frac{\text{kg}}{\text{m}^3}$ ,  $C = 490 \frac{\text{J}}{\text{kg} \cdot \text{K}}$

The initial temperature of the rod is  $20^{\circ}\text{C}$ .



# Example I: Explicit Method

Recall,

$$\alpha = \frac{k}{\rho C}$$

therefore,

$$\begin{aligned}\alpha &= \frac{54}{7800 \times 490} \\ &= 1.4129 \times 10^{-5} \text{ m}^2 / \text{s}.\end{aligned}$$

Then,

$$\begin{aligned}\lambda &= \alpha \frac{\Delta t}{(\Delta x)^2} \\ &= 1.4129 \times 10^{-5} \frac{3}{(0.01)^2} \\ &= 0.4239.\end{aligned}$$

Number of time steps,

$$\begin{aligned}&= \frac{t_{\text{final}} - t_{\text{initial}}}{\Delta t} \\ &= \frac{9 - 0}{3} \\ &= 3.\end{aligned}$$

Boundary Conditions

$$\left. \begin{aligned}T_0^j &= 100^\circ\text{C} \\ T_5^j &= 25^\circ\text{C}\end{aligned} \right\} \text{ for all } j = 0, 1, 2, 3$$

All internal nodes are at  $20^\circ\text{C}$  for  $t = 0$  sec. This can be represented as,

$$T_i^0 = 20^\circ\text{C}, \text{ for all } i = 1, 2, 3, 4$$

# Example I: Explicit Method

**Nodal temperatures when  $t = 0 \text{ sec}$ ,  $j = 0$ :**

$$\left. \begin{array}{l} T_0^0 = 100^\circ\text{C} \\ T_1^0 = 20^\circ\text{C} \\ T_2^0 = 20^\circ\text{C} \\ T_3^0 = 20^\circ\text{C} \\ T_4^0 = 20^\circ\text{C} \\ T_5^0 = 25^\circ\text{C} \end{array} \right\} \text{Interior nodes}$$

**We can now calculate the temperature at each node explicitly using the equation formulated earlier,**

$$T_i^{j+1} = T_i^j + \lambda \left( T_{i+1}^j - 2T_i^j + T_{i-1}^j \right)$$

# Example I: Explicit Method

## Nodal temperatures when $t = 3 \text{ sec}$ (Example Calculations)

$$\underline{i = 0} \quad T_0^1 = 100^\circ\text{C} - \text{Boundary Condition}$$

**setting**  $j = 0$

$$\begin{aligned} \underline{i = 1} \quad T_1^1 &= T_1^0 + \lambda(T_2^0 - 2T_1^0 + T_0^0) \\ &= 20 + 0.4239(20 - 2(20) + 100) \\ &= 20 + 0.4239(80) \\ &= 20 + 33.912 \\ &= 53.912^\circ\text{C} \end{aligned}$$

$$\begin{aligned} \underline{i = 2} \quad T_2^1 &= T_2^0 + \lambda(T_3^0 - 2T_2^0 + T_1^0) \\ &= 20 + 0.4239(20 - 2(20) + 20) \\ &= 20 + 0.4239(0) \\ &= 20 + 0 \\ &= 20^\circ\text{C} \end{aligned}$$

## Nodal temperatures when $t = 3 \text{ sec}$ , $j = 1$ :

$$T_0^1 = 100^\circ\text{C} - \text{Boundary Condition}$$

$$\left. \begin{aligned} T_1^1 &= 53.912^\circ\text{C} \\ T_2^1 &= 20^\circ\text{C} \\ T_3^1 &= 20^\circ\text{C} \\ T_4^1 &= 22.120^\circ\text{C} \end{aligned} \right\} \text{Interior nodes}$$

$$T_5^1 = 25^\circ\text{C} - \text{Boundary Condition}$$

# Example I: Explicit Method

**Nodal temperatures when  $t = 6 \text{ sec}$  (Example Calculations)**

$i = 0$      $T_0^2 = 100^\circ\text{C}$  – Boundary Condition  
                  **setting**     $j = 1$  ,

$i = 1$      $T_1^2 = T_1^1 + \lambda(T_2^1 - 2T_1^1 + T_0^1)$   
                   $= 53.912 + 0.4239(20 - 2(53.912) + 100)$   
                   $= 53.912 + 0.4239(12.176)$   
                   $= 53.912 + 5.1614$   
                   $= 59.073^\circ\text{C}$

$i = 2$      $T_2^2 = T_2^1 + \lambda(T_3^1 - 2T_2^1 + T_1^1)$   
                   $= 20 + 0.4239(20 - 2(20) + 53.912)$   
                   $= 20 + 0.4239(33.912)$   
                   $= 20 + 14.375$   
                   $= 34.375^\circ\text{C}$

**Nodal temperatures when  $t = 6 \text{ sec}$  ,  $j = 2$ :**

$T_0^2 = 100^\circ\text{C}$  – Boundary Condition

$T_1^2 = 59.073^\circ\text{C}$

$T_2^2 = 34.375^\circ\text{C}$

$T_3^2 = 20.889^\circ\text{C}$

$T_4^2 = 22.442^\circ\text{C}$

$T_5^2 = 25^\circ\text{C}$  – Boundary Condition

} Interior nodes

# Example I: Explicit Method

**Nodal temperatures when  $t = 9 \text{ sec}$  (Example Calculations)**

$i = 0$      $T_0^3 = 100^\circ\text{C}$  – Boundary Condition  
                  **setting  $j = 2$  ,**

$i = 1$   
$$T_1^3 = T_1^2 + \lambda(T_2^2 - 2T_1^2 + T_0^2)$$
$$= 59.073 + 0.4239(34.375 - 2(59.073) + 100)$$
$$= 59.073 + 0.4239(16.229)$$
$$= 59.073 + 6.8795$$
$$= 65.953^\circ\text{C}$$

