Chapter 10.03
Elliptic Partial Differential Equations

After reading this chapter, you should be able to:

1. use numerical methods to solve elliptic partial differential equations by direct method, Gauss-Seidel method, and Gauss-Seidel method with over relaxation.

The general second order linear PDE with two independent variables and one dependent variable is given by

\[ A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D = 0 \]  

(1)

where \( A, B, C \) are functions of the independent variables \( x \) and \( y \), and \( D \) can be a function of \( x, y, u, \frac{\partial u}{\partial x} \) and \( \frac{\partial u}{\partial y} \). Equation (1) is considered to be elliptic if

\[ B^2 - 4AC < 0 \]  

(2)

One popular example of an elliptic second order linear partial differential equation is the Laplace equation which is of the form

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]  

(3)

As

\[ A = 1, B = 0, C = 1, D = 0 \]

then

\[ B^2 - 4AC = 0 - 4(1)(1) \]

\[ = -4 < 0 \]

Hence equation (3) is elliptic.

The Direct Method of Solving Elliptic PDEs

Let’s find the solution via a specific physical example. Take a rectangular plate as shown in Fig. 1 where each side of the plate is maintained at a specific temperature. We are interested in finding the temperature within the plate at steady state. No heat sinks or sources exist in the problem.
The partial differential equation that governs the temperature $T(x, y)$ is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

(4)

To find the temperature within the plate, we divide the plate area by a grid as shown in Figure 2.
The length \( L \) along the \( x \)-axis is divided into \( m \) equal segments, while the width \( W \) along the \( y \)-axis is divided into \( n \) equal segments, hence giving

\[
\Delta x = \frac{L}{m}
\]
\[
\Delta y = \frac{W}{n}
\]

Now we will apply the finite difference approximation of the partial derivatives at a general interior node \((i, j)\).

\[
\frac{\partial^2 T}{\partial x^2}
\approx \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2}
\]

\[
\frac{\partial^2 T}{\partial y^2}
\approx \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}
\]

Equations (7) and (8) are central divided difference approximations of the second derivatives. Substituting Equations (7) and (8) in Equation (4), we get

\[
\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0
\]

For a grid with \( \Delta x = \Delta y \)

Equation (9) can be simplified as

\[
T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0
\]

Now we can write this equation at all the interior nodes of the plate, that is \((m - 1) \times (n - 1)\) nodes. This will result in an equal number of equations and unknowns. The unknowns are the temperatures at the interior \((m - 1) \times (n - 1)\) nodes. Solving these equations will give us the two-dimensional profile of the temperature inside the plate.

**Example 1**

A plate \(2.4 \text{ m} \times 3.0 \text{ m}\) is subjected to temperatures as shown in Figure 3. Use a square grid length of \(0.6 \text{ m}\). Using the direct method, find the temperature at the interior nodes.
Figure 3: Plate with dimension and boundary temperatures

Solution

\[ \Delta x = \Delta y = 0.6m \]

Re-writing Equations (5) and (6) we have

\[ m = \frac{L}{\Delta x} \]
\[ = \frac{2.4}{0.6} \]
\[ = 4 \]

\[ n = \frac{W}{\Delta y} \]
\[ = \frac{3}{0.6} \]
\[ = 5 \]

The nodes are shown in Figure 4.
All the nodes on the left and right boundary have an $i$ value of zero and $m$, respectively. While all the nodes on the top and bottom boundary have a $j$ value of zero and $n$, respectively.

From the boundary conditions

\[
\begin{align*}
T_{0,j} &= 75, j = 1, 2, 3, 4 \\
T_{4,j} &= 100, j = 1, 2, 3, 4 \\
T_{i,0} &= 50, i = 1, 2, 3 \\
T_{i,5} &= 300, i = 1, 2, 3
\end{align*}
\]

(E1.1)

The corner nodal temperature of $T_{0,5}, T_{4,5}, T_{4,0}$ and $T_{0,0}$ are not needed. Now to get the temperature at the interior nodes we have to write Equation (10) for all the combinations of $i$ and $j$, $i = 1, \ldots, m - 1; j = 1, \ldots, n - 1$.

\[i = 1\) and \(j = 1\]

\[
\begin{align*}
T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0} - 4T_{1,1} &= 0 \\
T_{2,1} + 75 + T_{1,2} + 50 - 4T_{1,1} &= 0 \\
-4T_{1,1} + T_{1,2} + T_{2,1} &= -125
\end{align*}
\]

(E1.2)

\[i = 1\) and \(j = 2\]

\[
\begin{align*}
T_{2,2} + T_{0,2} + T_{1,3} + T_{1,1} - 4T_{1,2} &= 0 \\
T_{2,2} + 75 + T_{1,3} + T_{1,1} - 4T_{1,2} &= 0 \\
T_{1,1} - 4T_{1,2} + T_{1,3} + T_{2,2} &= -75
\end{align*}
\]

