Chapter 10.03
Elliptic Partial Differential Equations

After reading this chapter, you should be able to:

1. *use numerical methods to solve elliptic partial differential equations by direct method, Gauss-Seidel method, and Gauss-Seidel method with over relaxation.*

The general second order linear PDE with two independent variables and one dependent variable is given by

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0 \qquad (1)$$

where A, B, C are functions of the independent variables x and y, and D can be a function of $x, y, u, \frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. Equation (1) is considered to be elliptic if

$$B^2 - 4AC < 0 \qquad (2)$$

One popular example of an elliptic second order linear partial differential equation is the Laplace equation which is of the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (3)$$

As

$$A = 1,\ B = 0, C = 1, D = 0$$

then

$$\begin{aligned} B^2 - 4AC &= 0 - 4(1)(1) \\ &= -4 < 0 \end{aligned}$$

Hence equation (3) is elliptic.

The Direct Method of Solving Elliptic PDEs

Let's find the solution via a specific physical example. Take a rectangular plate as shown in Fig. 1 where each side of the plate is maintained at a specific temperature. We are interested in finding the temperature within the plate at steady state. No heat sinks or sources exist in the problem.

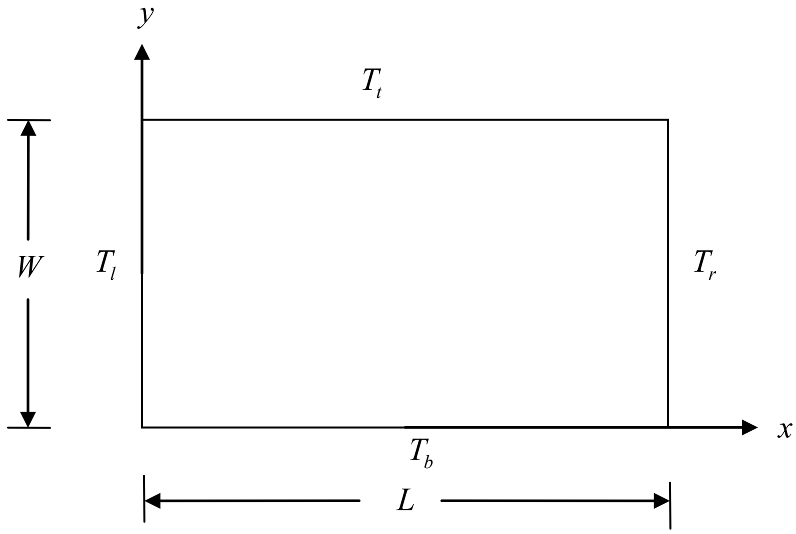

Figure 1: Schematic diagram of the plate with the temperature boundary conditions

The partial differential equation that governs the temperature $T(x,y)$ is given by

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 \tag{4}$$

To find the temperature within the plate, we divide the plate area by a grid as shown in Figure 2.

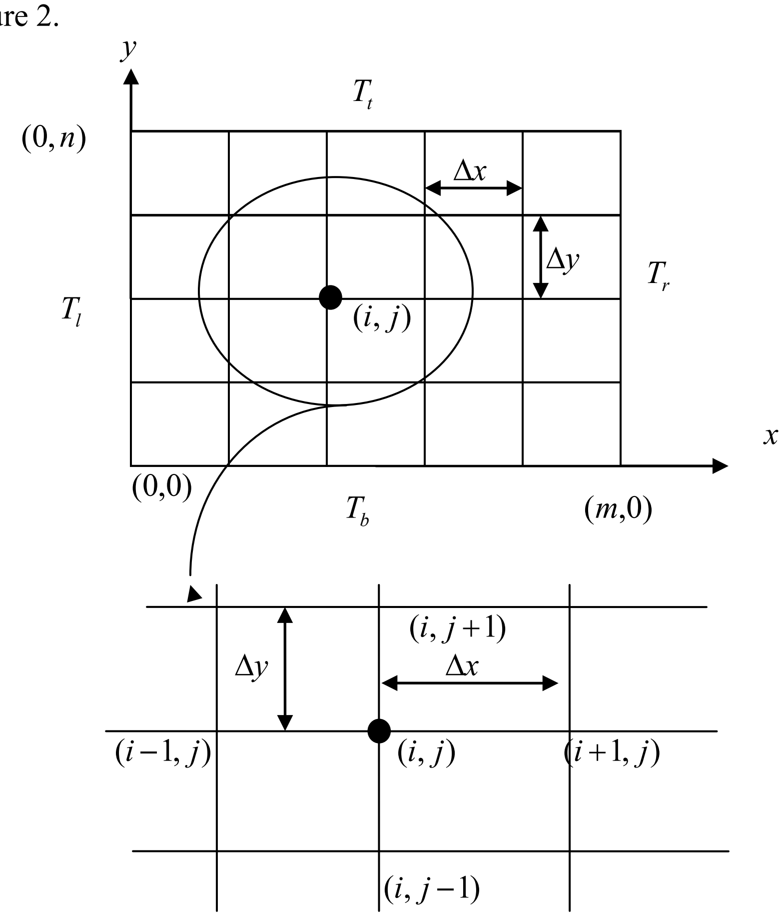

Figure 2: Plate area divided into a grid

The length L along the $x-$axis is divided into m equal segments, while the width W along the $y-$axis is divided into n equal segments, hence giving

$$\Delta x = \frac{L}{m} \tag{5}$$

$$\Delta y = \frac{W}{n} \tag{6}$$

Now we will apply the finite difference approximation of the partial derivatives at a general interior node (i, j).

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{i,j} \cong \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} \tag{7}$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{i,j} \cong \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} \tag{8}$$

Equations (7) and (8) are central divided difference approximations of the second derivatives. Substituting Equations (7) and (8) in Equation (4), we get

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0 \tag{9}$$

For a grid with

$$\Delta x = \Delta y$$

Equation (9) can be simplified as

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0 \tag{10}$$

Now we can write this equation at all the interior nodes of the plate, that is $(m-1)\times(n-1)$ nodes. This will result in an equal number of equations and unknowns. The unknowns are the temperatures at the interior $(m-1)\times(n-1)$ nodes. Solving these equations will give us the two-dimensional profile of the temperature inside the plate.

Example 1

A plate $2.4\,m \times 3.0\,m$ is subjected to temperatures as shown in Figure 3. Use a square grid length of $0.6\,m$. Using the direct method, find the temperature at the interior nodes.

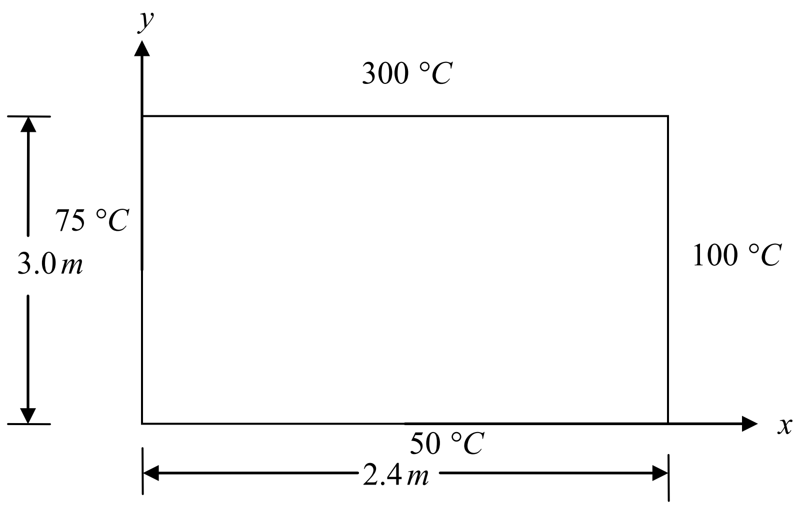

Figure 3: Plate with dimension and boundary temperatures

Solution

$$\Delta x = \Delta y = 0.6m$$

Re-writing Equations (5) and (6) we have

$$m = \frac{L}{\Delta x}$$
$$= \frac{2.4}{0.6}$$
$$= 4$$
$$n = \frac{W}{\Delta y}$$
$$= \frac{3}{0.6}$$
$$= 5$$

The nodes are shown in Figure 4.

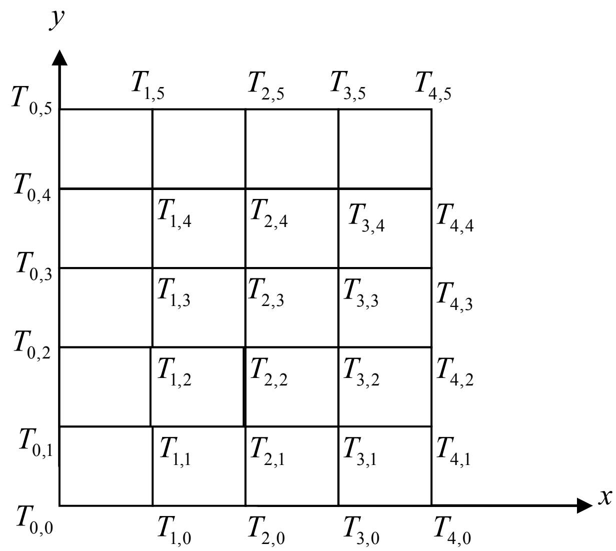

Figure 4: Plate with nodes

All the nodes on the left and right boundary have an i value of zero and m, respectively. While all the nodes on the top and bottom boundary have a j value of zero and n, respectively.

From the boundary conditions

$$\left.\begin{aligned} T_{0,j} &= 75, j = 1,2,3,4 \\ T_{4,j} &= 100, j = 1,2,3,4 \\ T_{i,0} &= 50, i = 1,2,3 \\ T_{i,5} &= 300, i = 1,2,3 \end{aligned}\right\} \tag{E1.1}$$

The corner nodal temperature of $T_{0,5}, T_{4,5}, T_{4,0}$ and $T_{0,0}$ are not needed. Now to get the temperature at the interior nodes we have to write Equation (10) for all the combinations of i and j, $i = 1,....,m-1; j = 1,....,n-1$.