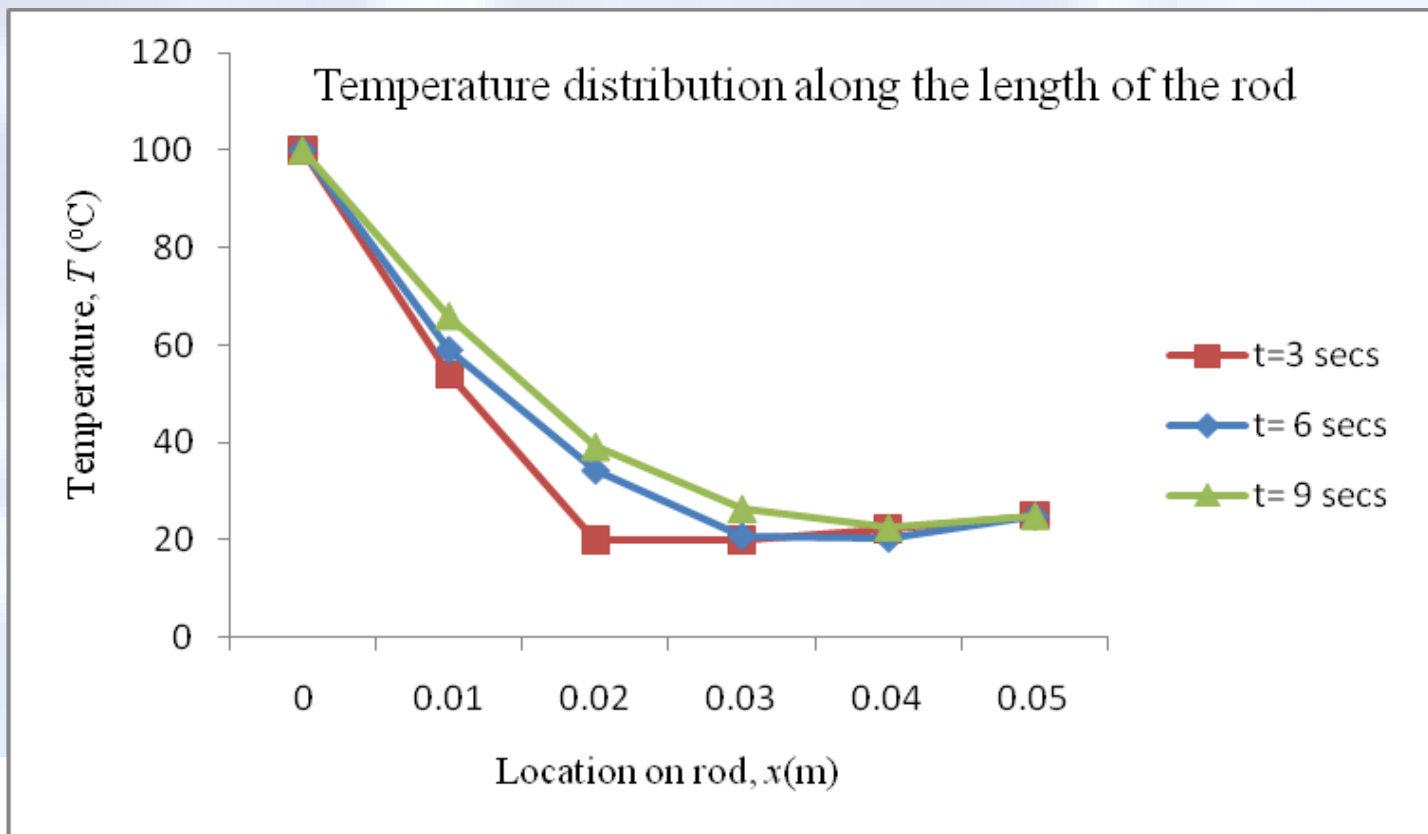
$i = 2$   
$$T_2^3 = T_2^2 + \lambda(T_3^2 - 2T_2^2 + T_1^2)$$
$$= 34.375 + 0.4239(20.899 - 2(34.375) + 59.073)$$
$$= 34.375 + 0.4239(11.222)$$
$$= 34.375 + 4.7570$$
$$= 39.132^\circ\text{C}$$

**Nodal temperatures when  $t = 9 \text{ sec}, j = 3$  :**

$$\left. \begin{array}{l} T_0^3 = 100^\circ\text{C} \text{ – Boundary Condition} \\ T_1^3 = 65.953^\circ\text{C} \\ T_2^3 = 39.132^\circ\text{C} \\ T_3^3 = 27.266^\circ\text{C} \\ T_4^3 = 22.872^\circ\text{C} \end{array} \right\} \text{Interior nodes}$$
$$T_5^3 = 25^\circ\text{C} \text{ – Boundary Condition}$$

# Example I: Explicit Method

To better visualize the temperature variation at different locations at different times, the temperature distribution along the length of the rod at different times is plotted below.



# The Implicit Method

## WHY:

- Using the explicit method, we were able to find the temperature at each node, one equation at a time.
- However, the temperature at a specific node was only dependent on the temperature of the neighboring nodes from the previous time step. This is contrary to what we expect from the physical problem.
- The implicit method allows us to solve this and other problems by developing a system of simultaneous linear equations for the temperature at all interior nodes at a particular time.



# The Implicit Method

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

The second derivative on the left hand side of the equation is approximated by the CDD scheme at time level  $j+1$  at node ( $i$ ) as

$$\left. \frac{\partial^2 T}{\partial x^2} \right|_{i,j+1} \approx \frac{T_{i+1}^{j+1} - 2T_i^{j+1} + T_{i-1}^{j+1}}{(\Delta x)^2}$$

# The Implicit Method

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

The first derivative on the right hand side of the equation is approximated by the BDD scheme at time level  $j+1$  at node ( $i$ ) as

$$\left. \frac{\partial T}{\partial t} \right|_{i,j+1} \approx \frac{T_i^{j+1} - T_i^j}{\Delta t}$$

# The Implicit Method

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

Substituting these approximations into the heat conduction equation yields

$$\alpha \frac{T_{i+1}^{j+1} - 2T_i^{j+1} + T_{i-1}^{j+1}}{(\Delta x)^2} = \frac{T_i^{j+1} - T_i^j}{\Delta t}$$

# The Implicit Method

From the previous slide,

$$\alpha \frac{T_{i+1}^{j+1} - 2T_i^{j+1} + T_{i-1}^{j+1}}{(\Delta x)^2} = \frac{T_i^{j+1} - T_i^j}{\Delta t}$$