(E1.3)

\[i = 1\) and \(j = 3\]

\[
\begin{align*}
T_{2,3} + T_{0,3} + T_{1,4} + T_{1,2} - 4T_{1,3} &= 0 \\
T_{2,3} + 75 + T_{1,4} + T_{1,2} - 4T_{1,3} &= 0
\end{align*}
\]
\[ T_{1,2} - 4T_{1,3} + T_{1,4} + T_{2,3} = -75 \]  \hspace{1cm} (E1.4)

\[ T_{i,j} = 1 \text{ and } j = 4 \]
\[ T_{2,4} + T_{0,4} + T_{1,4} + T_{2,1} - 4T_{1,4} = 0 \]
\[ T_{2,4} + 75 + 300 + T_{1,4} - 4T_{1,4} = 0 \]
\[ T_{1,4} - 4T_{1,4} + T_{2,4} = -375 \]  \hspace{1cm} (E1.5)

\[ T_{i,j} = 2 \text{ and } j = 1 \]
\[ T_{2,3} + T_{1,1} + T_{2,2} + T_{2,0} - 4T_{2,1} = 0 \]
\[ T_{2,3} + T_{1,1} + T_{2,2} + 50 - 4T_{2,1} = 0 \]
\[ T_{1,1} - 4T_{2,1} + T_{2,2} + T_{3,1} = -50 \]  \hspace{1cm} (E1.6)

\[ T_{i,j} = 2 \text{ and } j = 2 \]
\[ T_{3,2} + T_{1,2} + T_{2,3} + T_{2,1} - 4T_{2,2} = 0 \]
\[ T_{1,2} + T_{2,2} - 4T_{2,2} + T_{2,3} + T_{3,2} = 0 \]  \hspace{1cm} (E1.7)

\[ T_{i,j} = 2 \text{ and } j = 3 \]
\[ T_{3,3} + T_{1,3} + T_{2,4} + T_{2,2} - 4T_{2,3} = 0 \]
\[ T_{1,3} + T_{2,2} - 4T_{2,2} + T_{2,4} + T_{3,3} = 0 \]  \hspace{1cm} (E1.8)

\[ T_{i,j} = 2 \text{ and } j = 4 \]
\[ T_{3,4} + T_{1,4} + T_{2,5} + T_{2,3} - 4T_{2,4} = 0 \]
\[ T_{3,4} + T_{1,4} + 300 + T_{2,3} - 4T_{2,4} = 0 \]
\[ T_{1,4} + T_{2,3} - 4T_{2,4} + T_{3,4} = -300 \]  \hspace{1cm} (E1.9)

\[ T_{i,j} = 3 \text{ and } j = 1 \]
\[ T_{4,3} + T_{2,1} + T_{3,2} + T_{3,0} - 4T_{3,1} = 0 \]
\[ 100 + T_{2,1} + T_{3,2} + 50 - 4T_{3,1} = 0 \]
\[ T_{2,1} - 4T_{3,1} + T_{3,2} = -150 \]  \hspace{1cm} (E1.10)

\[ T_{i,j} = 3 \text{ and } j = 2 \]
\[ T_{4,2} + T_{2,2} + T_{3,3} + T_{3,1} - 4T_{3,2} = 0 \]
\[ 100 + T_{2,2} + T_{3,3} + T_{3,1} - 4T_{3,2} = 0 \]
\[ T_{2,2} + T_{3,1} - 4T_{3,2} + T_{3,3} = -100 \]  \hspace{1cm} (E1.11)

\[ T_{i,j} = 3 \text{ and } j = 3 \]
\[ T_{4,3} + T_{2,3} + T_{3,4} + T_{3,2} - 4T_{3,3} = 0 \]
\[ 100 + T_{2,3} + T_{3,4} + T_{3,2} - 4T_{3,3} = 0 \]
\[ T_{2,3} + T_{3,2} - 4T_{3,3} + T_{3,4} = -100 \]  \hspace{1cm} (E1.12)

\[ T_{i,j} = 3 \text{ and } j = 4 \]
\[ T_{4,4} + T_{2,4} + T_{3,5} + T_{3,3} - 4T_{3,4} = 0 \]
\[ 100 + T_{2,4} + 300 + T_{3,3} - 4T_{3,4} = 0 \]
Equations (E1.2) to (E1.13) represent a set of twelve simultaneous linear equations and solving them gives the temperature at the twelve interior nodes. The solution is

\[
\begin{bmatrix}
T_{1,1} \\
T_{1,2} \\
T_{1,3} \\
T_{1,4} \\
T_{2,1} \\
T_{2,2} \\
T_{2,3} \\
T_{2,4} \\
T_{3,1} \\
T_{3,2} \\
T_{3,3} \\
T_{3,4}
\end{bmatrix} = \begin{bmatrix}
73.8924 \\
93.0252 \\
119.907 \\
173.355 \\
77.5443 \\
103.302 \\
138.248 \\
198.512 \\
82.9833 \\
104.389 \\
131.271 \\
182.446
\end{bmatrix} \text{ °C}
\]

\[T_{2,4} + T_{3,3} - 4T_{3,4} = -400\] 

(E1.13)

**Figure 5:** Temperatures at the interior nodes of the plate
Gauss-Seidel Method

To take advantage of the sparseness of the coefficient matrix as seen in Example 1, the Gauss-Seidel method may provide a more efficient way of finding the solution. In this case, Equation (10) is written for all interior nodes as

\[
T_{i,j} = \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}}{4}, i = 1, 2, 3, 4; j = 1, 2, 3, 4, 5
\]  

(11)

Now Equation (11) is solved iteratively for all interior nodes until all the temperatures at the interior nodes are within a pre-specified tolerance.