<u>i=1 and j=1</u>

$$T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0} - 4T_{1,1} = 0$$

$$T_{2,1} + 75 + T_{1,2} + 50 - 4T_{1,1} = 0$$

$$-4T_{1,1} + T_{1,2} + T_{2,1} = -125 \tag{E1.2}$$

<u>i=1 and j=2</u>

$$T_{2,2} + T_{0,2} + T_{1,3} + T_{1,1} - 4T_{1,2} = 0$$

$$T_{2,2} + 75 + T_{1,3} + T_{1,1} - 4T_{1,2} = 0$$

$$T_{1,1} - 4T_{1,2} + T_{1,3} + T_{2,2} = -75 \tag{E1.3}$$

<u>i=1 and j=3</u>

$$T_{2,3} + T_{0,3} + T_{1,4} + T_{1,2} - 4T_{1,3} = 0$$

$$T_{2,3} + 75 + T_{1,4} + T_{1,2} - 4T_{1,3} = 0$$

$$T_{1,2} - 4T_{1,3} + T_{1,4} + T_{2,3} = -75 \quad \text{(E1.4)}$$

<u>i=1 and j=4</u>

$$T_{2,4} + T_{0,4} + T_{1,5} + T_{1,3} - 4T_{1,4} = 0$$

$$T_{2,4} + 75 + 300 + T_{1,3} - 4T_{1,4} = 0$$

$$T_{1,3} - 4T_{1,4} + T_{2,4} = -375 \quad \text{(E1.5)}$$

<u>i=2 and j=1</u>

$$T_{3,1} + T_{1,1} + T_{2,2} + T_{2,0} - 4T_{2,1} = 0$$

$$T_{3,1} + T_{1,1} + T_{2,2} + 50 - 4T_{2,1} = 0$$

$$T_{1,1} - 4T_{2,1} + T_{2,2} + T_{3,1} = -50 \quad \text{(E1.6)}$$

<u>i=2 and j=2</u>

$$T_{3,2} + T_{1,2} + T_{2,3} + T_{2,1} - 4T_{2,2} = 0$$

$$T_{1,2} + T_{2,1} - 4T_{2,2} + T_{2,3} + T_{3,2} = 0 \quad \text{(E1.7)}$$

<u>i=2 and j=3</u>

$$T_{3,3} + T_{1,3} + T_{2,4} + T_{2,2} - 4T_{2,3} = 0$$

$$T_{1,3} + T_{2,2} - 4T_{2,3} + T_{2,4} + T_{3,3} = 0 \quad \text{(E1.8)}$$

<u>i=2 and j=4</u>

$$T_{3,4} + T_{1,4} + T_{2,5} + T_{2,3} - 4T_{2,4} = 0$$

$$T_{3,4} + T_{1,4} + 300 + T_{2,3} - 4T_{2,4} = 0$$

$$T_{1,4} + T_{2,3} - 4T_{2,4} + T_{3,4} = -300 \quad \text{(E1.9)}$$

<u>i=3 and j=1</u>

$$T_{4,1} + T_{2,1} + T_{3,2} + T_{3,0} - 4T_{3,1} = 0$$

$$100 + T_{2,1} + T_{3,2} + 50 - 4T_{3,1} = 0$$

$$T_{2,1} - 4T_{3,1} + T_{3,2} = -150 \quad \text{(E1.10)}$$

<u>i=3 and j=2</u>

$$T_{4,2} + T_{2,2} + T_{3,3} + T_{3,1} - 4T_{3,2} = 0$$

$$100 + T_{2,2} + T_{3,3} + T_{3,1} - 4T_{3,2} = 0$$

$$T_{2,2} + T_{3,1} - 4T_{3,2} + T_{3,3} = -100 \quad \text{(E1.11)}$$

<u>i=3 and j=3</u>

$$T_{4,3} + T_{2,3} + T_{3,4} + T_{3,2} - 4T_{3,3} = 0$$

$$100 + T_{2,3} + T_{3,4} + T_{3,2} - 4T_{3,3} = 0$$

$$T_{2,3} + T_{3,2} - 4T_{3,3} + T_{3,4} = -100 \quad \text{(E1.12)}$$

<u>i=3 and j=4</u>

$$T_{4,4} + T_{2,4} + T_{3,5} + T_{3,3} - 4T_{3,4} = 0$$

$$100 + T_{2,4} + 300 + T_{3,3} - 4T_{3,4} = 0$$

$$T_{2,4} + T_{3,3} - 4T_{3,4} = -400 \tag{E1.13}$$

Equations (E1.2) to (E1.13) represent a set of twelve simultaneous linear equations and solving them gives the temperature at the twelve interior nodes. The solution is

$$\begin{bmatrix} T_{1,1} \\ T_{1,2} \\ T_{1,3} \\ T_{1,4} \\ T_{2,1} \\ T_{2,2} \\ T_{2,3} \\ T_{2,4} \\ T_{3,1} \\ T_{3,2} \\ T_{3,3} \\ T_{3,4} \end{bmatrix} = \begin{bmatrix} 73.8924 \\ 93.0252 \\ 119.907 \\ 173.355 \\ 77.5443 \\ 103.302 \\ 138.248 \\ 198.512 \\ 82.9833 \\ 104.389 \\ 131.271 \\ 182.446 \end{bmatrix} {}^{\circ}C$$

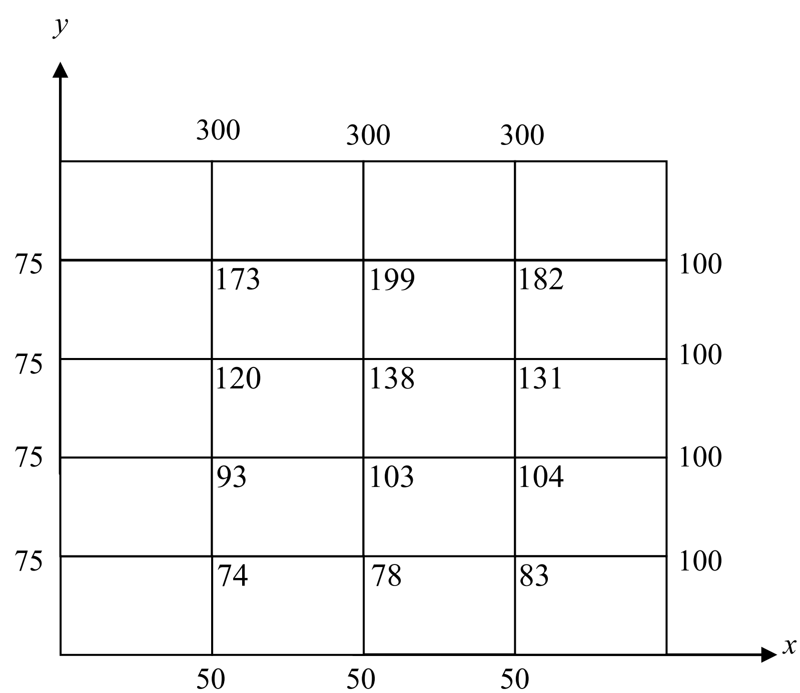

Figure 5: Temperatures at the interior nodes of the plate

Gauss-Seidel Method

To take advantage of the sparseness of the coefficient matrix as seen in Example 1, the Gauss-Seidel method may provide a more efficient way of finding the solution. In this case, Equation (10) is written for all interior nodes as

$$T_{i,j} = \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}}{4}, i = 1,2,3,4; j = 1,2,3,4,5 \tag{11}$$

Now Equation (11) is solved iteratively for all interior nodes until all the temperatures at the interior nodes are within a pre-specified tolerance.

Example 2

A plate $2.4\,m \times 3.0\,m$ is subjected to the temperatures as shown in Fig. 6. Use a square grid length of $0.6\,m$. Using the Gauss-Seidel method, find the temperature at the interior nodes. Conduct two iterations at all interior nodes. Find the maximum absolute relative error at the end of the second iteration. Assume the initial temperature at all interior nodes to be $0\,°C$.

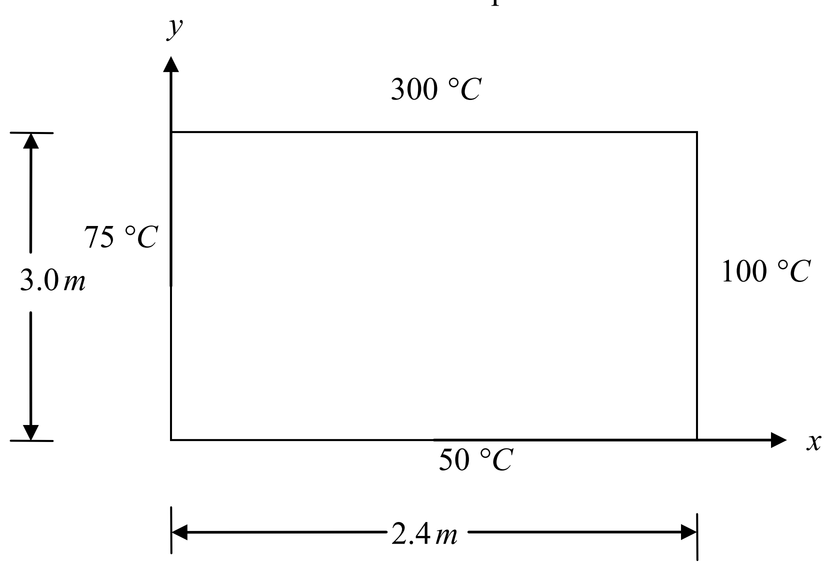

Figure 6: A rectangular plate with thedimensions and boundary temperatures

Solution

$$\Delta x = \Delta y = 0.6m$$

Re-writing Equations (5) and (6) we have

$$m = \frac{L}{\Delta x}$$

$$= \frac{2.4}{0.6}$$

$$= 4$$

$$n = \frac{W}{\Delta y}$$

$$= \frac{3}{0.6}$$
$$= 5$$

The interior nodes are shown in Figure 7.

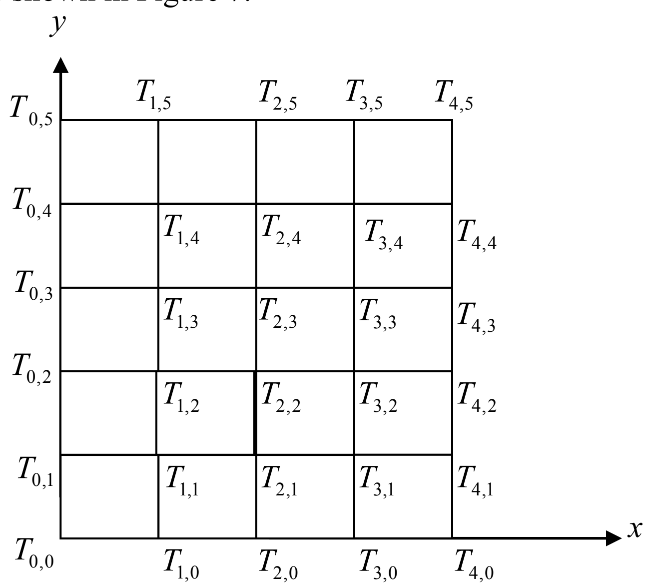

Figure 7: Plate with nodes

All the nodes on the left and right boundary have an i value of zero and m, respectively. All of the nodes on the top or bottom boundary have a j value of either zero or n, respectively.

From the boundary conditions

$$\left.\begin{aligned} T_{0,j} &= 75, j = 1,2,3,4 \\ T_{4,j} &= 100, j = 1,2,3,4 \\ T_{i,0} &= 50, i = 1,2,3 \\ T_{i,5} &= 300, i = 1,2,3 \end{aligned}\right\} \quad \text{(E2.1)}$$

The corner nodal temperature of $T_{0,5}, T_{4,5}, T_{4,0}$ and $T_{0,0}$ are not needed. Now to get the temperature at the interior nodes we have to write Equation (11) for all of the combinations of i and j, $i = 1,....,m-1; j = 1,...,n-1$.