Rearranging yields

$$-\lambda T_{i-1}^{j+1} + (1 + 2\lambda)T_i^{j+1} - \lambda T_{i+1}^{j+1} = T_i^j$$

given that,

$$\lambda = \alpha \frac{\Delta t}{(\Delta x)^2}$$

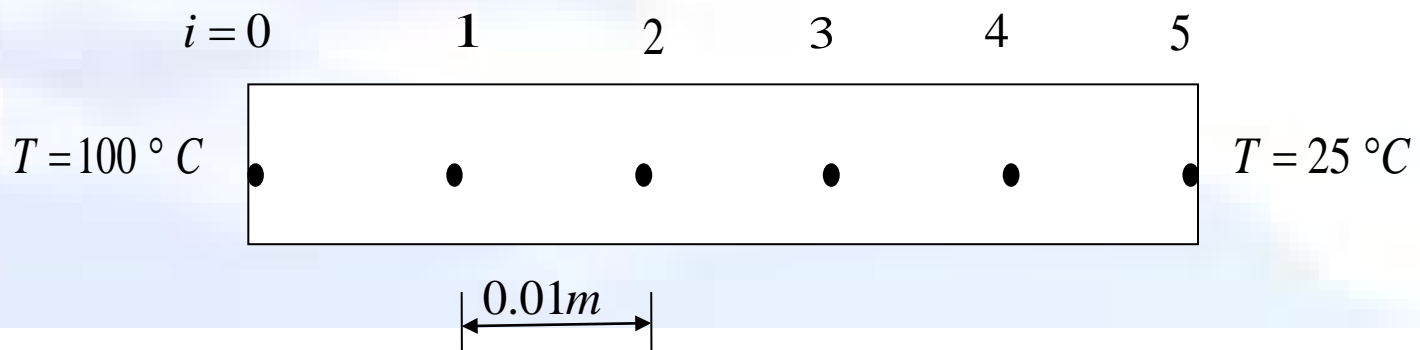
The rearranged equation can be written for every node during each time step. These equations can then be solved as a simultaneous system of linear equations to find the nodal temperatures at a particular time.

# Example 2: Implicit Method

Consider a steel rod that is subjected to a temperature of  $100^{\circ}\text{C}$  on the left end and  $25^{\circ}\text{C}$  on the right end. If the rod is of length  $0.05\text{m}$ , use the implicit method to find the temperature distribution in the rod from  $t = 0$  and  $t = 9$  seconds. Use  $\Delta x = 0.01\text{m}$ ,  $\Delta t = 3\text{s}$ .

**Given:**  $k = 54 \frac{\text{W}}{\text{m} \cdot \text{K}}$ ,  $\rho = 7800 \frac{\text{kg}}{\text{m}^3}$ ,  $C = 490 \frac{\text{J}}{\text{kg} \cdot \text{K}}$

The initial temperature of the rod is  $20^{\circ}\text{C}$ .



# Example 2: Implicit Method

Recall,

$$\alpha = \frac{k}{\rho C}$$

therefore,

$$\begin{aligned}\alpha &= \frac{54}{7800 \times 490} \\ &= 1.4129 \times 10^{-5} \text{ m}^2 / \text{s}.\end{aligned}$$

Then,

$$\begin{aligned}\lambda &= \alpha \frac{\Delta t}{(\Delta x)^2} \\ &= 1.4129 \times 10^{-5} \frac{3}{(0.01)^2} \\ &= 0.4239.\end{aligned}$$

Number of time steps,

$$\begin{aligned}&= \frac{t_{\text{final}} - t_{\text{initial}}}{\Delta t} \\ &= \frac{9 - 0}{3} \\ &= 3.\end{aligned}$$

Boundary Conditions

$$\left. \begin{aligned}T_0^j &= 100^\circ\text{C} \\ T_5^j &= 25^\circ\text{C}\end{aligned} \right\} \text{ for all } j = 0, 1, 2, 3$$

**All internal nodes are at  $20^\circ\text{C}$  for  $t = 0$  sec. This can be represented as,**

$$T_i^0 = 20^\circ\text{C}, \text{ for all } i = 1, 2, 3, 4$$

# Example 2: Implicit Method

**Nodal temperatures when  $t = 0\text{sec}$ ,  $j = 0$ :**

$$\left. \begin{array}{l} T_0^0 = 100^\circ\text{C} \\ T_1^0 = 20^\circ\text{C} \\ T_2^0 = 20^\circ\text{C} \\ T_3^0 = 20^\circ\text{C} \\ T_4^0 = 20^\circ\text{C} \\ T_5^0 = 25^\circ\text{C} \end{array} \right\} \text{Interior nodes}$$

**We can now form our system of equations for the first time step by writing the approximated heat conduction equation for each node.**

$$-\lambda T_{i-1}^{j+1} + (1 + 2\lambda)T_i^{j+1} - \lambda T_{i+1}^{j+1} = T_i^j$$

# Example 2: Implicit Method

**Nodal temperatures when  $t = 3 \text{ sec}$ , (Example Calculations)**

$i = 0$   $T_0^1 = 100^\circ\text{C}$  – Boundary Condition

**For the interior nodes setting  $j = 0$  and  $i = 1, 2, 3, 4$  gives the following,**

$$\begin{aligned} \underline{i = 1} \quad & -\lambda T_0^1 + (1 + 2\lambda)T_1^1 - \lambda T_2^1 = T_1^0 \\ & (-0.4239 \times 100) + (1 + 2 \times 0.4239)T_1^1 - (0.4239T_2^1) = 20 \\ & -42.39 + 1.8478T_1^1 - 0.4239T_2^1 = 20 \\ & 1.8478T_1^1 - 0.4239T_2^1 = 62.390 \end{aligned}$$

$$\begin{aligned} \underline{i = 2} \quad & -\lambda T_1^1 + (1 + 2\lambda)T_2^1 - \lambda T_3^1 = T_2^0 \\ & -0.4239T_1^1 + 1.8478T_2^1 - 0.4239T_3^1 = 20 \end{aligned}$$