Example 2

A plate 2.4 m × 3.0 m is subjected to the temperatures as shown in Fig. 6. Use a square grid length of 0.6 m. Using the Gauss-Seidel method, find the temperature at the interior nodes. Conduct two iterations at all interior nodes. Find the maximum absolute relative error at the end of the second iteration. Assume the initial temperature at all interior nodes to be 0°C.

![Figure 6: A rectangular plate with the dimensions and boundary temperatures](image)

Solution

\[
\Delta x = \Delta y = 0.6 m
\]

Re-writing Equations (5) and (6) we have

\[
m = \frac{L}{\Delta x} = \frac{2.4}{0.6} = 4
\]

\[
n = \frac{W}{\Delta y}
\]
The interior nodes are shown in Figure 7.

![Figure 7: Plate with nodes](image)

All the nodes on the left and right boundary have an $i$ value of zero and $m$, respectively. All of the nodes on the top or bottom boundary have a $j$ value of either zero or $n$, respectively.

From the boundary conditions

\[
\begin{align*}
    T_{0,j} & = 75, \quad j = 1,2,3,4 \\
    T_{4,j} & = 100, \quad j = 1,2,3,4 \\
    T_{i,0} & = 50, \quad i = 1,2,3 \\
    T_{i,5} & = 300, \quad i = 1,2,3
\end{align*}
\]  

(E2.1)

The corner nodal temperature of $T_{0,5}, T_{4,5}, T_{4,0}$ and $T_{0,0}$ are not needed. Now to get the temperature at the interior nodes we have to write Equation (11) for all of the combinations of $i$ and $j$, $i = 1,\ldots,m-1$; $j = 1,\ldots,n-1$.

**Iteration 1**

For iteration 1, we start with all of the interior nodes having a temperature of $0^\circ C$.

For $i=1$ and $j=1$

\[
T_{1,1} = \frac{T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0}}{4} = \frac{0 + 75 + 0 + 50}{4}
\]
\[ C° = 2500.31 \]  
\[ i=1 \text{ and } j=2 \]  
\[ T_{1,2} = \frac{T_{2,2} + T_{0,2} + T_{1,1} + T_{1,1}}{4} \]  
\[ = \frac{0 + 75 + 0 + 31.2500}{4} \]  
\[ = 26.5625°C \]  
\[ i=1 \text{ and } j=3 \]  
\[ T_{1,3} = \frac{T_{2,3} + T_{0,3} + T_{1,4} + T_{1,2}}{4} \]  
\[ = \frac{0 + 75 + 0 + 26.5625}{4} \]  
\[ = 25.3906°C \]  
\[ i=1 \text{ and } j=4 \]  
\[ T_{1,4} = \frac{T_{2,4} + T_{0,4} + T_{1,5} + T_{1,3}}{4} \]  
\[ = \frac{0 + 75 + 300 + 25.3906}{4} \]  
\[ = 100.098°C \]  
\[ i=2 \text{ and } j=1 \]  
\[ T_{2,1} = \frac{T_{3,1} + T_{1,1} + T_{2,2} + T_{2,0}}{4} \]  
\[ = \frac{0 + 31.2500 + 0 + 50}{4} \]  
\[ = 20.3125°C \]  
\[ i=2 \text{ and } j=2 \]  
\[ T_{2,2} = \frac{T_{3,2} + T_{1,2} + T_{2,3} + T_{2,1}}{4} \]  
\[ = \frac{0 + 26.5625 + 0 + 20.3125}{4} \]  
\[ = 11.7188°C \]  
\[ i=2 \text{ and } j=3 \]  
\[ T_{2,3} = \frac{T_{3,3} + T_{1,3} + T_{2,4} + T_{2,2}}{4} \]  
\[ = \frac{0 + 25.3906 + 0 + 11.7188}{4} \]  
\[ = 9.27735°C \]
\(i=2\) and \(j=4\)
\[
T_{2,4} = \frac{T_{3,4} + T_{1,4} + T_{2,5} + T_{2,3}}{4} = \frac{0 + 100.098 + 300 + 9.27735}{4} = 102.344^\circ C
\]

\(i=3\) and \(j=1\)
\[
T_{3,1} = \frac{T_{4,1} + T_{2,1} + T_{3,2} + T_{3,0}}{4} = \frac{100 + 20.3125 + 0 + 50}{4} = 42.5781^\circ C
\]

\(i=3\) and \(j=2\)
\[
T_{3,2} = \frac{T_{4,2} + T_{2,2} + T_{3,3} + T_{3,1}}{4} = \frac{100 + 11.7188 + 0 + 42.5781}{4} = 38.5742^\circ C
\]

\(i=3\) and \(j=3\)
\[
T_{3,3} = \frac{T_{4,3} + T_{2,3} + T_{3,4} + T_{3,2}}{4} = \frac{100 + 9.27735 + 0 + 38.5742}{4} = 36.9629^\circ C
\]

\(i=3\) and \(j=4\)
\[
T_{3,4} = \frac{T_{4,4} + T_{2,4} + T_{3,5} + T_{3,3}}{4} = \frac{100 + 102.344 + 300 + 36.9629}{4} = 134.827^\circ C
\]

**Iteration 2**

For iteration 2, we use the temperatures from iteration 1.