Iteration 1

For iteration 1, we start with all of the interior nodes having a temperature of $0°C$.

i=1 and j=1

$$T_{1,1} = \frac{T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0}}{4}$$
$$= \frac{0 + 75 + 0 + 50}{4}$$

$$= 31.2500°C$$

<u>i=1 and j=2</u>

$$T_{1,2} = \frac{T_{2,2} + T_{0,2} + T_{1,3} + T_{1,1}}{4}$$
$$= \frac{0 + 75 + 0 + 31.2500}{4}$$
$$= 26.5625°C$$

<u>i=1 and j=3</u>

$$T_{1,3} = \frac{T_{2,3} + T_{0,3} + T_{1,4} + T_{1,2}}{4}$$
$$= \frac{0 + 75 + 0 + 26.5625}{4}$$
$$= 25.3906°C$$

<u>i=1 and j=4</u>

$$T_{1,4} = \frac{T_{2,4} + T_{0,4} + T_{1,5} + T_{1,3}}{4}$$
$$= \frac{0 + 75 + 300 + 25.3906}{4}$$
$$= 100.098°C$$

<u>i=2 and j=1</u>

$$T_{2,1} = \frac{T_{3,1} + T_{1,1} + T_{2,2} + T_{2,0}}{4}$$
$$= \frac{0 + 31.2500 + 0 + 50}{4}$$
$$= 20.3125°C$$

<u>i=2 and j=2</u>

$$T_{2,2} = \frac{T_{3,2} + T_{1,2} + T_{2,3} + T_{2,1}}{4}$$
$$= \frac{0 + 26.5625 + 0 + 20.3125}{4}$$
$$= 11.7188°C$$

<u>i=2 and j=3</u>

$$T_{2,3} = \frac{T_{3,3} + T_{1,3} + T_{2,4} + T_{2,2}}{4}$$
$$= \frac{0 + 25.3906 + 0 + 11.7188}{4}$$
$$= 9.27735°C$$

$\underline{i\text{=2 and } j\text{=4}}$

$$T_{2,4} = \frac{T_{3,4} + T_{1,4} + T_{2,5} + T_{2,3}}{4}$$
$$= \frac{0 + 100.098 + 300 + 9.27735}{4}$$
$$= 102.344°C$$

$\underline{i\text{=3 and } j\text{=1}}$

$$T_{3,1} = \frac{T_{4,1} + T_{2,1} + T_{3,2} + T_{3,0}}{4}$$
$$= \frac{100 + 20.3125 + 0 + 50}{4}$$
$$= 42.5781°C$$

$\underline{i\text{=3 and } j\text{=2}}$

$$T_{3,2} = \frac{T_{4,2} + T_{2,2} + T_{3,3} + T_{3,1}}{4}$$
$$= \frac{100 + 11.7188 + 0 + 42.5781}{4}$$
$$= 38.5742°C$$

$\underline{i\text{=3 and } j\text{=3}}$

$$T_{3,3} = \frac{T_{4,3} + T_{2,3} + T_{3,4} + T_{3,2}}{4}$$
$$= \frac{100 + 9.27735 + 0 + 38.5742}{4}$$
$$= 36.9629°C$$

$\underline{i\text{=3 and } j\text{=4}}$

$$T_{3,4} = \frac{T_{4,4} + T_{2,4} + T_{3,5} + T_{3,3}}{4}$$
$$= \frac{100 + 102.344 + 300 + 36.9629}{4}$$
$$= 134.827°C$$

Iteration 2

For iteration 2, we use the temperatures from iteration 1.

$\underline{i\text{=1 and } j\text{=1}}$

$$T_{1,1} = \frac{T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0}}{4}$$
$$= \frac{20.3125 + 75 + 26.5625 + 50}{4}$$
$$= 42.9688°C$$

$$\left|\varepsilon_a\right|_{1,1} = \left|\frac{T_{1,1}^{present} - T_{1,1}^{previous}}{T_{1,1}^{present}}\right| \times 100$$
$$= \left|\frac{42.9688 - 31.2500}{42.9688}\right| \times 100$$
$$= 27.27\%$$

<u>i=1 and j=2</u>

$$T_{1,2} = \frac{T_{2,2} + T_{0,2} + T_{1,3} + T_{1,1}}{4}$$
$$= \frac{11.7188 + 75 + 25.3906 + 42.9688}{4}$$
$$= 38.7696°C$$
$$\left|\varepsilon_a\right|_{1,2} = \left|\frac{T_{1,2}^{present} - T_{1,2}^{previous}}{T_{1,2}^{present}}\right| \times 100$$
$$= \left|\frac{38.7696 - 26.5625}{38.7696}\right| \times 100$$
$$= 31.49\%$$

<u>i=1 and j=3</u>

$$T_{1,3} = \frac{T_{2,3} + T_{0,3} + T_{1,4} + T_{1,2}}{4}$$
$$= \frac{9.27735 + 75 + 100.098 + 38.7696}{4}$$
$$= 55.7862°C$$
$$\left|\varepsilon_a\right|_{1,3} = \left|\frac{T_{1,3}^{present} - T_{1,3}^{previous}}{T_{1,3}^{present}}\right| \times 100$$
$$= \left|\frac{55.7862 - 25.3906}{55.7862}\right| \times 100$$
$$= 54.49\%$$

<u>i=1 and j=4</u>

$$T_{1,4} = \frac{T_{2,4} + T_{0,4} + T_{1,5} + T_{1,3}}{4}$$
$$= \frac{102.344 + 75 + 300 + 55.7862}{4}$$
$$= 133.283°C$$

$$\left|\varepsilon_a\right|_{1,4} = \left|\frac{T_{1,4}^{present} - T_{1,4}^{previous}}{T_{1,4}^{present}}\right| \times 100$$

$$= \left|\frac{133.283 - 100.098}{133.283}\right| \times 100$$

$$= 24.90\%$$

<u>i=2 and j=1</u>

$$T_{2,1} = \frac{T_{3,1} + T_{1,1} + T_{2,2} + T_{2,0}}{4}$$

$$= \frac{42.5781 + 42.9688 + 11.7188 + 50}{4}$$

$$= 36.8164°C$$

$$\left|\varepsilon_a\right|_{2,1} = \left|\frac{T_{2,1}^{present} - T_{2,1}^{previous}}{T_{2,1}^{present}}\right| \times 100$$

$$= \left|\frac{36.8164 - 20.3125}{36.8164}\right| \times 100$$

$$= 44.83\%$$

<u>i=2 and j=2</u>

$$T_{2,2} = \frac{T_{3,2} + T_{1,2} + T_{2,3} + T_{2,1}}{4}$$

$$= \frac{38.5742 + 38.7696 + 9.27735 + 36.8164}{4}$$

$$= 30.8594°C$$

$$\left|\varepsilon_a\right|_{2,2} = \left|\frac{T_{2,2}^{present} - T_{2,2}^{previous}}{T_{2,2}^{present}}\right| \times 100$$

$$= \left|\frac{30.8594 - 11.7188}{30.8594}\right| \times 100$$

$$= 62.03\%$$

<u>i=2 and j=3</u>

$$T_{2,3} = \frac{T_{3,3} + T_{1,3} + T_{2,4} + T_{2,2}}{4}$$

$$= \frac{36.9629 + 55.7862 + 102.344 + 30.8594}{4}$$

$$= 56.4881°C$$

$$\left|\varepsilon_a\right|_{2,3} = \left|\frac{T_{2,3}^{present} - T_{2,3}^{previous}}{T_{2,3}^{present}}\right| \times 100$$

$$= \left|\frac{56.4881 - 9.27735}{56.4881}\right| \times 100$$

$$= 83.58\%$$

<u>*i*=2 and *j*=4</u>

$$T_{2,4} = \frac{T_{3,4} + T_{1,4} + T_{2,5} + T_{2,3}}{4}$$

$$= \frac{134.827 + 133.283 + 300 + 56.4881}{4}$$

$$= 156.150^\circ C$$

$$\left|\varepsilon_a\right|_{2,4} = \left|\frac{T_{2,4}^{present} - T_{2,4}^{previous}}{T_{2,4}^{present}}\right| \times 100$$

$$= \left|\frac{156.150 - 102.344}{156.150}\right| \times 100$$

$$= 34.46\%$$

<u>*i*=3 and *j*=1</u>

$$T_{3,1} = \frac{T_{4,1} + T_{2,1} + T_{3,2} + T_{3,0}}{4}$$

$$= \frac{100 + 36.8164 + 38.5742 + 50}{4}$$

$$= 56.3477^\circ C$$

$$\left|\varepsilon_a\right|_{3,1} = \left|\frac{T_{3,1}^{present} - T_{3,1}^{previous}}{T_{3,1}^{present}}\right| \times 100$$

$$= \left|\frac{56.3477 - 42.5781}{56.3477}\right| \times 100$$

$$= 24.44\%$$

<u>*i*=3 and *j*=2</u>

$$T_{3,2} = \frac{T_{4,2} + T_{2,2} + T_{3,3} + T_{3,1}}{4}$$

$$= \frac{100 + 30.8594 + 36.9629 + 56.3477}{4}$$

$$= 56.0425^\circ C$$

$$\left|\varepsilon_a\right|_{3,2} = \left|\frac{T_{3,2}^{present} - T_{3,2}^{previous}}{T_{3,2}^{present}}\right| \times 100$$
$$= \left|\frac{56.0425 - 38.5742}{56.0425}\right| \times 100$$
$$= 31.70\%$$

<u>*i*=3 and *j*=3</u>

$$T_{3,3} = \frac{T_{4,3} + T_{2,3} + T_{3,4} + T_{3,2}}{4}$$
$$= \frac{100 + 56.4881 + 134.827 + 56.0425}{4}$$
$$= 86.8394°C$$
$$\left|\varepsilon_a\right|_{3,3} = \left|\frac{T_{3,3}^{present} - T_{3,3}^{previous}}{T_{3,3}^{present}}\right| \times 100$$
$$= \left|\frac{86.8394 - 36.9629}{86.8394}\right| \times 100$$
$$= 57.44\%$$

<u>*i*=3 and *j*=4</u>

$$T_{3,4} = \frac{T_{4,4} + T_{2,4} + T_{3,5} + T_{3,3}}{4}$$
$$= \frac{100 + 156.150 + 300 + 86.8394}{4}$$
$$= 160.747°C$$
$$\left|\varepsilon_a\right|_{3,4} = \left|\frac{T_{3,4}^{present} - T_{3,4}^{previous}}{T_{3,4}^{present}}\right| \times 100$$
$$= \left|\frac{160.747 - 134.827}{160.747}\right| \times 100$$
$$= 16.12\%$$

The maximum absolute relative error at the end of iteration 2 is 83%.