**For the first time step we can write four such equations with four unknowns, expressing them in matrix form yields**

$$\begin{bmatrix} 1.8478 & -0.4239 & 0 & 0 \\ -0.4239 & 1.8478 & -0.4239 & 0 \\ 0 & -0.4239 & 1.8478 & -0.4239 \\ 0 & 0 & -0.4239 & 1.8478 \end{bmatrix} \begin{bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \end{bmatrix} = \begin{bmatrix} 62.390 \\ 20 \\ 20 \\ 30.598 \end{bmatrix}$$



# Example 2: Implicit Method

$$\begin{bmatrix} 1.8478 & -0.4239 & 0 & 0 \\ -0.4239 & 1.8478 & -0.4239 & 0 \\ 0 & -0.4239 & 1.8478 & -0.4239 \\ 0 & 0 & -0.4239 & 1.8478 \end{bmatrix} \begin{bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \end{bmatrix} = \begin{bmatrix} 62.390 \\ 20 \\ 20 \\ 30.598 \end{bmatrix}$$

The above coefficient matrix is tri-diagonal. Special algorithms such as Thomas' algorithm can be used to solve simultaneous linear equation with tri-diagonal coefficient matrices. The solution is given by

$$\begin{bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \end{bmatrix} = \begin{bmatrix} 39.451 \\ 24.792 \\ 21.438 \\ 21.477 \end{bmatrix}$$

Hence, the nodal  
temps at  $t = 3$  sec are  $\longrightarrow$

$$\begin{bmatrix} T_0^1 \\ T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \\ T_5^1 \end{bmatrix} = \begin{bmatrix} 100 \\ 39.451 \\ 24.792 \\ 21.438 \\ 21.477 \\ 25 \end{bmatrix}$$

# Example 2: Implicit Method

**Nodal temperatures when  $t = 6 \text{ sec}$ , (Example Calculations)**

$i = 0$   $T_0^2 = 100^\circ\text{C}$  – Boundary Condition

**For the interior nodes setting  $j = 1$  and  $i = 1, 2, 3, 4$  gives the following,**

$$\underline{i = 1} \quad -\lambda T_0^2 + (1 + 2\lambda)T_1^2 - \lambda T_2^2 = T_1^1$$

$$(-0.4239 \times 100) + (1 + 2 \times 0.4239)T_1^2 - 0.4239T_2^2 = 39.451$$

$$-42.39 + 1.8478T_1^2 - 0.4239T_2^2 = 39.451$$

$$1.8478T_1^2 - 0.4239T_2^2 = 81.841$$

$$\underline{i = 2} \quad -\lambda T_1^2 + (1 + 2\lambda)T_2^2 - \lambda T_3^2 = T_2^1$$

$$-0.4239T_1^2 + 1.8478T_2^2 - 0.4239T_3^2 = 24.792$$

**For the second time step we can write four such equations with four unknowns, expressing them in matrix form yields**

$$\begin{bmatrix} 1.8478 & -0.4239 & 0 & 0 \\ -0.4239 & 1.8478 & -0.4239 & 0 \\ 0 & -0.4239 & 1.8478 & -0.4239 \\ 0 & 0 & -0.4239 & 1.8478 \end{bmatrix} \begin{bmatrix} T_1^2 \\ T_2^2 \\ T_3^2 \\ T_4^2 \end{bmatrix} = \begin{bmatrix} 81.841 \\ 24.792 \\ 21.438 \\ 32.075 \end{bmatrix}$$

# Example 2: Implicit Method

$$\begin{bmatrix} 1.8478 & -0.4239 & 0 & 0 \\ -0.4239 & 1.8478 & -0.4239 & 0 \\ 0 & -0.4239 & 1.8478 & -0.4239 \\ 0 & 0 & -0.4239 & 1.8478 \end{bmatrix} \begin{bmatrix} T_1^2 \\ T_2^2 \\ T_3^2 \\ T_4^2 \end{bmatrix} = \begin{bmatrix} 81.841 \\ 24.792 \\ 21.438 \\ 32.075 \end{bmatrix}$$

The above coefficient matrix is tri-diagonal. Special algorithms such as Thomas' algorithm can be used to solve simultaneous linear equation with tri-diagonal coefficient matrices. The solution is given by

$$\begin{bmatrix} T_1^2 \\ T_2^2 \\ T_3^2 \\ T_4^2 \end{bmatrix} = \begin{bmatrix} 51.326 \\ 30.669 \\ 23.876 \\ 22.836 \end{bmatrix} \quad \xrightarrow{\text{Hence, the nodal temps at } t = 6 \text{ sec are}} \quad \begin{bmatrix} T_0^2 \\ T_1^2 \\ T_2^2 \\ T_3^2 \\ T_4^2 \\ T_5^2 \end{bmatrix} = \begin{bmatrix} 100 \\ 51.326 \\ 30.669 \\ 23.876 \\ 22.836 \\ 25 \end{bmatrix}$$

# Example 2: Implicit Method

**Nodal temperatures when  $t = 9 \text{ sec}$  , (Example Calculations)**

$i = 0$   $T_0^3 = 100^\circ\text{C}$  – Boundary Condition

**For the interior nodes setting  $j = 2$  and  $i = 1, 2, 3, 4$  gives the following,**

$$\underline{i = 1} \quad -\lambda T_0^3 + (1 + 2\lambda)T_1^3 - \lambda T_2^3 = T_1^2$$

$$(-0.4239 \times 100) + (1 + 2 \times 0.4239)T_1^3 - (0.4239T_2^3) = 51.326$$

$$-42.39 + 1.8478T_1^3 - 0.4239T_2^3 = 51.326$$

$$1.8478T_1^3 - 0.4239T_2^3 = 93.716$$

$$\underline{i = 2} \quad -\lambda T_1^3 + (1 + 2\lambda)T_2^3 - \lambda T_3^3 = T_2^2$$

$$-0.4239T_1^3 + 1.8478T_2^3 - 0.4239T_3^3 = 30.669$$

**For the third time step we can write four such equations with four unknowns, expressing them in matrix form yields**

$$\begin{bmatrix} 1.8478 & -0.4239 & 0 & 0 \\ -0.4239 & 1.8478 & -0.4239 & 0 \\ 0 & -0.4239 & 1.8478 & -0.4239 \\ 0 & 0 & -0.4239 & 1.8478 \end{bmatrix} \begin{bmatrix} T_1^3 \\ T_2^3 \\ T_3^3 \\ T_4^3 \end{bmatrix} = \begin{bmatrix} 93.716 \\ 30.669 \\ 23.876 \\ 33.434 \end{bmatrix}$$

