

# Chapter 10.01

## Introduction to Partial Differential Equations

*After reading this chapter, you should be able to:*

1. *identify the difference between ordinary and partial differential equations.*
2. *identify different types of partial differential equations.*

### What is a Partial Differential Equation (PDE)

A differential equation with one independent variable is called an ordinary differential equation. An example of such an equation would be

$$3\frac{dy}{dx} + 5y^2 = 3e^{-x}, y(0) = 5$$

where  $y$  is the dependent variable, and  $x$  is the independent variable.

What if there is more than one independent variable? Then the differential equation is called a partial differential equation. An example of such an equation would be

$$3\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x^2 + y^2$$

subject to certain conditions: where  $u$  is the dependent variable, and  $x$  and  $y$  are the independent variables.

### From Ordinary to a Partial Differential Equation

Assume we put a spherical steel ball that is at room temperature in hot water. The temperature of the ball is going to increase with time. What if we wish to find what this temperature vs. time profile would look like for the ball? We would develop a mathematical model for this based on the law of conservation of heat energy. From an energy balance,

$$\text{Heat gained} - \text{Heat lost} = \text{Heat stored} \quad (1)$$

The energy stored in the mass is given by

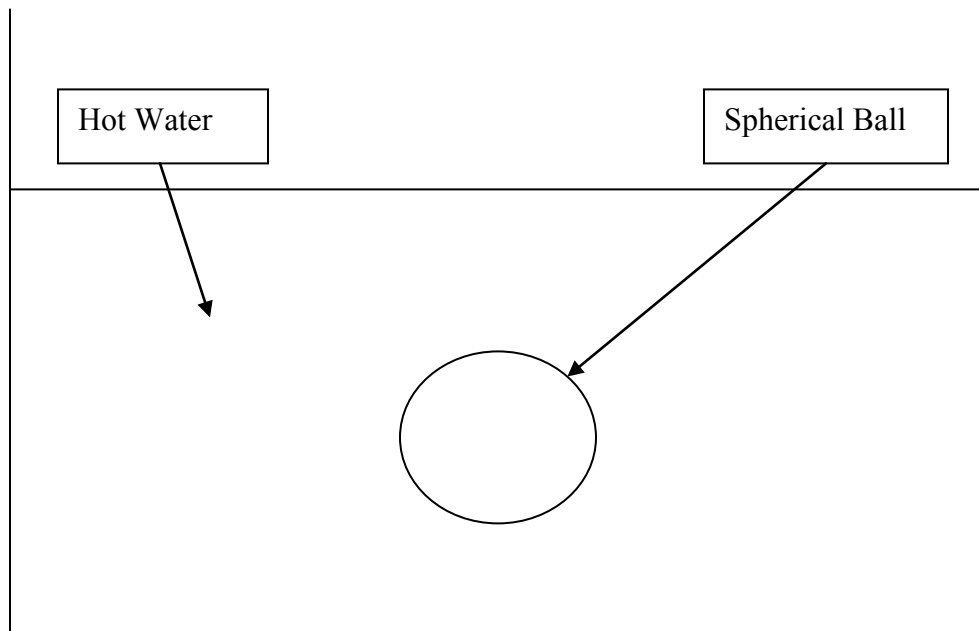
$$\text{Heat stored in the ball} = mC\theta \quad (2)$$

where

$m$  = mass of ball,  $kg$

$C$  = specific heat of the ball,  $J/(kg - K)$

$\theta$  = temperature of the ball at a given time,  $K$



The rate of heat gained by the ball due to convection is

$$\text{Rate of heat gained due to convection} = hA(\theta - \theta_a), \quad (3)$$

where

$h$  = the convective cooling coefficient,  $W/(m^2 \cdot K)$ .

$A$  = surface area of ball,  $m^2$

$\theta_a$  = ambient temperature of the hot water,  $K$

As you can see we have the expression for the rate at which heat is gained (not the heat gained), so we rewrite the heat energy balance as

$$\begin{aligned} \text{Rate at which heat is gained} - \text{Rate at which heat is lost} \\ = \text{Rate at which heat is stored} \end{aligned} \quad (4)$$