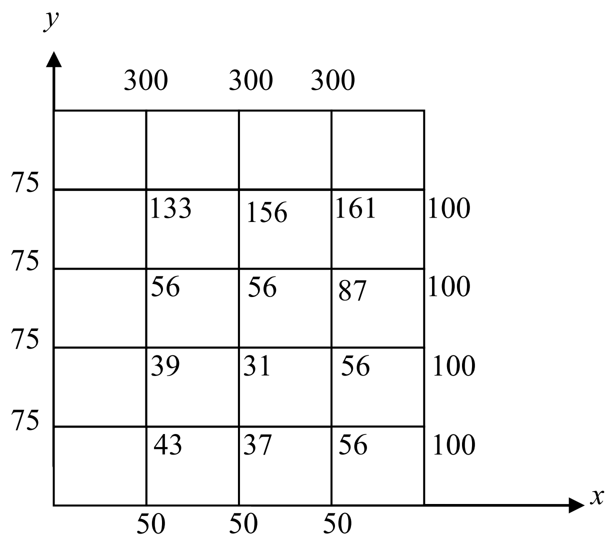

Figure 8: Temperature distribution after two iterations

It took ten iterations to get all of the temperature values within 1% error. The table below lists the temperature values at the interior nodes at the end of each iteration:

Node	Number of Iterations				
	1	2	3	4	5
$T_{1,1}$	31.2500	42.9688	50.1465	56.1966	61.6376
$T_{1,2}$	26.5625	38.7695	52.9480	65.9264	76.5753
$T_{1,3}$	25.3906	55.7861	79.4296	96.8614	106.8163
$T_{1,4}$	100.0977	133.2825	152.6447	162.1695	167.1287
$T_{2,1}$	20.3125	36.8164	46.8384	55.6240	63.6980
$T_{2,2}$	11.7188	30.8594	53.0792	72.8024	85.3707
$T_{2,3}$	9.2773	56.4880	93.8744	113.5205	124.2410
$T_{2,4}$	102.3438	156.1493	176.8166	186.6986	191.8910
$T_{3,1}$	42.5781	56.3477	63.2202	70.3522	75.3468
$T_{3,2}$	38.5742	56.0425	75.7847	87.6890	94.6990
$T_{3,3}$	36.9629	86.8393	107.6015	118.0785	123.7836
$T_{3,4}$	134.8267	160.7471	171.1045	176.1943	178.9186

Node	Number of Iterations				
	6	7	8	9	10
$T_{1,1}$	66.3183	69.4088	71.2832	72.3848	73.0239
$T_{1,2}$	83.3763	87.4348	89.8017	91.1701	91.9585
$T_{1,3}$	112.4365	115.6295	117.4532	118.4980	119.0976
$T_{1,4}$	169.8319	171.3450	172.2037	172.6943	172.9755
$T_{2,1}$	69.2590	72.6980	74.7374	75.9256	76.6127
$T_{2,2}$	92.8938	97.2939	99.8423	102.3119	102.1577
$T_{2,3}$	130.2512	133.6661	135.6184	136.7377	137.3802
$T_{2,4}$	194.7504	196.3616	197.2791	197.8043	198.1055
$T_{3,1}$	78.4895	80.3724	81.4754	82.1148	82.4837
$T_{3,2}$	98.7917	101.1642	102.5335	103.3221	103.7757
$T_{3,3}$	126.9904	128.8164	129.8616	130.4612	130.8056
$T_{3,4}$	180.4352	181.2945	181.7852	182.0664	182.2278

Successive Over Relaxation Method

The coefficient matrix for solving for temperatures given in Example 1 is diagonally dominant. Hence the Gauss-Siedel method is guaranteed to converge. To accelerate convergence to the solution, over relaxation is used. In this case

$$T_{i,j}^{relaxed} = \lambda T_{i,j}^{new} + (1-\lambda)T_{i,j}^{old} \tag{12}$$

where

$T_{i,j}^{new} =$ value of temperature from current iteration,

$T_{i,j}^{old} =$ value of temperature from previous iteration,

$\lambda =$ weighting factor, $1 < \lambda < 2$.

Again, these iterations are continued till the pre-specified tolerance is met for all nodal temperatures. This method is also called the Lieberman method.

Example 3

A plate $2.4\,m \times 3.0\,m$ is subjected to the temperatures as shown in Fig. 6. Use a square grid length of $0.6\,m$. Use the Gauss-Seidel with successive over relaxation method with a weighting factor of 1.4 to find the temperature at the interior nodes. Conduct two iterations at all interior nodes. Find the maximum absolute relative error at the end of the second iteration. Assume the initial temperature at all interior nodes to be $0\,°C$.

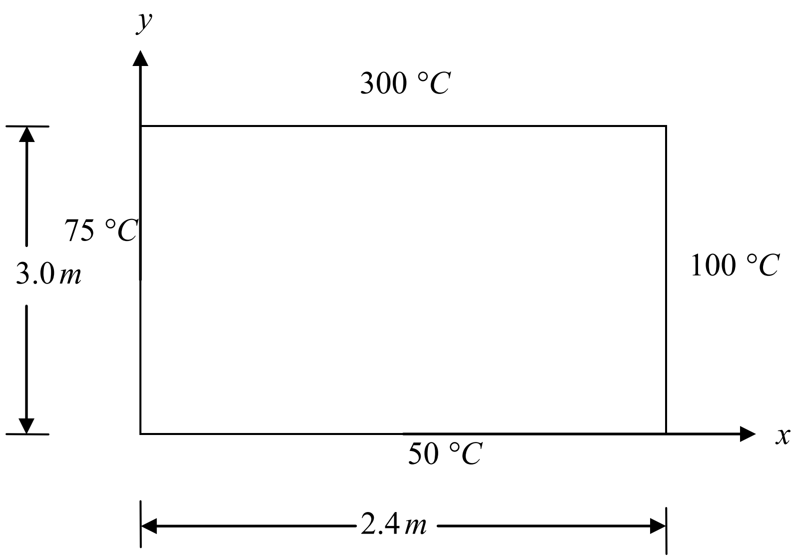

Figure 9: A rectangular plate with the dimensions and boundary temperatures

Solution

$$\Delta x = \Delta y = 0.6m$$

Re-writing Equations (5) and (6) we have

$$m = \frac{L}{\Delta x}$$
$$= \frac{2.4}{0.6}$$
$$= 4$$
$$n = \frac{W}{\Delta y}$$
$$= \frac{3}{0.6}$$
$$= 5$$

The interior nodes are shown in the Figure 10.

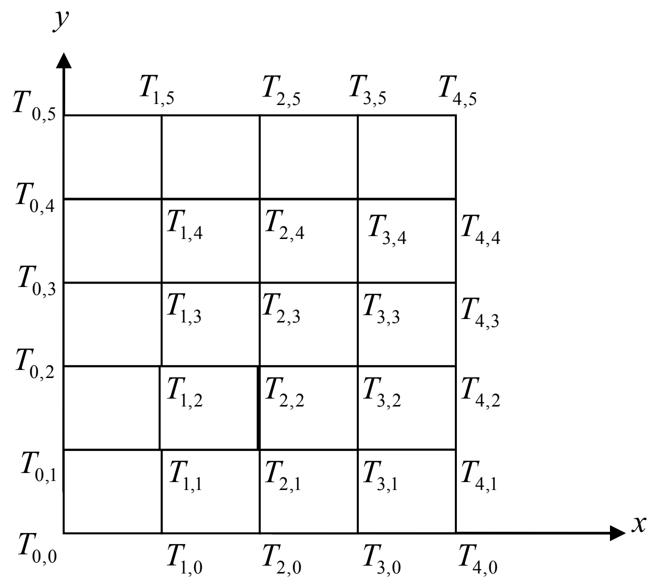

Figure 10: Plate with nodes

All of the nodes on the left and right boundary have an i value of zero and m, respectively. All of the nodes on the top or bottom boundary have a j value of either zero or n, respectively.

From the boundary conditions

$$\left.\begin{aligned} T_{0,j} &= 75, j = 1,2,3,4 \\ T_{4,j} &= 100, j = 1,2,3,4 \\ T_{i,0} &= 50, i = 1,2,3 \\ T_{i,5} &= 300, i = 1,2,3 \end{aligned}\right\} \tag{E3.1}$$

The corner nodal temperature of $T_{0,5}, T_{4,5}, T_{4,0}$ and $T_{0,0}$ are not needed. Now to get the temperature at the interior nodes, we have to write Equation (11) for all of the combinations of i and j, $i = 1$ to $m-1$, $j = 1$ to $n-1$. After getting the temperature from Equation (11), we have to use Equation (12) to apply the over relaxation method.

<u>Iteration 1</u>

For iteration 1, we start with all of the interior nodes having a temperature of $0°C$.

<u>i=1 and j=1</u>

$$\begin{aligned} T_{1,1} &= \frac{T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0}}{4} \\ &= \frac{0 + 75 + 0 + 50}{4} \\ &= 31.2500°C \end{aligned}$$

$$T_{1,1}^{relaxed} = \lambda T_{1,1}^{new} + (1-\lambda) T_{1,1}^{old}$$

$$= 1.4(31.2500) + (1-1.4)0$$
$$= 43.7500°C$$

<u>i=1 and j=2</u>

$$T_{1,2} = \frac{T_{2,2} + T_{0,2} + T_{1,3} + T_{1,1}}{4}$$
$$= \frac{0 + 75 + 0 + 43.75}{4}$$
$$= 29.6875°C$$
$$T_{1,2}^{relaxed} = \lambda T_{1,2}^{new} + (1-\lambda)T_{1,2}^{old}$$
$$= 1.4(29.6875) + (1-1.4)0$$
$$= 41.5625°C$$

<u>i=1 and j=3</u>

$$T_{1,3} = \frac{T_{2,3} + T_{0,3} + T_{1,4} + T_{1,2}}{4}$$
$$= \frac{0 + 75 + 0 + 41.5625}{4}$$
$$= 29.1406°C$$
$$T_{1,3}^{relaxed} = \lambda T_{1,3}^{new} + (1-\lambda)T_{1,3}^{old}$$
$$= 1.4(29.1406) + (1-1.4)0$$
$$= 40.7969°C$$

<u>i=1 and j=4</u>

$$T_{1,4} = \frac{T_{2,4} + T_{0,4} + T_{1,5} + T_{1,3}}{4}$$
$$= \frac{0 + 75 + 300 + 40.7969}{4}$$
$$= 103.949°C$$
$$T_{1,4}^{relaxed} = \lambda T_{1,4}^{new} + (1-\lambda)T_{1,4}^{old}$$
$$= 1.4(103.949) + (1-1.4)0$$
$$= 145.529°C$$

<u>i=2 and j=1</u>

$$T_{2,1} = \frac{T_{3,1} + T_{1,1} + T_{2,2} + T_{2,0}}{4}$$
$$= \frac{0 + 43.75 + 0 + 50}{4}$$
$$= 23.4375°C$$
$$T_{2,1}^{relaxed} = \lambda T_{2,1}^{new} + (1-\lambda)T_{2,1}^{old}$$

$$= 1.4(23.4375) + (1-1.4)0$$
$$= 32.8215°C$$

<u>i=2 and j=2</u>

$$T_{2,2} = \frac{T_{3,2} + T_{1,2} + T_{2,3} + T_{2,1}}{4}$$
$$= \frac{0 + 41.5625 + 0 + 32.8125}{4}$$
$$= 18.5938°C$$
$$T_{2,2}^{relaxed} = \lambda T_{2,2}^{new} + (1-\lambda)T_{2,2}^{old}$$
$$= 1.4(18.5938) + (1-1.4)0$$
$$= 26.0313°C$$