\(i=1\) and \(j=1\)
\[
T_{1,1} = \frac{T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0}}{4} = \frac{20.3125 + 75 + 26.5625 + 50}{4} = 42.9688^\circ C
\]
\[ |\varepsilon_{i,1}^{a}| = \left| \frac{T_{i,1}^{\text{present}} - T_{i,1}^{\text{previous}}}{T_{i,1}^{\text{present}}} \right| \times 100 \]
\[ = \left| \frac{42.9688 - 31.2500}{42.9688} \right| \times 100 \]
\[ = 27.27\% \]

\( i=1 \) and \( j=2 \)
\[ T_{1,2} = \frac{T_{2,2} + T_{0,2} + T_{1,3} + T_{1,1}}{4} \]
\[ = \frac{11.7188 + 75 + 25.3906 + 42.9688}{4} \]
\[ = 38.7696^\circ C \]
\[ |\varepsilon_{i,2}^{a}| = \left| \frac{T_{i,2}^{\text{present}} - T_{i,2}^{\text{previous}}}{T_{i,2}^{\text{present}}} \right| \times 100 \]
\[ = \left| \frac{38.7696 - 26.5625}{38.7696} \right| \times 100 \]
\[ = 31.49\% \]

\( i=1 \) and \( j=3 \)
\[ T_{1,3} = \frac{T_{2,3} + T_{0,3} + T_{1,4} + T_{1,2}}{4} \]
\[ = \frac{9.27735 + 75 + 100.098 + 38.7696}{4} \]
\[ = 55.7862^\circ C \]
\[ |\varepsilon_{i,3}^{a}| = \left| \frac{T_{i,3}^{\text{present}} - T_{i,3}^{\text{previous}}}{T_{i,3}^{\text{present}}} \right| \times 100 \]
\[ = \left| \frac{55.7862 - 25.3906}{55.7862} \right| \times 100 \]
\[ = 54.49\% \]

\( i=1 \) and \( j=4 \)
\[ T_{1,4} = \frac{T_{2,4} + T_{0,4} + T_{1,5} + T_{1,3}}{4} \]
\[ = \frac{102.344 + 75 + 300 + 55.7862}{4} \]
\[ = 133.283^\circ C \]
\[
\varepsilon_{*,2,4} = \left| \frac{T_{*,4}^{\text{present}} - T_{*,4}^{\text{previous}}}{T_{*,4}^{\text{present}}} \right| \times 100
\]
\[
= \left| \frac{133.283 - 100.098}{133.283} \right| \times 100
\]
\[
= 24.90\%
\]

\[i=2 \text{ and } j=1\]
\[
T_{2,1} = \frac{T_{2,1}^{\text{present}} + T_{2,2}^{\text{previous}} + T_{2,0}^{\text{previous}}}{4}
\]
\[
= \frac{42.5781 + 42.9688 + 11.7188 + 50}{4}
\]
\[
= 36.8164^\circ C
\]
\[
\varepsilon_{*,2,1} = \left| \frac{T_{*,1}^{\text{present}} - T_{*,1}^{\text{previous}}}{T_{*,1}^{\text{present}}} \right| \times 100
\]
\[
= \left| \frac{36.8164 - 20.3125}{36.8164} \right| \times 100
\]
\[
= 44.83\%
\]

\[i=2 \text{ and } j=2\]
\[
T_{2,2} = \frac{T_{2,2}^{\text{present}} + T_{2,2}^{\text{previous}} + T_{2,3}^{\text{previous}} + T_{2,1}^{\text{previous}}}{4}
\]
\[
= \frac{38.5742 + 38.7696 + 9.27735 + 36.8164}{4}
\]
\[
= 30.8594^\circ C
\]
\[
\varepsilon_{*,2,2} = \left| \frac{T_{*,2}^{\text{present}} - T_{*,2}^{\text{previous}}}{T_{*,2}^{\text{present}}} \right| \times 100
\]
\[
= \left| \frac{30.8594 - 11.7188}{30.8594} \right| \times 100
\]
\[
= 62.03\%
\]

\[i=2 \text{ and } j=3\]
\[
T_{2,3} = \frac{T_{2,3}^{\text{present}} + T_{2,2}^{\text{previous}} + T_{2,4}^{\text{previous}} + T_{2,2}^{\text{previous}}}{4}
\]
\[
= \frac{36.9629 + 55.7862 + 102.344 + 30.8594}{4}
\]
\[
= 56.4881^\circ C
\]
\[
|\varepsilon_{a,2,3}| = \left| \frac{T_{2,3}^{\text{present}} - T_{2,3}^{\text{previous}}}{T_{2,3}^{\text{present}}} \right| \times 100 \\
= \left| \frac{56.4881 - 9.27735}{56.4881} \right| \times 100 \\
= 83.58\%
\]

\(i=2\) and \(j=4\)

\[
T_{2,4} = \frac{T_{3,4} + T_{1,4} + T_{2,5} + T_{2,3}}{4} \\
= \frac{134.827 + 133.283 + 300 + 56.4881}{4} \\
= 156.150^\circ C
\]

\[
|\varepsilon_{a,2,4}| = \left| \frac{T_{2,4}^{\text{present}} - T_{2,4}^{\text{previous}}}{T_{2,4}^{\text{present}}} \right| \times 100 \\
= \left| \frac{156.150 - 102.344}{156.150} \right| \times 100 \\
= 34.46\%
\]