# Example 2: Implicit Method

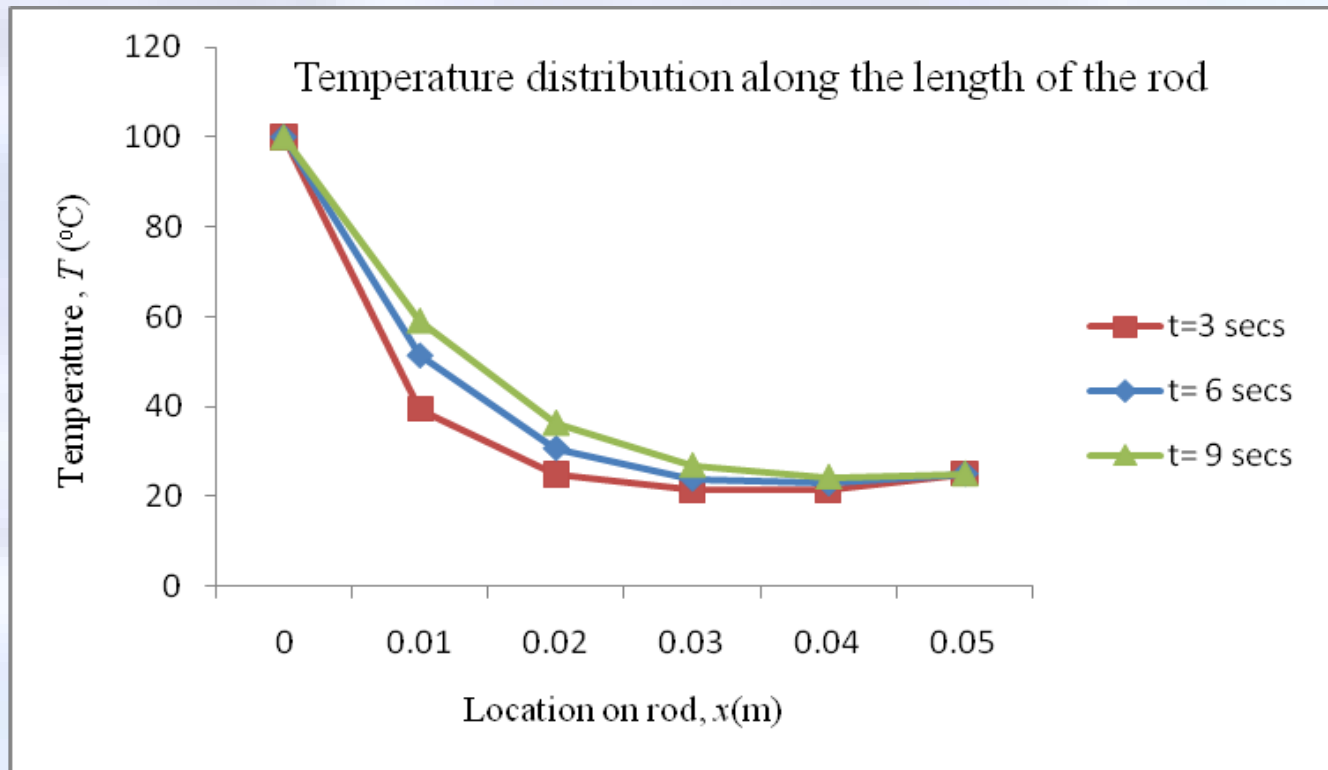
$$\begin{bmatrix} 1.8478 & -0.4239 & 0 & 0 \\ -0.4239 & 1.8478 & -0.4239 & 0 \\ 0 & -0.4239 & 1.8478 & -0.4239 \\ 0 & 0 & -0.4239 & 1.8478 \end{bmatrix} \begin{bmatrix} T_1^3 \\ T_2^3 \\ T_3^3 \\ T_4^3 \end{bmatrix} = \begin{bmatrix} 93.716 \\ 30.669 \\ 23.876 \\ 33.434 \end{bmatrix}$$

The above coefficient matrix is tri-diagonal. Special algorithms such as Thomas' algorithm can be used to solve simultaneous linear equation with tri-diagonal coefficient matrices. The solution is given by

$$\begin{bmatrix} T_1^3 \\ T_2^3 \\ T_3^3 \\ T_4^3 \end{bmatrix} = \begin{bmatrix} 59.043 \\ 36.292 \\ 26.809 \\ 24.243 \end{bmatrix} \quad \text{Hence, the nodal temps at } t = 9 \text{ sec are} \quad \begin{bmatrix} T_0^3 \\ T_1^3 \\ T_2^3 \\ T_3^3 \\ T_4^3 \\ T_5^3 \end{bmatrix} = \begin{bmatrix} 100 \\ 59.043 \\ 36.292 \\ 26.809 \\ 24.243 \\ 25 \end{bmatrix}$$

# Example 2: Implicit Method

To better visualize the temperature variation at different locations at different times, the temperature distribution along the length of the rod at different times is plotted below.



# The Crank-Nicolson Method

**WHY:**

Using the implicit method our approximation of  $\frac{\partial^2 T}{\partial x^2}$  was of  $O(\Delta x)^2$  accuracy, while our approximation of  $\frac{\partial T}{\partial t}$  was of  $O(\Delta t)$  accuracy.