<u>i=2 and j=3</u>

$$T_{2,3} = \frac{T_{3,3} + T_{1,3} + T_{2,4} + T_{2,2}}{4}$$
$$= \frac{0 + 40.7969 + 0 + 26.0313}{4}$$
$$= 16.7071°C$$
$$T_{2,3}^{relaxed} = \lambda T_{2,3}^{new} + (1-\lambda)T_{2,3}^{old}$$
$$= 1.4(16.7071) + (1-1.4)0$$
$$= 23.3899°C$$

<u>i=2 and j=4</u>

$$T_{2,4} = \frac{T_{3,4} + T_{1,4} + T_{2,5} + T_{2,3}}{4}$$
$$= \frac{0 + 145.529 + 300 + 23.3899}{4}$$
$$= 117.230°C$$
$$T_{2,4}^{relaxed} = \lambda T_{2,4}^{new} + (1-\lambda)T_{2,4}^{old}$$
$$= 1.4(117.230) + (1-1.4)0$$
$$= 164.122°C$$

<u>i=3 and j=1</u>

$$T_{3,1} = \frac{T_{4,1} + T_{2,1} + T_{3,2} + T_{3,0}}{4}$$
$$= \frac{100 + 32.8125 + 0 + 50}{4}$$
$$= 45.7031°C$$
$$T_{3,1}^{relaxed} = \lambda T_{3,1}^{new} + (1-\lambda)T_{3,1}^{old}$$

$$= 1.4(45.7031) + (1-1.4)0$$
$$= 63.9844°C$$

<u>i=3 and j=2</u>

$$T_{3,2} = \frac{T_{4,2} + T_{2,2} + T_{3,3} + T_{3,1}}{4}$$
$$= \frac{100 + 26.0313 + 0 + 63.9844}{4}$$
$$= 47.5039°C$$
$$T_{3,2}^{relaxed} = \lambda T_{3,2}^{new} + (1-\lambda)T_{3,2}^{old}$$
$$= 1.4(47.5039) + (1-1.4)0$$
$$= 66.5055°C$$

<u>i=3 and j=3</u>

$$T_{3,3} = \frac{T_{4,3} + T_{2,3} + T_{3,4} + T_{3,2}}{4}$$
$$= \frac{100 + 23.3899 + 0 + 66.5055}{4}$$
$$= 47.4739°C$$
$$T_{3,3}^{relaxed} = \lambda T_{3,3}^{new} + (1-\lambda)T_{3,3}^{old}$$
$$= 1.4(47.4739) + (1-1.4)0$$
$$= 66.4634°C$$

<u>i=3 and j=4</u>

$$T_{3,4} = \frac{T_{4,4} + T_{2,4} + T_{3,5} + T_{3,3}}{4}$$
$$= \frac{100 + 164.122 + 300 + 66.4634}{4}$$
$$= 157.646°C$$
$$T_{3,4}^{relaxed} = \lambda T_{3,4}^{new} + (1-\lambda)T_{3,4}^{old}$$
$$= 1.4(157.646) + (1-1.4)0$$
$$= 220.704°C$$

<u>Iteration 2</u>

For iteration 2, we take the temperatures from iteration 1.

<u>i=1 and j=1</u>

$$T_{1,1} = \frac{T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0}}{4}$$
$$= \frac{32.8125 + 75 + 41.5625 + 50}{4}$$
$$= 49.8438°C$$

$$\begin{aligned} T_{1,1}^{relaxed} &= \lambda T_{1,1}^{new} + (1-\lambda)T_{1,1}^{old} \\ &= 1.4(49.8438) + (1-1.4)43.75 \\ &= 52.2813°C \end{aligned}$$

$$\begin{aligned} \left|\varepsilon_a\right|_{1,1} &= \left|\frac{T_{1,1}^{present} - T_{1,1}^{previous}}{T_{1,1}^{present}}\right| \times 100 \\ &= \left|\frac{52.2813 - 43.7500}{52.2813}\right| \times 100 \\ &= 16.32\% \end{aligned}$$

<u>*i*=1 and *j*=2</u>

$$\begin{aligned} T_{1,2} &= \frac{T_{2,2} + T_{0,2} + T_{1,3} + T_{1,1}}{4} \\ &= \frac{26.0313 + 75 + 40.7969 + 52.2813}{4} \\ &= 48.5274°C \end{aligned}$$

$$\begin{aligned} T_{1,2}^{relaxed} &= \lambda T_{1,2}^{new} + (1-\lambda)T_{1,2}^{old} \\ &= 1.4(48.5274) + (1-1.4)41.5625 \\ &= 51.3133°C \end{aligned}$$

$$\begin{aligned} \left|\varepsilon_a\right|_{1,2} &= \left|\frac{T_{1,2}^{present} - T_{1,2}^{previous}}{T_{1,2}^{present}}\right| \times 100 \\ &= \left|\frac{51.3133 - 41.5625}{51.3133}\right| \times 100 \\ &= 19.00\% \end{aligned}$$

<u>*i*=1 and *j*=3</u>

$$\begin{aligned} T_{1,3} &= \frac{T_{2,3} + T_{0,3} + T_{1,4} + T_{1,2}}{4} \\ &= \frac{23.3899 + 75 + 145.529 + 51.3133}{4} \\ &= 73.8103°C \end{aligned}$$

$$\begin{aligned} T_{1,3}^{relaxed} &= \lambda T_{1,3}^{new} + (1-\lambda)T_{1,3}^{old} \\ &= 1.4(73.8103) + (1-1.4)40.7969 \\ &= 87.0157°C \end{aligned}$$

$$\left|\varepsilon_a\right|_{1,3} = \left|\frac{T_{1,3}^{present} - T_{1,3}^{previous}}{T_{1,3}^{present}}\right| \times 100$$

$$= \left|\frac{87.0157 - 40.7969}{87.0157}\right| \times 100$$

$$= 53.12\%$$

<u>*i*=1 and *j*=4</u>

$$T_{1,4} = \frac{T_{2,4} + T_{0,4} + T_{1,5} + T_{1,3}}{4}$$

$$= \frac{164.122 + 75 + 300 + 87.0157}{4}$$

$$= 156.534°C$$

$$T_{1,4}^{relaxed} = \lambda T_{1,4}^{new} + (1-\lambda)T_{1,4}^{old}$$

$$= 1.4(156.534) + (1-1.4)145.529$$

$$= 160.936°C$$

$$\left|\varepsilon_a\right|_{1,4} = \left|\frac{T_{1,4}^{present} - T_{1,4}^{previous}}{T_{1,4}^{present}}\right| \times 100$$

$$= \left|\frac{160.936 - 145.529}{160.936}\right| \times 100$$

$$= 9.57\%$$

<u>*i*=2 and *j*=1</u>

$$T_{2,1} = \frac{T_{3,1} + T_{1,1} + T_{2,2} + T_{2,0}}{4}$$

$$= \frac{63.9844 + 52.2813 + 26.0313 + 50.000}{4}$$

$$= 48.0743°C$$

$$T_{2,1}^{relaxed} = \lambda T_{2,1}^{new} + (1-\lambda)T_{2,1}^{old}$$

$$= 1.4(48.0743) + (1-1.4)32.8125$$

$$= 54.1790°C$$

$$\left|\varepsilon_a\right|_{2,1} = \left|\frac{T_{2,1}^{present} - T_{2,1}^{previous}}{T_{2,1}^{present}}\right| \times 100$$

$$= \left|\frac{54.1790 - 32.8125}{54.1790}\right| \times 100$$

$$= 39.44\%$$

<u>i=2 and j=2</u>

$$T_{2,2} = \frac{T_{3,2} + T_{1,2} + T_{2,3} + T_{2,1}}{4}$$

$$= \frac{66.5055 + 51.3133 + 23.3899 + 54.1790}{4}$$

$$= 48.8469°C$$

$$T_{2,2}^{relaxed} = \lambda T_{2,2}^{new} + (1-\lambda)T_{2,2}^{old}$$

$$= 1.4(48.8469) + (1-1.4)26.0313$$

$$= 57.9732°C$$

$$\left|\varepsilon_a\right|_{2,2} = \left|\frac{T_{2,2}^{present} - T_{2,2}^{previous}}{T_{2,2}^{present}}\right| \times 100$$

$$= \left|\frac{57.9732 - 26.0313}{57.9732}\right| \times 100$$

$$= 55.10\%$$

<u>i=2 and j=3</u>

$$T_{2,3} = \frac{T_{3,3} + T_{1,3} + T_{2,4} + T_{2,2}}{4}$$

$$= \frac{66.4634 + 87.0157 + 164.122 + 57.9732}{4}$$

$$= 93.8936°C$$

$$T_{2,3}^{relaxed} = \lambda T_{2,3}^{new} + (1-\lambda)T_{2,3}^{old}$$

$$= 1.4(93.8936) + (1-1.4)23.3899$$

$$= 122.095°C$$

$$\left|\varepsilon_a\right|_{2,3} = \left|\frac{T_{2,3}^{present} - T_{2,3}^{previous}}{T_{2,3}^{present}}\right| \times 100$$

$$= \left|\frac{122.095 - 23.3899}{122.095}\right| \times 100$$

$$= 80.84\%$$

<u>i=2 and j=4</u>

$$T_{2,4} = \frac{T_{3,4} + T_{1,4} + T_{2,5} + T_{2,3}}{4}$$

$$= \frac{220.704 + 160.936 + 300 + 122.095}{4}$$

$$= 200.934°C$$

$$T_{2,4}^{relaxed} = \lambda T_{2,4}^{new} + (1-\lambda)T_{2,4}^{old}$$

$$= 1.4(200.934) + (1-1.4)164.122$$
$$= 215.659°C$$
$$\left|\varepsilon_a\right|_{2,4} = \left|\frac{T_{2,4}^{present} - T_{2,4}^{previous}}{T_{2,4}^{present}}\right| \times 100$$
$$= \left|\frac{215.659 - 164.122}{215.659}\right| \times 100$$
$$= 23.90\%$$

<u>*i*=3 and *j*=1</u>

$$T_{3,1} = \frac{T_{4,1} + T_{2,1} + T_{3,2} + T_{3,0}}{4}$$
$$= \frac{100 + 54.1790 + 66.5055 + 50}{4}$$
$$= 67.6711°C$$
$$T_{3,1}^{relaxed} = \lambda T_{3,1}^{new} + (1-\lambda)T_{3,1}^{old}$$
$$= 1.4(67.6711) + (1-1.4)63.9844$$
$$= 69.1458°C$$
$$\left|\varepsilon_a\right|_{3,1} = \left|\frac{T_{3,1}^{present} - T_{3,1}^{previous}}{T_{3,1}^{present}}\right| \times 100$$
$$= \left|\frac{69.1458 - 63.9844}{69.1458}\right| \times 100$$
$$= 7.46\%$$