\(i=3\) and \(j=1\)

\[
T_{3,1} = \frac{T_{4,1} + T_{2,1} + T_{3,2} + T_{3,0}}{4} \\
= \frac{100 + 36.8164 + 38.5742 + 50}{4} \\
= 56.3477^\circ C
\]

\[
|\varepsilon_{a,3,1}| = \left| \frac{T_{3,1}^{\text{present}} - T_{3,1}^{\text{previous}}}{T_{3,1}^{\text{present}}} \right| \times 100 \\
= \left| \frac{56.3477 - 42.5781}{56.3477} \right| \times 100 \\
= 24.44\%
\]

\(i=3\) and \(j=2\)

\[
T_{3,2} = \frac{T_{4,2} + T_{2,2} + T_{3,3} + T_{3,1}}{4} \\
= \frac{100 + 30.8594 + 36.9629 + 56.3477}{4} \\
= 56.0425^\circ C
\]
The maximum absolute relative error at the end of iteration 2 is 83%.
Figure 8: Temperature distribution after two iterations

It took ten iterations to get all of the temperature values within 1% error. The table below lists the temperature values at the interior nodes at the end of each iteration:

<table>
<thead>
<tr>
<th>Node</th>
<th>Number of Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>$T_{1,1}$</td>
<td>31.2500</td>
</tr>
<tr>
<td>$T_{1,2}$</td>
<td>26.5625</td>
</tr>
<tr>
<td>$T_{1,3}$</td>
<td>25.3906</td>
</tr>
<tr>
<td>$T_{1,4}$</td>
<td>100.0977</td>
</tr>
<tr>
<td>$T_{2,1}$</td>
<td>20.3125</td>
</tr>
<tr>
<td>$T_{2,2}$</td>
<td>11.7188</td>
</tr>
<tr>
<td>$T_{2,3}$</td>
<td>9.2773</td>
</tr>
<tr>
<td>$T_{2,4}$</td>
<td>102.3438</td>
</tr>
<tr>
<td>$T_{3,1}$</td>
<td>42.5781</td>
</tr>
<tr>
<td>$T_{3,2}$</td>
<td>38.5742</td>
</tr>
<tr>
<td>$T_{3,3}$</td>
<td>36.9629</td>
</tr>
<tr>
<td>$T_{3,4}$</td>
<td>134.8267</td>
</tr>
</tbody>
</table>
### Successive Over Relaxation Method

The coefficient matrix for solving for temperatures given in Example 1 is diagonally dominant. Hence the Gauss-Siedel method is guaranteed to converge. To accelerate convergence to the solution, over relaxation is used. In this case

$$T_{relaxed}^{i,j} = \lambda T_{new}^{i,j} + (1 - \lambda)T_{old}^{i,j}$$

(12)

where

- $T_{new}^{i,j} = $ value of temperature from current iteration,
- $T_{old}^{i,j} = $ value of temperature from previous iteration,
- $\lambda = $ weighting factor, $1 < \lambda < 2$.

Again, these iterations are continued till the pre-specified tolerance is met for all nodal temperatures. This method is also called the Lieberman method.

### Example 3

A plate $2.4 \text{ m} \times 3.0 \text{ m}$ is subjected to the temperatures as shown in Fig. 6. Use a square grid length of $0.6 \text{ m}$. Use the Gauss-Seidel with successive over relaxation method with a weighting factor of 1.4 to find the temperature at the interior nodes. Conduct two iterations at all interior nodes. Find the maximum absolute relative error at the end of the second iteration. Assume the initial temperature at all interior nodes to be $0^\circ C$. 

<table>
<thead>
<tr>
<th>Node</th>
<th>Number of Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6</td>
</tr>
<tr>
<td>$T_{6,1}$</td>
<td>66.3183</td>
</tr>
<tr>
<td>$T_{6,2}$</td>
<td>83.3763</td>
</tr>
<tr>
<td>$T_{6,3}$</td>
<td>112.4365</td>
</tr>
<tr>
<td>$T_{6,4}$</td>
<td>169.8319</td>
</tr>
<tr>
<td>$T_{7,1}$</td>
<td>69.2590</td>
</tr>
<tr>
<td>$T_{7,2}$</td>
<td>92.8938</td>
</tr>
<tr>
<td>$T_{7,3}$</td>
<td>130.2512</td>
</tr>
<tr>
<td>$T_{7,4}$</td>
<td>194.7504</td>
</tr>
<tr>
<td>$T_{8,1}$</td>
<td>78.4895</td>
</tr>
<tr>
<td>$T_{8,2}$</td>
<td>98.7917</td>
</tr>
<tr>
<td>$T_{8,3}$</td>
<td>126.9904</td>
</tr>
<tr>
<td>$T_{8,4}$</td>
<td>180.4352</td>
</tr>
</tbody>
</table>
Figure 9: A rectangular plate with the dimensions and boundary temperatures

Solution

$\Delta x = \Delta y = 0.6m$

Re-writing Equations (5) and (6) we have

$m = \frac{L}{\Delta x} = \frac{2.4}{0.6} = 4$

$n = \frac{W}{\Delta y} = \frac{3}{0.6} = 5$

The interior nodes are shown in the Figure 10.
All of the nodes on the left and right boundary have an $i$ value of zero and $m$, respectively. All of the nodes on the top or bottom boundary have a $j$ value of either zero or $n$, respectively.