# The Crank-Nicolson Method

One can achieve similar orders of accuracy by approximating the second derivative, on the left hand side of the heat equation, at the midpoint of the time step. Doing so yields

$$\left. \frac{\partial^2 T}{\partial x^2} \right|_{i,j} \approx \frac{\alpha}{2} \left[ \frac{T_{i+1}^j - 2T_i^j + T_{i-1}^j}{(\Delta x)^2} + \frac{T_{i+1}^{j+1} - 2T_i^{j+1} + T_{i-1}^{j+1}}{(\Delta x)^2} \right]$$



# The Crank-Nicolson Method

The first derivative, on the right hand side of the heat equation, is approximated using the forward divided difference method at time level  $j+1$ ,

$$\frac{\partial T}{\partial t} \Big|_{i,j} \approx \frac{T_i^{j+1} - T_i^j}{\Delta t}$$

# The Crank-Nicolson Method

- Substituting these approximations into the governing equation for heat conductance yields

$$\frac{\alpha}{2} \left[ \frac{T_{i+1}^j - 2T_i^j + T_{i-1}^j}{(\Delta x)^2} + \frac{T_{i+1}^{j+1} - 2T_i^{j+1} + T_{i-1}^{j+1}}{(\Delta x)^2} \right] = \frac{T_i^{j+1} - T_i^j}{\Delta t}$$

giving

$$-\lambda T_{i-1}^{j+1} + 2(1 + \lambda)T_i^{j+1} - \lambda T_{i+1}^{j+1} = \lambda T_{i-1}^j + 2(1 - \lambda)T_i^j + \lambda T_{i+1}^j$$

where

$$\lambda = \alpha \frac{\Delta t}{(\Delta x)^2}$$

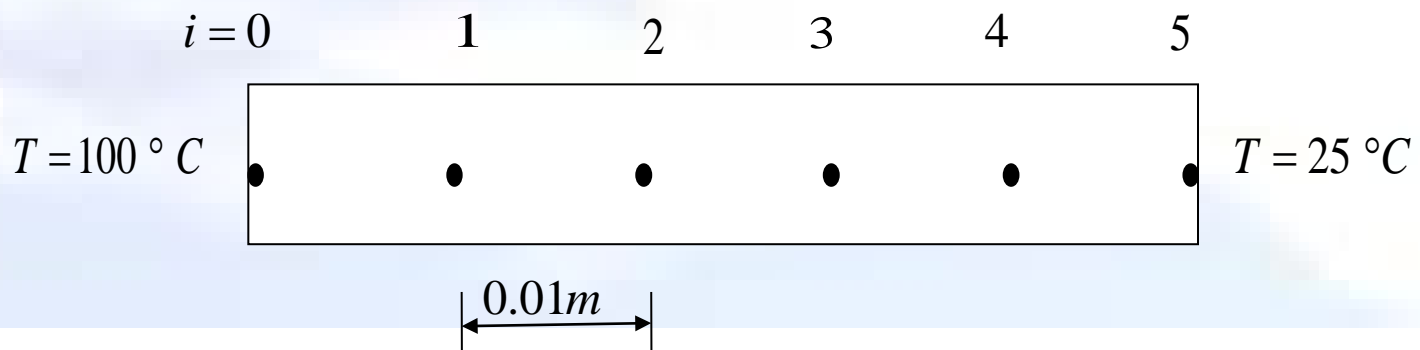
- Having rewritten the equation in this form allows us to discretize the physical problem. We then solve a system of simultaneous linear equations to find the temperature at every node at any point in time.

# Example 3: Crank-Nicolson

Consider a steel rod that is subjected to a temperature of  $100^{\circ}\text{C}$  on the left end and  $25^{\circ}\text{C}$  on the right end. If the rod is of length  $0.05\text{m}$ , use the Crank-Nicolson method to find the temperature distribution in the rod from  $t = 0$  to  $t = 9$  seconds. Use  $\Delta x = 0.01\text{m}$ ,  $\Delta t = 3\text{s}$ .

**Given:**  $k = 54 \frac{\text{W}}{\text{m} \cdot \text{K}}$ ,  $\rho = 7800 \frac{\text{kg}}{\text{m}^3}$ ,  $C = 490 \frac{\text{J}}{\text{kg} \cdot \text{K}}$

The initial temperature of the rod is  $20^{\circ}\text{C}$ .



# Example 3: Crank-Nicolson

Recall,

$$\alpha = \frac{k}{\rho C}$$

therefore,

$$\begin{aligned}\alpha &= \frac{54}{7800 \times 490} \\ &= 1.4129 \times 10^{-5} \text{ m}^2 / \text{s}.\end{aligned}$$

Then,

$$\begin{aligned}\lambda &= \alpha \frac{\Delta t}{(\Delta x)^2} \\ &= 1.4129 \times 10^{-5} \frac{3}{(0.01)^2} \\ &= 0.4239.\end{aligned}$$

Number of time steps,

$$\begin{aligned}&= \frac{t_{\text{final}} - t_{\text{initial}}}{\Delta t} \\ &= \frac{9 - 0}{3} \\ &= 3.\end{aligned}$$

Boundary Conditions

$$\left. \begin{aligned}T_0^j &= 100^\circ\text{C} \\ T_5^j &= 25^\circ\text{C}\end{aligned} \right\} \text{ for all } j = 0, 1, 2, 3$$

**All internal nodes are at  $20^\circ\text{C}$  for  $t = 0$  sec. This can be represented as,**

$$T_i^0 = 20^\circ\text{C}, \text{ for all } i = 1, 2, 3, 4$$

# Example 3: Crank-Nicolson

**Nodal temperatures when  $t = 0\text{sec}$ ,  $j = 0$ :**

$$\left. \begin{array}{l} T_0^0 = 100^\circ\text{C} \\ T_1^0 = 20^\circ\text{C} \\ T_2^0 = 20^\circ\text{C} \\ T_3^0 = 20^\circ\text{C} \\ T_4^0 = 20^\circ\text{C} \\ T_5^0 = 25^\circ\text{C} \end{array} \right\} \text{Interior nodes}$$

**We can now form our system of equations for the first time step by writing the approximated heat conduction equation for each node.**

$$-\lambda T_{i-1}^{j+1} + 2(1 + \lambda)T_i^{j+1} - \lambda T_{i+1}^{j+1} = \lambda T_{i-1}^j + 2(1 - \lambda)T_i^j + \lambda T_{i+1}^j$$