<u>*i*=3 and *j*=2</u>

$$T_{3,2} = \frac{T_{4,2} + T_{2,2} + T_{3,3} + T_{3,1}}{4}$$
$$= \frac{100 + 57.9732 + 66.4634 + 69.1458}{4}$$
$$= 73.3956°C$$
$$T_{3,2}^{relaxed} = \lambda T_{3,2}^{new} + (1-\lambda)T_{3,2}^{old}$$
$$= 1.4(73.3956) + (1-1.4)66.5055$$
$$= 76.1516°C$$
$$\left|\varepsilon_a\right|_{3,2} = \left|\frac{T_{3,2}^{present} - T_{3,2}^{previous}}{T_{3,2}^{present}}\right| \times 100$$
$$= \left|\frac{76.1516 - 66.5055}{76.1516}\right| \times 100$$
$$= 12.67\%$$

<u>*i*=3 and *j*=3</u>

$$T_{3,3}=\frac{T_{4,3}+T_{2,3}+T_{3,4}+T_{3,2}}{4}$$
$$=\frac{100+122.095+220.704+76.1516}{4}$$
$$=129.738°C$$
$$T_{3,3}^{relaxed}=\lambda T_{3,3}^{new}+(1-\lambda)T_{3,3}^{old}$$
$$=1.4(129.738)+(1-1.4)66.4634$$
$$=155.048°C$$
$$\left|\varepsilon_a\right|_{3,3}=\left|\frac{T_{3,3}^{present}-T_{3,3}^{previous}}{T_{3,3}^{present}}\right|\times 100$$
$$=\left|\frac{155.048-66.4634}{155.048}\right|\times 100$$
$$=57.13\%$$

<u>*i*=3 and *j*=4</u>

$$T_{3,4}=\frac{T_{4,4}+T_{2,4}+T_{3,5}+T_{3,3}}{4}$$
$$=\frac{100+215.659+300+155.048}{4}$$
$$=192.677°C$$
$$T_{3,4}^{relaxed}=\lambda T_{3,4}^{new}+(1-\lambda)T_{3,4}^{old}$$
$$=1.4(192.677)+(1-1.4)220.704$$
$$=181.466°C$$
$$\left|\varepsilon_a\right|_{3,4}=\left|\frac{T_{3,4}^{present}-T_{3,4}^{previous}}{T_{3,4}^{present}}\right|\times 100$$
$$=\left|\frac{181.466-220.704}{181.466}\right|\times 100$$
$$=21.62\%$$

The maximum absolute relative error at the end of iteration 2 is 81%.

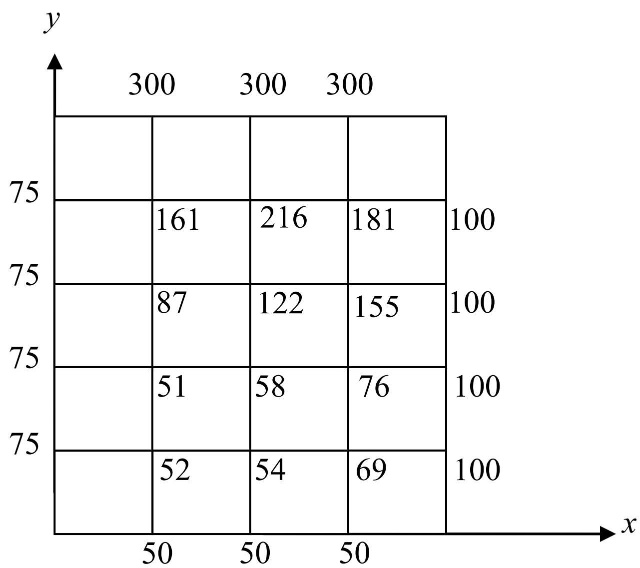

Figure 11: Temperature distribution after two iterations

It took nine iterations to get all of the temperature values within 1% error. The table below lists the temperature values at all nodes after each iteration.

Node	Number of Iterations				
	1	2	3	4	5
$T_{1,1}$	43.7500	52.2813	59.7598	68.3636	75.6025
$T_{1,2}$	41.5625	51.3133	77.3856	93.5293	101.8402
$T_{1,3}$	40.7969	87.0125	117.5901	130.5043	119.8434
$T_{1,4}$	145.5289	160.9353	183.5128	173.8030	173.3888
$T_{2,1}$	32.8125	54.1789	61.2360	75.6074	86.4009
$T_{2,2}$	26.0313	57.9731	94.7142	116.7560	105.9062
$T_{2,3}$	23.3898	122.0937	155.2159	140.9145	139.0181
$T_{2,4}$	164.1216	215.6582	200.8045	199.1851	198.6561
$T_{3,1}$	63.9844	69.1458	72.9273	90.9098	83.7806
$T_{3,2}$	66.5055	76.1516	117.4804	106.8690	105.2995
$T_{3,3}$	66.4634	155.0472	131.9376	133.3050	131.1769
$T_{3,4}$	220.7047	181.4650	183.8737	182.8220	182.3127

Node	Number of Iterations			
	6	7	8	9
$T_{1,1}$	79.3934	71.2937	74.2346	73.7832
$T_{1,2}$	92.3140	92.1224	93.0388	92.9758
$T_{1,3}$	119.9649	119.388	119.8366	119.9378
$T_{1,4}$	173.4118	173.0515	173.3665	173.3937
$T_{2,1}$	77.1177	76.4550	77.6097	77.5449
$T_{2,2}$	102.4498	102.4844	103.3554	103.3285
$T_{2,3}$	137.6794	137.7443	138.2932	138.3236
$T_{2,4}$	198.2290	198.2693	198.6060	198.5498
$T_{3,1}$	82.8338	82.4002	83.1150	82.9805
$T_{3,2}$	103.6414	104.0334	104.5308	104.3815
$T_{3,3}$	130.8010	131.0842	131.3876	131.2525
$T_{3,4}$	182.2354	182.3796	182.5459	182.4230

Alternative Boundary Conditions

In Examples 1-3, the boundary conditions on the plate had a specified temperature on each edge. What if the conditions are different? For example; what if one of the edges of the plate is insulated? In this case, the boundary condition would be the derivative of the temperature (called the Neuman boundary condition). If the right edge of the plate is insulated, then the temperatures on the right edge nodes also become unknowns. The finite difference Equation (10) in this case for the right edge for the nodes (m, j), $j = 1,2,3,..n-1;\quad i = 1,2,..,m$

$$T_{m+1,j} + T_{m-1,j} + T_{m,j-1} + T_{m,j+1} - 4T_{m,j} = 0 \tag{13}$$

However, the node $(m+1, j)$ is not inside the plate. The derivative boundary condition needs to be used to account for these additional unknown nodal temperatures on the right edge. This is done by approximating the derivative at the edge node (m, j) as

$$\left.\frac{\partial T}{\partial x}\right|_{m,j} \cong \frac{T_{m+1,j} - T_{m-1,j}}{2(\Delta x)} \tag{14}$$

giving

$$T_{m+1,j} = T_{m-1,j} + 2(\Delta x)\left.\frac{\partial T}{\partial x}\right|_{m,j} \tag{15}$$

substituting Equation (15) in Equation (13), gives

$$2T_{m-1,j} + 2(\Delta x)\left.\frac{\partial T}{\partial x}\right|_{m,j} + T_{m,j-1} + T_{m,j+1} - 4T_{m,j} = 0 \tag{16}$$

Now if the edge is insulated,

$$\left.\frac{\partial T}{\partial x}\right|_{m,j} = 0 \tag{17}$$

substituting Equation (17) in Equation (16), gives an equation to use at the Neuman Boundary condition

$$2T_{m-1,j} + T_{m,j-1} + T_{m,j+1} - 4T_{m,j} = 0 \tag{18}$$

Example 4

A plate $2.4\,m \times 3.0\,m$ is subjected to the temperatures and insulated boundary conditions as shown in Fig. 12. Use a square grid length of $0.6\,m$. Assume the initial temperatures at all of the interior nodes to be $0\,°C$. Find the temperatures at the interior nodes using the direct method.

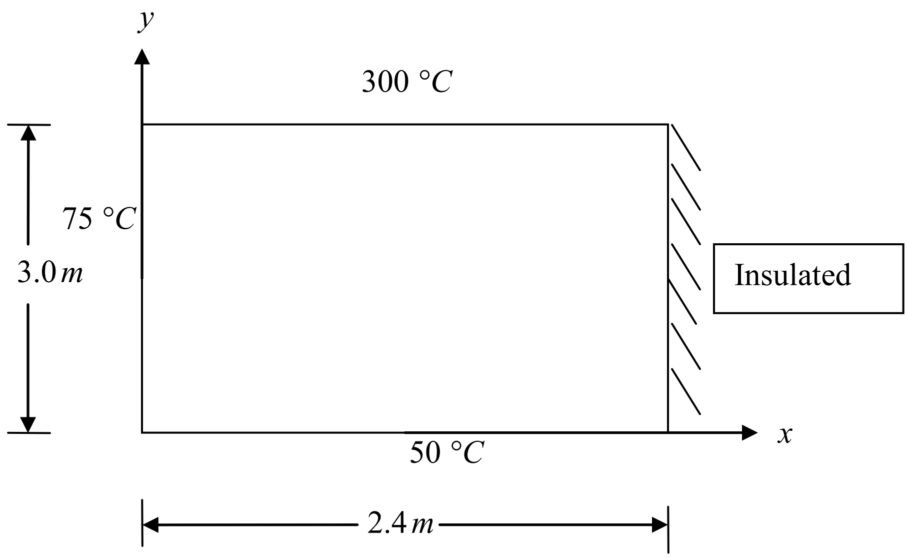

Figure 12: Plate with the dimensions and boundary conditions

Solution

$$\Delta x = \Delta y = 0.6m$$

Re-writing Equations (5) and (6) we have

$$m = \frac{L}{\Delta x}$$
$$= \frac{2.4}{0.6}$$
$$= 4$$
$$n = \frac{W}{\Delta y}$$
$$= \frac{3}{0.6}$$
$$= 5$$

The unknown temperature nodes are shown in Figure 13.

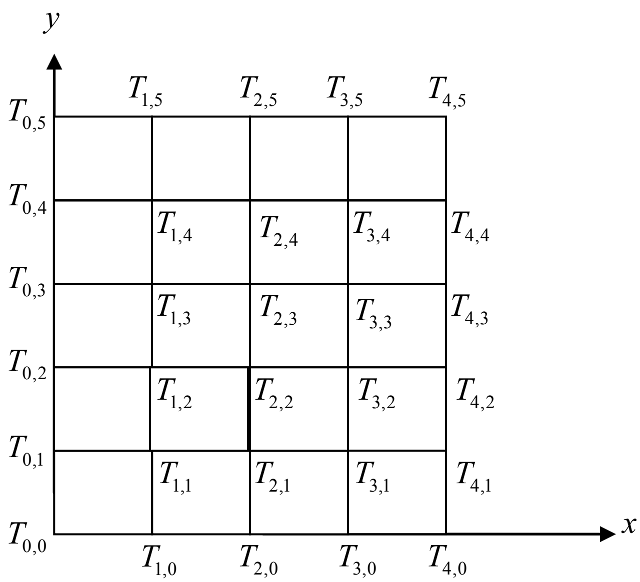

Figure 13: Plate with the nodes labeled

All of the nodes on the boundary have an i value of either zero or m. All of the nodes on the boundary have a j value of either zero or n.
From the boundary conditions

$$\left.\begin{aligned} &T_{0,j} = 75; j = 1,2,3,4 \\ &T_{i,0} = 50; i = 1,2,3,4 \\ &T_{i,5} = 300; i = 1,2,3,4 \\ &\left.\frac{\partial T}{\partial x}\right|_{4,j} = 0; j = 1,2,3,4 \end{aligned}\right\} \quad \text{(E4.1)}$$

Now in order to find the temperatures at the interior nodes, we have to write Equation (10) for all of the combinations of i and j. We express this using i from 1 to $m-1$ and j from 1 to $n-1$. For the right side boundary nodes, where $i = m = 4$, we have to write Equation (18) for $j = 1,2,3,4$. This would give $m \times n - 1$ simultaneous linear equations with $m \times n - 1$ unknowns.