From the boundary conditions
\[
\begin{align*}
T_{0,j} &= 75, \quad j = 1,2,3,4 \\
T_{4,j} &= 100, \quad j = 1,2,3,4 \\
T_{i,0} &= 50, \quad i = 1,2,3 \\
T_{i,5} &= 300, \quad i = 1,2,3
\end{align*}
\]  

(E3.1)

The corner nodal temperature of $T_{0,5}, T_{4,5}, T_{4,0}$ and $T_{0,0}$ are not needed. Now to get the temperature at the interior nodes, we have to write Equation (11) for all of the combinations of $i$ and $j$, $i = 1$ to $m - 1$, $j = 1$ to $n - 1$. After getting the temperature from Equation (11), we have to use Equation (12) to apply the over relaxation method.

**Iteration 1**
For iteration 1, we start with all of the interior nodes having a temperature of $0^\circ C$.

$i = 1$ and $j = 1$
\[
T_{1,1}^{\text{new}} = \frac{T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0}}{4} \\
= \frac{0 + 75 + 0 + 50}{4} \\
= \frac{125}{4} \\
= 31.250^\circ C
\]

$T_{1,1}^{\text{relaxed}} = \lambda T_{1,1}^{\text{new}} + (1 - \lambda)T_{1,1}^{\text{old}}$
\[ C = 1.4(31.2500) + (1 - 1.4)0 = 43.7500^\circ C \]

\( i=1 \) and \( j=2 \)

\[ T_{1,2} = \frac{T_{2,2} + T_{0,2} + T_{1,3} + T_{1,1}}{4} = \frac{0 + 75 + 0 + 43.75}{4} = 29.6875^\circ C \]

\[ T'_{relaxed}^{1,2} = \lambda T_{1,2}^{new} + (1 - \lambda)T_{1,2}^{old} = 1.4(29.6875) + (1 - 1.4)0 = 41.5625^\circ C \]

\( i=1 \) and \( j=3 \)

\[ T_{1,3} = \frac{T_{2,3} + T_{0,3} + T_{1,4} + T_{1,2}}{4} = \frac{0 + 75 + 0 + 41.5625}{4} = 29.1406^\circ C \]

\[ T'_{relaxed}^{1,3} = \lambda T_{1,3}^{new} + (1 - \lambda)T_{1,3}^{old} = 1.4(29.1406) + (1 - 1.4)0 = 40.7969^\circ C \]

\( i=1 \) and \( j=4 \)

\[ T_{1,4} = \frac{T_{2,4} + T_{0,4} + T_{1,5} + T_{1,3}}{4} = \frac{0 + 75 + 300 + 40.7969}{4} = 103.949^\circ C \]

\[ T'_{relaxed}^{1,4} = \lambda T_{1,4}^{new} + (1 - \lambda)T_{1,4}^{old} = 1.4(103.949) + (1 - 1.4)0 = 145.529^\circ C \]

\( i=2 \) and \( j=1 \)

\[ T_{2,1} = \frac{T_{3,1} + T_{1,1} + T_{2,2} + T_{2,0}}{4} = \frac{0 + 43.75 + 0 + 50}{4} = 23.4375^\circ C \]

\[ T'_{relaxed}^{2,1} = \lambda T_{2,1}^{new} + (1 - \lambda)T_{2,1}^{old} \]
\[ T_{2,2} = \frac{T_{3,2} + T_{1,2} + T_{2,3} + T_{2,1}}{4} = \frac{0 + 41.5625 + 0 + 32.8125}{4} = 18.5938^\circ C \]

\[ T_{2,2}^{relaxed} = \lambda T_{2,2}^{new} + (1 - \lambda)T_{2,2}^{old} = 1.4(18.5938) + (1-1.4)0 = 26.0313^\circ C \]

\[ T_{2,3} = \frac{T_{3,3} + T_{1,3} + T_{2,4} + T_{2,2}}{4} = \frac{0 + 40.7969 + 0 + 26.0313}{4} = 16.7071^\circ C \]

\[ T_{2,3}^{relaxed} = \lambda T_{2,3}^{new} + (1 - \lambda)T_{2,3}^{old} = 1.4(16.7071) + (1-1.4)0 = 23.3899^\circ C \]

\[ T_{2,4} = \frac{T_{3,4} + T_{1,4} + T_{2,5} + T_{2,3}}{4} = \frac{0 + 145.529 + 300 + 23.3899}{4} = 117.230^\circ C \]

\[ T_{2,4}^{relaxed} = \lambda T_{2,4}^{new} + (1 - \lambda)T_{2,4}^{old} = 1.4(117.230) + (1-1.4)0 = 164.122^\circ C \]

\[ T_{3,1} = \frac{T_{4,1} + T_{2,1} + T_{3,2} + T_{3,0}}{4} = \frac{100 + 32.8125 + 0 + 50}{4} = 45.7031^\circ C \]

\[ T_{3,1}^{relaxed} = \lambda T_{3,1}^{new} + (1 - \lambda)T_{3,1}^{old} \]
\[ C^\circ = 1.4(45.7031) + (1-1.4)0 \]
\[ = 63.9844^\circ C \]