# Example 3: Crank-Nicolson

**Nodal temperatures when  $t = 3 \text{ sec}$ , (Example Calculations)**

$i = 0$      $T_0^1 = 100^\circ\text{C}$  – Boundary Condition

**For the interior nodes setting  $j = 0$  and  $i = 1, 2, 3, 4$  gives the following**

$i = 1$

$$-\lambda T_0^1 + 2(1 + \lambda)T_1^1 - \lambda T_2^1 = \lambda T_0^0 + 2(1 - \lambda)T_1^0 + \lambda T_2^0$$

$$(-0.4239 \times 100) + 2(1 + 0.4239)T_1^1 - 0.4239T_2^1 = (0.4239)100 + 2(1 - 0.4239)20 + (0.4239)20$$

$$-42.39 + 2.8478T_1^1 - 0.4239T_2^1 = 42.39 + 23.044 + 8.478$$

$$2.8478T_1^1 - 0.4239T_2^1 = 116.30$$

**For the first time step we can write four such equations with four unknowns, expressing them in matrix form yields**

$$\begin{bmatrix} 2.8478 & -0.4239 & 0 & 0 \\ -0.4239 & 2.8478 & -0.4239 & 0 \\ 0 & -0.4239 & 2.8478 & -0.4239 \\ 0 & 0 & -0.4239 & 2.8478 \end{bmatrix} \begin{bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \end{bmatrix} = \begin{bmatrix} 116.30 \\ 40.000 \\ 40.000 \\ 52.718 \end{bmatrix}$$

# Example 3: Crank-Nicolson

$$\begin{bmatrix} 2.8478 & -0.4239 & 0 & 0 \\ -0.4239 & 2.8478 & -0.4239 & 0 \\ 0 & -0.4239 & 2.8478 & -0.4239 \\ 0 & 0 & -0.4239 & 2.8478 \end{bmatrix} \begin{bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \end{bmatrix} = \begin{bmatrix} 116.30 \\ 40.000 \\ 40.000 \\ 52.718 \end{bmatrix}$$

The above coefficient matrix is tri-diagonal. Special algorithms such as Thomas' algorithm can be used to solve simultaneous linear equation with tri-diagonal coefficient matrices. The solution is given by

$$\begin{bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \end{bmatrix} = \begin{bmatrix} 44.372 \\ 23.746 \\ 20.797 \\ 21.607 \end{bmatrix} \xrightarrow{\text{Hence, the nodal temps at } t = 3 \text{ sec are}} \begin{bmatrix} T_0^1 \\ T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \\ T_5^1 \end{bmatrix} = \begin{bmatrix} 100 \\ 44.372 \\ 23.746 \\ 20.797 \\ 21.607 \\ 25 \end{bmatrix}$$

# Example 3: Crank-Nicolson

**Nodal temperatures when  $t = 6 \text{ sec}$ , (Example Calculations)**

$i = 0$   $T_0^2 = 100^\circ\text{C}$  – Boundary Condition

**For the interior nodes setting  $j = 1$  and  $i = 1, 2, 3, 4$  gives the following,**  
 $i = 1$

$$-\lambda T_0^2 + 2(1 + \lambda)T_1^2 - \lambda T_2^2 = \lambda T_0^1 + 2(1 - \lambda)T_1^1 + \lambda T_2^1$$

$$(-0.4239 \times 100) + 2(1 + 0.4239)T_1^2 - 0.4239T_2^2 =$$

$$(0.4239)100 + 2(1 - 0.4239)44.372 + (0.4239)23.746$$

$$-42.39 + 2.8478T_1^2 - 0.4239T_2^2 = 42.39 + 51.125 + 10.066$$

$$\boxed{2.8478T_1^2 - 0.4239T_2^2 = 145.971}$$

**For the second time step we can write four such equations with four unknowns, expressing them in matrix form yields**

$$\begin{bmatrix} 2.8478 & -0.4239 & 0 & 0 \\ -0.4239 & 2.8478 & -0.4239 & 0 \\ 0 & -0.4239 & 2.8478 & -0.4239 \\ 0 & 0 & -0.4239 & 2.8478 \end{bmatrix} \begin{bmatrix} T_1^2 \\ T_2^2 \\ T_3^2 \\ T_4^2 \end{bmatrix} = \begin{bmatrix} 145.971 \\ 54.985 \\ 43.187 \\ 54.908 \end{bmatrix}$$



# Example 3: Crank-Nicolson

$$\begin{bmatrix} 2.8478 & -0.4239 & 0 & 0 \\ -0.4239 & 2.8478 & -0.4239 & 0 \\ 0 & -0.4239 & 2.8478 & -0.4239 \\ 0 & 0 & -0.4239 & 2.8478 \end{bmatrix} \begin{bmatrix} T_1^2 \\ T_2^2 \\ T_3^2 \\ T_4^2 \end{bmatrix} = \begin{bmatrix} 145.971 \\ 54.985 \\ 43.187 \\ 54.908 \end{bmatrix}$$

The above coefficient matrix is tri-diagonal. Special algorithms such as Thomas' algorithm can be used to solve simultaneous linear equation with tri-diagonal coefficient matrices. The solution is given by

$$\begin{bmatrix} T_1^2 \\ T_2^2 \\ T_3^2 \\ T_4^2 \end{bmatrix} = \begin{bmatrix} 55.883 \\ 31.075 \\ 23.174 \\ 22.730 \end{bmatrix} \xrightarrow{\text{Hence, the nodal temps at } t = 6 \text{ sec are}} \begin{bmatrix} T_0^2 \\ T_1^2 \\ T_2^2 \\ T_3^2 \\ T_4^2 \\ T_5^2 \end{bmatrix} = \begin{bmatrix} 100 \\ 55.883 \\ 31.075 \\ 23.174 \\ 22.730 \\ 25 \end{bmatrix}$$

# Example 3: Crank-Nicolson

**Nodal temperatures when  $t = 9 \text{ sec}$  , (Example Calculations)**

$i = 0$      $T_0^3 = 100^\circ\text{C}$  – Boundary Condition

**For the interior nodes setting  $j = 2$  and  $i = 1, 2, 3, 4$  gives the following,**  
 $i = 1$

$$-\lambda T_0^3 + 2(1 + \lambda)T_1^3 - \lambda T_2^3 = \lambda T_0^2 + 2(1 - \lambda)T_1^2 + \lambda T_2^2$$

$$(-0.4239 \times 100) + 2(1 + 0.4239)T_1^3 - 0.4239T_2^3 =$$

$$(0.4239)100 + 2(1 - 0.4239)55.883 + (0.4239)31.075$$

$$-42.39 + 2.8478T_1^3 - 0.4239T_2^3 = 42.39 + 64.388 + 13.173$$

$$\boxed{2.8478T_1^3 - 0.4239T_2^3 = 162.34}$$

**For the third time step we can write four such equations with four unknowns, expressing them in matrix form yields**

$$\begin{bmatrix} 2.8478 & -0.4239 & 0 & 0 \\ -0.4239 & 2.8478 & -0.4239 & 0 \\ 0 & -0.4239 & 2.8478 & -0.4239 \\ 0 & 0 & -0.4239 & 2.8478 \end{bmatrix} \begin{bmatrix} T_1^3 \\ T_2^3 \\ T_3^3 \\ T_4^3 \end{bmatrix} = \begin{bmatrix} 162.34 \\ 69.318 \\ 49.509 \\ 57.210 \end{bmatrix}$$