<u>i=1 and j=1</u>

$$T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0} - 4T_{1,1} = 0$$

$$T_{2,1} + 75 + T_{1,2} + 50 - 4T_{1,1} = 0$$

$$-4T_{1,1} + T_{1,2} + T_{2,1} = -125 \quad \text{(E4.2)}$$

<u>i=1 and j=2</u>

$$T_{2,2} + T_{0,2} + T_{1,3} + T_{1,1} - 4T_{1,2} = 0$$

$$T_{2,2} + 75 + T_{1,3} + T_{1,1} - 4T_{1,2} = 0$$

$$T_{1,1} - 4T_{1,2} + T_{1,3} + T_{2,2} = -75 \quad \text{(E4.3)}$$

$\underline{i=1 \text{ and } j=3}$

$$T_{2,3} + T_{0,3} + T_{1,4} + T_{1,2} - 4T_{1,3} = 0$$

$$T_{2,3} + 75 + T_{1,4} + T_{1,2} - 4T_{1,3} = 0$$

$$T_{1,2} - 4T_{1,3} + T_{1,4} + T_{2,3} = -75 \qquad \text{(E4.4)}$$

$\underline{i=1 \text{ and } j=4}$

$$T_{2,4} + T_{0,4} + T_{1,5} + T_{1,3} - 4T_{1,4} = 0$$

$$T_{2,4} + 75 + 300 + T_{1,3} - 4T_{1,4} = 0$$

$$T_{1,3} - 4T_{1,4} + T_{2,4} = -375 \qquad \text{(E4.5)}$$

$\underline{i=2 \text{ and } j=1}$

$$T_{3,1} + T_{1,1} + T_{2,2} + T_{2,0} - 4T_{2,1} = 0$$

$$T_{3,1} + T_{1,1} + T_{2,2} + 50 - 4T_{2,1} = 0$$

$$T_{1,1} - 4T_{2,1} + T_{2,2} + T_{3,1} = -50 \qquad \text{(E4.6)}$$

$\underline{i=2 \text{ and } j=2}$

$$T_{3,2} + T_{1,2} + T_{2,3} + T_{2,1} - 4T_{2,2} = 0$$

$$T_{1,2} + T_{2,1} - 4T_{2,2} + T_{2,3} + T_{3,2} = 0 \qquad \text{(E4.7)}$$

$\underline{i=2 \text{ and } j=3}$

$$T_{3,3} + T_{1,3} + T_{2,4} + T_{2,2} - 4T_{2,3} = 0$$

$$T_{1,3} + T_{2,2} - 4T_{2,3} + T_{2,4} + T_{3,3} = 0 \qquad \text{(E4.8)}$$

$\underline{i=2 \text{ and } j=4}$

$$T_{3,4} + T_{1,4} + T_{2,5} + T_{2,3} - 4T_{2,4} = 0$$

$$T_{3,4} + T_{1,4} + 300 + T_{2,3} - 4T_{2,4} = 0$$

$$T_{1,4} + T_{2,3} - 4T_{2,4} + T_{3,4} = -300 \qquad \text{(E4.9)}$$

$\underline{i=3 \text{ and } j=1}$

$$T_{4,1} + T_{2,1} + T_{3,2} + T_{3,0} - 4T_{3,1} = 0$$

$$T_{4,1} + T_{2,1} + T_{3,2} + 50 - 4T_{3,1} = 0$$

$$T_{2,1} - 4T_{3,1} + T_{3,2} + T_{4,1} = -50 \qquad \text{(E4.10)}$$

$\underline{i=3 \text{ and } j=2}$

$$T_{4,2} + T_{2,2} + T_{3,3} + T_{3,1} - 4T_{3,2} = 0$$

$$T_{2,2} + T_{3,1} - 4T_{3,2} + T_{3,3} + T_{4,2} = 0 \qquad \text{(E4.11)}$$

$\underline{i=3 \text{ and } j=3}$

$$T_{4,3} + T_{2,3} + T_{3,4} + T_{3,2} - 4T_{3,3} = 0$$

$$T_{2,3} + T_{3,2} - 4T_{3,3} + T_{3,4} + T_{4,3} = 0 \qquad \text{(E4.12)}$$

$\underline{i=3 \text{ and } j=4}$

$$T_{4,4} + T_{2,4} + T_{3,5} + T_{3,3} - 4T_{3,4} = 0$$

$$T_{4,4} + T_{2,4} + 300 + T_{3,3} - 4T_{3,4} = 0$$

$$T_{2,4} + T_{3,3} - 4T_{3,4} + T_{4,4} = -300 \quad \text{(E4.13)}$$

Now for $i = 4$ (for this problem $m = 4$), all of these nodes are on the right hand side boundary which is insulated, so we use Equation (18) for $j = 1,2,3 \text{ and } 4$ .Substituting i for m variables gives

i=4 and j=1

$$2T_{3,1} + T_{4,0} + T_{4,2} - 4T_{4,1} = 0 \quad \text{(E4.14)}$$

$$2T_{3,1} + 50 + T_{4,2} - 4T_{4,1} = 0$$

$$2T_{3,1} - 4T_{4,1} + T_{4,2} = -50$$

i=4 and j=2

$$2T_{3,2} + T_{4,1} + T_{4,3} - 4T_{4,2} = 0 \quad \text{(E4.15)}$$

$$2T_{3,2} + T_{4,1} - 4T_{4,2} + T_{4,3} = 0$$

i=4 and j=3

$$2T_{3,3} + T_{4,2} + T_{4,4} - 4T_{4,3} = 0 \quad \text{(E4.16)}$$

$$2T_{3,3} + T_{4,2} - 4T_{4,3} + T_{4,4} = 0$$

i=4 and j=4

$$2T_{3,4} + T_{4,3} + T_{4,5} - 4T_{4,4} = 0 \quad \text{(E4.17)}$$

$$2T_{3,4} + T_{4,3} + 300 - 4T_{4,4} = 0$$

$$2T_{3,4} + T_{4,3} - 4T_{4,4} = -300$$

Equations (E4.2) to (E4.17) represent a set of sixteen simultaneous linear equations, and solving them gives the temperature at sixteen interior nodes. The solution is

$$\begin{bmatrix} T_{1,1} \\ T_{1,2} \\ T_{1,3} \\ T_{1,4} \\ T_{2,1} \\ T_{2,2} \\ T_{2,3} \\ T_{2,4} \\ T_{3,1} \\ T_{3,2} \\ T_{3,3} \\ T_{3,4} \\ T_{4,1} \\ T_{4,2} \\ T_{4,3} \\ T_{4,4} \end{bmatrix} = \begin{bmatrix} 76.8254 \\ 99.4444 \\ 128.617 \\ 180.410 \\ 82.8571 \\ 117.335 \\ 159.614 \\ 218.021 \\ 87.2678 \\ 127.426 \\ 174.483 \\ 232.060 \\ 88.7882 \\ 130.617 \\ 178.830 \\ 232.738 \end{bmatrix} {}^\circ C$$

APPENDIX A

Analytical Solution of Example 1

The differential equation for Example 1 is

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 .$$

The temperature boundary conditions are given on the four sides of the plate (Dirichilet boundary conditions). This problem is too complex to solve analytically. To make this simple, we split the problem into two problems and using the principle of superposition. We then superimpose the solutions of the two simple problems to get the final solution. How the total problem is split is shown in Figure A.1.

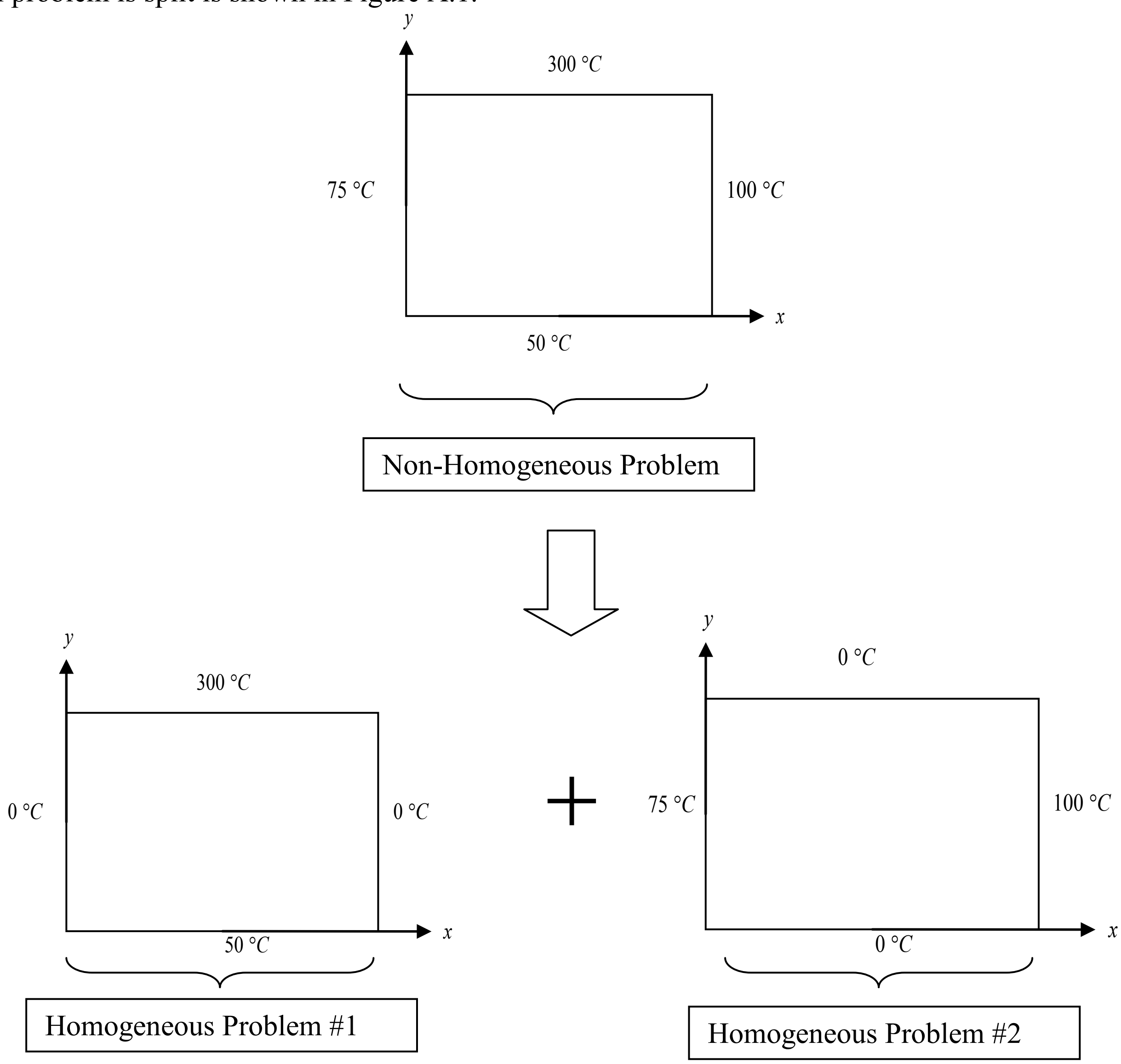

Figure A.1: Splitting of non-homogeneous problem into two homogeneous problems

From Figure A.1, the total solution of the problem is obtained by the summation of the solutions of Problem 1 and Problem 2.