*i=3* and *j=2*

\[ T_{3,2} = \frac{T_{4,2} + T_{2,2} + T_{3,3} + T_{3,2}}{4} \]
\[ = \frac{100 + 26.0313 + 0 + 63.9844}{4} \]
\[ = 47.5039^\circ C \]

\[ T_{3,2}^{relaxed} = \lambda T_{3,2}^{new} + (1-\lambda)T_{3,2}^{old} \]
\[ = 1.4(47.5039) + (1-1.4)0 \]
\[ = 66.5055^\circ C \]

*i=3* and *j=3*

\[ T_{3,3} = \frac{T_{4,3} + T_{2,3} + T_{3,4} + T_{3,2}}{4} \]
\[ = \frac{100 + 23.3899 + 0 + 66.5055}{4} \]
\[ = 47.4739^\circ C \]

\[ T_{3,3}^{relaxed} = \lambda T_{3,3}^{new} + (1-\lambda)T_{3,3}^{old} \]
\[ = 1.4(47.4739) + (1-1.4)0 \]
\[ = 66.4634^\circ C \]

*i=3* and *j=4*

\[ T_{3,4} = \frac{T_{4,4} + T_{2,4} + T_{3,5} + T_{3,3}}{4} \]
\[ = \frac{100 + 164.122 + 300 + 66.4634}{4} \]
\[ = 157.646^\circ C \]

\[ T_{3,4}^{relaxed} = \lambda T_{3,4}^{new} + (1-\lambda)T_{3,4}^{old} \]
\[ = 1.4(157.646) + (1-1.4)0 \]
\[ = 220.704^\circ C \]

**Iteration 2**

For iteration 2, we take the temperatures from iteration 1.

*i=1* and *j=1*

\[ T_{1,1} = \frac{T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0}}{4} \]
\[ = \frac{32.8125 + 75 + 41.5625 + 50}{4} \]
\[ = 49.8438^\circ C \]
\[
T_{1,1}^{\text{relaxed}} = \lambda T_{1,1}^{\text{new}} + (1 - \lambda)T_{1,1}^{\text{old}}
= 1.4(49.8438) + (1-1.4)43.75
= 52.2813^\circ C
\]

\[
\left|\varepsilon_{1,1}\right| = \left|\frac{T_{1,1}^{\text{present}} - T_{1,1}^{\text{previous}}}{T_{1,1}^{\text{present}}}\right| \times 100
= \left|\frac{52.2813 - 43.75}{52.2813}\right| \times 100
= 16.32\%
\]

\(i=1\) and \(j=2\)
\[
T_{1,2} = \frac{T_{2,2} + T_{0,2} + T_{1,1} + T_{1,1}}{4}
= \frac{26.0313 + 75 + 40.7969 + 52.2813}{4}
= 51.3133\%\%
\]

\[
T_{1,2}^{\text{relaxed}} = \lambda T_{1,2}^{\text{new}} + (1 - \lambda)T_{1,2}^{\text{old}}
= 1.4(48.5274) + (1-1.4)41.5625
= 51.3133^\circ C
\]

\[
\left|\varepsilon_{1,2}\right| = \left|\frac{T_{1,2}^{\text{present}} - T_{1,2}^{\text{previous}}}{T_{1,2}^{\text{present}}}\right| \times 100
= \left|\frac{51.3133 - 41.5625}{51.3133}\right| \times 100
= 19.00\%
\]

\(i=1\) and \(j=3\)
\[
T_{1,3} = \frac{T_{2,3} + T_{0,3} + T_{1,1} + T_{1,2}}{4}
= \frac{23.3899 + 75 + 145.529 + 51.3133}{4}
= 73.8103^\circ C
\]

\[
T_{1,3}^{\text{relaxed}} = \lambda T_{1,3}^{\text{new}} + (1 - \lambda)T_{1,3}^{\text{old}}
= 1.4(73.8103) + (1-1.4)40.7969
= 87.0157^\circ C
\]
\[ |\varepsilon_a|_{1,3} = \frac{T_{1,3}^{\text{present}} - T_{1,3}^{\text{previous}}}{T_{1,3}^{\text{present}}} \times 100 \]
\[ = \frac{87.0157 - 40.7969}{87.0157} \times 100 \]
\[ = 53.12\% \]

\( i=1 \) and \( j=4 \)
\[ T_{1,4} = \frac{T_{2,4} + T_{0,4} + T_{1,5} + T_{1,3}}{4} \]
\[ = \frac{164.122 + 75 + 300 + 87.0157}{4} \]
\[ = 156.534^\circ C \]
\[ T_{1,4}^{\text{relaxed}} = \lambda T_{1,4}^{\text{new}} + (1 - \lambda) T_{1,4}^{\text{old}} \]
\[ = 1.4(156.534) + (1-1.4)145.529 \]
\[ = 160.936^\circ C \]
\[ |\varepsilon_a|_{1,4} = \frac{T_{1,4}^{\text{present}} - T_{1,4}^{\text{previous}}}{T_{1,4}^{\text{present}}} \times 100 \]
\[ = \frac{160.936 - 145.529}{160.936} \times 100 \]
\[ = 9.57\% \]