This gives us

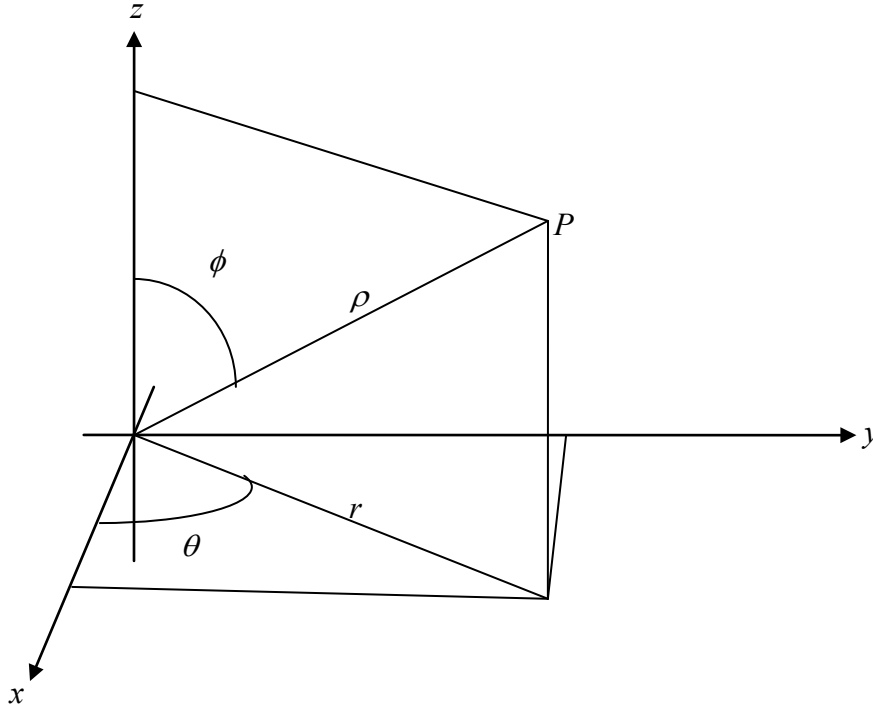
$$hA(\theta - \theta_a) = mC \frac{d\theta}{dt} \quad (5)$$

Equation (5) is a first order ordinary differential equation that when solved with the initial condition  $\theta(0) = \theta_0$ , would give us the temperature of the spherical ball as a function of time.

However, we made a large assumption in deriving Equation (5) - we assumed that the system is lumped. What does a lumped system mean? It implies that the internal conduction in the sphere is large enough that the temperature throughout the ball is uniform. This allows us to make the assumption that the temperature is only a function of time and not of the location in the spherical ball. The system being considered lumped for this case depends on: material of the ball, geometry, and heat exchange factor (convection coefficient) of the ball with its surroundings.

What happens if the system cannot be treated as a lumped system? In that case, the temperature of the ball will now be a function not only of time, but also the location.

In spherical co-ordinates, the location is given by  $r, \theta, \phi$  co-ordinates.



**Figure 1** Spherical Coordinate System.

The differential equation would now be a partial differential equation and is given as

$$\frac{k}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \theta}{\partial r} \right) + \frac{k}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \theta}{\partial \theta} \right) + \frac{k}{r^2 \sin^2 \theta} \frac{\partial^2 \theta}{\partial \phi^2} = \rho C \frac{\partial \theta}{\partial t}, t \geq 0, \theta(0) = \theta_a$$

$$\frac{\partial \theta}{\partial r} + h(\theta - \theta_a) = 0, \text{ at the surface} \quad (6)$$

where

$k$  = thermal conductivity of material,  $W / (m - K)$

$\rho$  = density of material,  $kg / m^3$

As an introduction to solve PDEs, most textbooks concentrate on linear second order PDEs with two independent variables and one dependent variable. The general form of such an equation is

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D = 0 \quad (7)$$

Where  $A, B,$  and  $C$  are functions of  $x$  and  $y$  and  $D$  is a function of  $x, y, u$  and  $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ .

Depending on the value of  $B^2 - 4AC$ , a 2<sup>nd</sup> order linear PDE can be classified into three categories.

1. if  $B^2 - 4AC < 0$ , it is called elliptic
2. if  $B^2 - 4AC = 0$ , it is called parabolic
3. if  $B^2 - 4AC > 0$ , it is called hyperbolic

### Elliptic Equation

The Laplace equation for steady state temperature in a plate is an example of an elliptic second order linear partial differential equation. The Laplace equation for steady state temperature in a plate is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad (8)$$

Using the general form of second order linear PDEs with one dependent variable and two independent variables,

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D = 0$$

$$A = 1, B = 0, C = 1, D = 0,$$

$$\begin{aligned} \text{gives } B^2 - 4AC &= 0 - 4(1)(1) \\ &= -4 \\ &= -4 < 0 \end{aligned}$$

This classifies Equation (8) as elliptic.

### Parabolic Equation

The heat conduction equation is an example of a parabolic second order linear partial differential equation. The heat conduction equation is given by

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \quad (9)$$

Using the general form of second order linear PDEs with one dependent variable and two independent variables,

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D = 0$$

$$A = k, B = 0, C = 0, D = -1,$$

$$\begin{aligned} \text{gives } B^2 - 4AC &= 0 - 4(0)(k) \\ &= 0 \end{aligned}$$

This classifies Equation (9) as parabolic.

Hyperbolic Equation

The wave equation is an example of a hyperbolic second order linear partial differential equation. The wave equation is given by

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} \quad (10)$$

Using the general form of second order linear PDEs with one dependent variable and two independent variables,

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D = 0$$

$$A = 1, B = 0, C = -\frac{1}{c^2}, D = 0$$

$$\text{gives } B^2 - 4AC = 0 - 4(1)\left(-\frac{1}{c^2}\right)$$

$$= \frac{4}{c^2}$$

$$= \frac{4}{c^2} > 0$$

This classifies Equation (10) as hyperbolic.

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