Solution to Problem 1

Let the solution to problem 1 be T_1.

Then the differential equation is

$$\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2} = 0 \quad 0 < x < L \;;\; 0 < y < W \tag{A.1}$$

with boundary conditions

$$T_1(0, y) = 0 \tag{A.2}$$

$$T_1(2.4, y) = 0 \tag{A.3}$$

$$T_1(x,0) = 50 \tag{A.4}$$

$$T_1(x,3.0) = 300 \tag{A.5}$$

Let T_1 be a function of $X(x)$ and $Y(y)$

$$T_1(x, y) = X(x).Y(y) \tag{A.6}$$

Substituting Equation (A.6) in Equation (A.1), we have

$$X''Y + Y''X = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\beta^2 \tag{A.7}$$

<u>Spatial Y solution</u>

Now from Equation (A.7) we can write

$$\frac{Y''}{Y} = \beta^2$$

$$Y'' - \beta^2 Y = 0 \tag{A.8}$$

Equation (A.8) is a homogeneous second order differential equation. These type of equations have the solution of the form $Y(y) = e^{my}$. Substituting $Y(y) = e^{my}$ in Equation (A.8) we get,

$$m^2 e^{my} - \beta^2 e^{my} = 0$$

$$e^{my}(m^2 - \beta^2) = 0$$

$$m^2 - \beta^2 = 0$$

$$m_1, m_2 = \beta, -\beta$$

From the values of m_1 and m_2, the solution of $Y(y)$ is written as

$$Y(y) = A\cosh(\beta y) + B\sinh(\beta y) \tag{A.9}$$

<u>Spatial X solution</u>

Now from Equation (A.7) we can write

$$\frac{X''}{X} = -\beta^2$$

$$X'' + \beta^2 X = 0 \tag{A.10}$$

Equation (A.10) is a homogeneous second order differential equation. These types of equations have the solution of the form $X(x) = e^{mx}$. Substituting $X(x) = e^{mx}$ in Equation (A.10), we get

$$m^2 e^{mx} + \beta^2 e^{mx} = 0$$

$$e^{mx}(m^2 + \beta^2) = 0$$

$$m^2 + \beta^2 = 0$$

$$m_1, m_2 = i\beta, \ -i\beta$$

From the values of m_1 and m_2, the solution of $X(x)$ is written as

$$X(x) = C\cos(\beta x) + D\sin(\beta x) \qquad \text{(A.11)}$$

Substituting Equation (A.9) and Equation (A.11) in Equation (A.6) gives

$$T_1(x, y) = \left[C\cos(\beta x) + D\sin(\beta x)\right]\left[A\cosh(\beta y) + B\sinh(\beta y)\right] \qquad \text{(A.12)}$$

To find the value of the constants we must use the boundary conditions. Applying boundary condition represented by Equation (A.2), we have

$$0 = C\left[A\cosh(\beta y) + B\sinh(\beta y)\right]$$

$$C = 0$$

Substituting $C = 0$ in Equation (A.12), we have

$$\begin{aligned} T_1(x, y) &= D\sin(\beta x)\left[A\cosh(\beta y) + B\sinh(\beta y)\right] \\ &= \sin(\beta x)\left[A\cosh(\beta y) + B\sinh(\beta y)\right] \end{aligned} \qquad \text{(A.13)}$$

Applying the boundary condition represented by Equation (A.13), we have

$$0 = \sin(2.4\beta)\left[A\cosh(\beta y) + B\sinh(\beta y)\right]$$

$$0 = \sin(2.4\beta)$$

$$2.4\beta = n\pi$$

$$\beta = \frac{n\pi}{2.4} \qquad \text{(A.14)}$$

Substituting Equation (A.14) in Equation (A.13)

$$T_1(x, y) = \sin\left(\frac{n\pi}{2.4}x\right)\left[A\cosh\left(\frac{n\pi}{2.4}y\right) + B\sinh\left(\frac{n\pi}{2.4}y\right)\right]$$

Since the general solution can have any value of n,

$$T_1(x, y) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{2.4}x\right)\left[A_n\cosh\left(\frac{n\pi}{2.4}y\right) + B_n\sinh\left(\frac{n\pi}{2.4}y\right)\right] \qquad \text{(A.15)}$$

Applying boundary condition represented by Equation (A.4), we have

$$50 = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{2.4}x\right) \qquad \text{(A.16)}$$

A half range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)$$

where

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Comparing Equation (A.16) with half range sine series, Equation (A.16) is a half-range expression of 50 in sine series with $L = 2.4$. Therefore

$$\begin{aligned}
A_n &= \frac{2}{2.4}\int_0^{2.4} 50\sin\left(\frac{n\pi}{2.4}x\right)dx \\
&= \frac{1}{1.2}50\int_0^{2.4}\sin\left(\frac{n\pi}{2.4}x\right)dx \\
&= \frac{50}{\frac{n\pi}{2.4}1.2}\left[-\cos\left(\frac{n\pi}{2.4}x\right)\right]_0^{2.4} \\
&= \frac{50\times 2.4}{1.2\times n\pi}\left[-\cos(n\pi)+1\right] \\
&= \frac{100}{n\pi}\left[-\cos(n\pi)+1\right] \\
&= \frac{100}{n\pi}\left[1-(-1)^n\right] \qquad \text{(A.17)}
\end{aligned}$$

Applying boundary condition represented by Equation (A.5) ,we have

$$300 = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{2.4}x\right)\left[A_n\cosh\left(\frac{n\pi}{2.4}3.0\right)+B_n\sinh\left(\frac{n\pi}{2.4}3.0\right)\right] \qquad \text{(A.18)}$$

Solving Equation (A.18) for B_n gives

$$\begin{aligned}
B_n &= \frac{1}{\sin\left(\frac{3n\pi}{2.4}\right)}\left\{\frac{2}{2.4}\int_0^{2.4}300\sin\left(\frac{n\pi}{2.4}x\right)dx - A_n\cos\left(\frac{3n\pi}{2.4}\right)\right\} \\
&= \frac{1}{\sin\left(\frac{3n\pi}{2.4}\right)}\left\{\frac{600}{2.4}\left[\frac{-\cos\left(\frac{n\pi x}{2.4}\right)}{\frac{n\pi}{2.4}}\right]_0^{2.4} - A_n\cos\left(\frac{3n\pi}{2.4}\right)\right\} \\
&= \frac{1}{\sin\left(\frac{3n\pi}{2.4}\right)}\left\{\frac{600}{2.4}\frac{2.4}{n\pi}\left[-\cos\left(\frac{n\pi x}{2.4}\right)\right]_0^{2.4} - A_n\cos\left(\frac{3n\pi}{2.4}\right)\right\} \\
&= \frac{1}{\sin\left(\frac{3n\pi}{2.4}\right)}\left\{\frac{600}{n\pi}\left[-\cos(n\pi)+1\right] - A_n\cos\left(\frac{3n\pi}{2.4}\right)\right\} \\
&= \frac{1}{\sin\left(\frac{3n\pi}{2.4}\right)}\left\{\frac{600}{n\pi}\left[1-(-1)^n\right] - A_n\cos\left(\frac{3n\pi}{2.4}\right)\right\} \qquad \text{(A.19)}
\end{aligned}$$

From Equations (A.15), (A.17) and (A.19), the solution T_1 is given as

$$T_1(x,y)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{2.4}x\right)\left[A_n\cosh\left(\frac{n\pi}{2.4}y\right)+B_n\sinh\left(\frac{n\pi}{2.4}y\right)\right] \tag{A.20}$$

where

$$A_n=\frac{100}{n\pi}\left[1-(-1)^n\right] \text{ and}$$

$$B_n=\frac{1}{\sin\left(\frac{3n\pi}{2.4}\right)}\left\{\frac{600}{n\pi}\left[1-(-1)^n\right]-A_n\cos\left(\frac{3n\pi}{2.4}\right)\right\}$$

Solution Problem 2

Let the solution to Problem 2 be T_2. Problem 2 can be solved similarly as Problem 1. The solution to Problem 2 is

$$T_2(x,y)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{3}y\right)\left[C_n\cosh\left(\frac{n\pi}{3}x\right)+D_n\sinh\left(\frac{n\pi}{3}x\right)\right] \tag{A.21}$$

where

$$C_n=\frac{150}{n\pi}\left[1-(-1)^n\right] \quad \text{and}$$

$$D_n=\frac{1}{\sin\left(\frac{2.4n\pi}{3}\right)}\left\{\frac{200}{n\pi}\left[1-(-1)^n\right]-C_n\cos\left(\frac{2.4n\pi}{3}\right)\right\}$$

Overall Solution

The overall solution to the problem is

$$T(x,y)=T_1(x,y)+T_2(x,y)$$

$$T(x,y)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{2.4}x\right)\left[A_n\cosh\left(\frac{n\pi}{2.4}y\right)+B_n\sinh\left(\frac{n\pi}{2.4}y\right)\right]+$$
$$\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{3}y\right)\left[C_n\cosh\left(\frac{n\pi}{3}x\right)+D_n\sinh\left(\frac{n\pi}{3}x\right)\right]$$

where

$$A_n=\frac{100}{n\pi}\left[1-(-1)^n\right],$$

$$B_n=\frac{1}{\sin\left(\frac{3n\pi}{2.4}\right)}\left\{\frac{600}{n\pi}\left[1-(-1)^n\right]-A_n\cos\left(\frac{3n\pi}{2.4}\right)\right\},$$

$$C_n=\frac{150}{n\pi}\left[1-(-1)^n\right] \quad ,$$

$$D_n=\frac{1}{\sin\left(\frac{2.4n\pi}{3}\right)}\left\{\frac{200}{n\pi}\left[1-(-1)^n\right]-C_n\cos\left(\frac{2.4n\pi}{3}\right)\right\}.$$

PARTIAL DIFFERENTIAL EQUATIONS	
Topic	Parabolic Differential Equations
Summary	Textbook notes for the parabolic partial differential equations
Major	All engineering majors
Authors	Autar Kaw, Sri Harsha Garapati, Frederik Schousboe
Date	February 21, 2011
Web Site	http://numericalmethods.eng.usf.edu