# Example 3: Crank-Nicolson

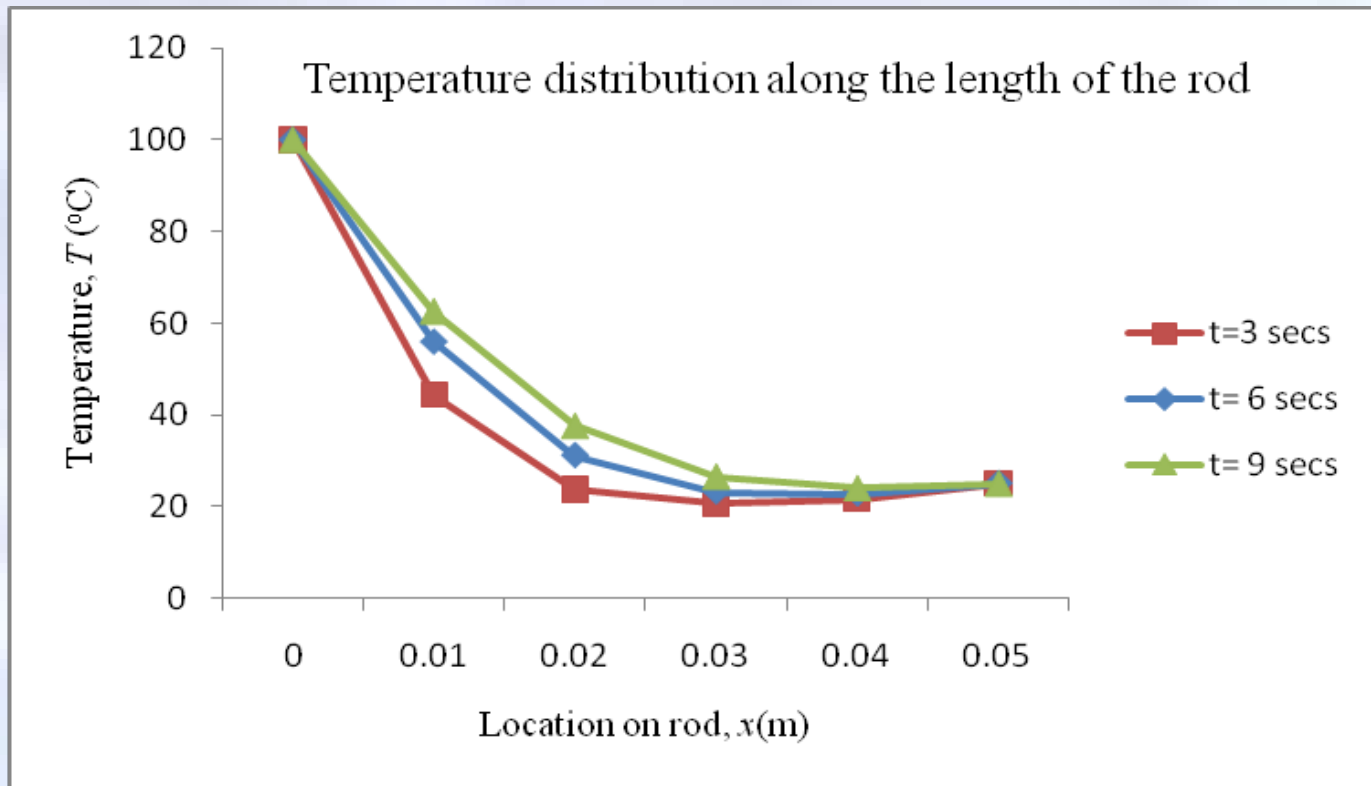
$$\begin{bmatrix} 2.8478 & -0.4239 & 0 & 0 \\ -0.4239 & 2.8478 & -0.4239 & 0 \\ 0 & -0.4239 & 2.8478 & -0.4239 \\ 0 & 0 & -0.4239 & 2.8478 \end{bmatrix} \begin{bmatrix} T_1^3 \\ T_2^3 \\ T_3^3 \\ T_4^3 \end{bmatrix} = \begin{bmatrix} 162.34 \\ 69.318 \\ 49.509 \\ 57.210 \end{bmatrix}$$

The above coefficient matrix is tri-diagonal. Special algorithms such as Thomas' algorithm can be used to solve simultaneous linear equation with tri-diagonal coefficient matrices. The solution is given by

$$\begin{bmatrix} T_1^3 \\ T_2^3 \\ T_3^3 \\ T_4^3 \end{bmatrix} = \begin{bmatrix} 62.604 \\ 37.613 \\ 26.562 \\ 24.042 \end{bmatrix} \xrightarrow{\text{Hence, the nodal temps at } t = 9 \text{ sec are}} \begin{bmatrix} T_0^3 \\ T_1^3 \\ T_2^3 \\ T_3^3 \\ T_4^3 \\ T_5^3 \end{bmatrix} = \begin{bmatrix} 100 \\ 62.604 \\ 37.613 \\ 26.562 \\ 24.042 \\ 25 \end{bmatrix}$$

# Example 3: Crank-Nicolson

To better visualize the temperature variation at different locations at different times, the temperature distribution along the length of the rod at different times is plotted below.



# Internal Temperatures at 9 sec.

The table below allows you to compare the results from all three methods discussed in juxtaposition with the analytical solution.

Node	Explicit	Implicit	Crank-Nicolson	Analytical
$T_1^3$	65.953	59.043	62.604	62.510
$T_2^3$	39.132	36.292	37.613	37.084
$T_3^3$	27.266	26.809	26.562	25.844
$T_4^3$	22.872	24.243	24.042	23.610

**THE END**