\( i=2 \) and \( j=1 \)
\[ T_{2,1} = \frac{T_{3,1} + T_{1,1} + T_{2,2} + T_{2,0}}{4} \]
\[ = \frac{63.9844 + 52.2813 + 26.0313 + 50.000}{4} \]
\[ = 48.0743^\circ C \]
\[ T_{2,1}^{\text{relaxed}} = \lambda T_{2,1}^{\text{new}} + (1 - \lambda) T_{2,1}^{\text{old}} \]
\[ = 1.4(48.0743) + (1-1.4)32.8125 \]
\[ = 54.1790^\circ C \]
\[ |\varepsilon_a|_{2,1} = \frac{T_{2,1}^{\text{present}} - T_{2,1}^{\text{previous}}}{T_{2,1}^{\text{present}}} \times 100 \]
\[ = \frac{54.1790 - 32.8125}{54.1790} \times 100 \]
\[ = 39.44\% \]
\(i=2\) and \(j=2\)

\[
T_{2,2} = \frac{T_{3,2} + T_{1,2} + T_{2,3} + T_{2,1}}{4}
\]

\[
= \frac{66.5055 + 51.3133 + 23.3899 + 54.1790}{4}
\]

\[
= 48.8469^\circ C
\]

\[
T_{2,2}^{relaxed} = \lambda T_{2,2}^{new} + (1-\lambda) T_{2,2}^{old}
\]

\[
= 1.4(48.8469) + (1-1.4)26.0313
\]

\[
= 57.9732^\circ C
\]

\[
|E_a|_{2,2} = \left| \frac{T_{2,2}^{present} - T_{2,2}^{previous}}{T_{2,2}^{present}} \right| \times 100
\]

\[
= \left| \frac{57.9732 - 26.0313}{57.9732} \right| \times 100
\]

\[
= 55.10%
\]

\(i=2\) and \(j=3\)

\[
T_{2,3} = \frac{T_{3,3} + T_{1,3} + T_{2,4} + T_{2,2}}{4}
\]

\[
= \frac{66.4634 + 87.0157 + 164.122 + 57.9732}{4}
\]

\[
= 93.8936^\circ C
\]

\[
T_{2,3}^{relaxed} = \lambda T_{2,3}^{new} + (1-\lambda) T_{2,3}^{old}
\]

\[
= 1.4(93.8936) + (1-1.4)23.3899
\]

\[
= 122.095^\circ C
\]

\[
|E_a|_{2,3} = \left| \frac{T_{2,3}^{present} - T_{2,3}^{previous}}{T_{2,3}^{present}} \right| \times 100
\]

\[
= \left| \frac{122.095 - 23.3899}{122.095} \right| \times 100
\]

\[
= 80.84%
\]

\(i=2\) and \(j=4\)

\[
T_{2,4} = \frac{T_{3,4} + T_{1,4} + T_{2,5} + T_{2,3}}{4}
\]

\[
= \frac{220.704 + 160.936 + 300 + 122.095}{4}
\]

\[
= 200.934^\circ C
\]

\[
T_{2,4}^{relaxed} = \lambda T_{2,4}^{new} + (1-\lambda) T_{2,4}^{old}
\]
\[ \varepsilon = 1.4(200.934) + (1-1.4)164.122 \]
\[ = 215.659°C \]
\[ \left| \varepsilon_{a,2,4} \right| = \frac{T_{\text{present}} - T_{\text{previous}}}{T_{\text{present}}} \times 100 \]
\[ = \frac{215.659 - 164.122}{215.659} \times 100 \]
\[ = 23.90% \]

\( i = 3 \) and \( j = 1 \)
\[ T_{3,1} = \frac{T_{4,1} + T_{2,1} + T_{3,2} + T_{3,0}}{4} \]
\[ = \frac{100 + 54.1790 + 66.5055 + 50}{4} \]
\[ = 67.6711°C \]
\[ T_{3,1}^{\text{relaxed}} = \lambda T_{3,1}^{\text{new}} + (1-\lambda)T_{3,1}^{\text{old}} \]
\[ = 1.4(67.6711) + (1-1.4)63.9844 \]
\[ = 69.1458°C \]
\[ \left| \varepsilon_{a,2,3} \right| = \frac{T_{\text{present}} - T_{\text{previous}}}{T_{\text{present}}} \times 100 \]
\[ = \frac{69.1458 - 63.9844}{69.1458} \times 100 \]
\[ = 7.46% \]

\( i = 3 \) and \( j = 2 \)
\[ T_{3,2} = \frac{T_{4,2} + T_{2,2} + T_{3,3} + T_{3,1}}{4} \]
\[ = \frac{100 + 57.9732 + 66.4634 + 69.1458}{4} \]
\[ = 73.3956°C \]
\[ T_{3,2}^{\text{relaxed}} = \lambda T_{3,2}^{\text{new}} + (1-\lambda)T_{3,2}^{\text{old}} \]
\[ = 1.4(73.3956) + (1-1.4)66.5055 \]
\[ = 76.1516°C \]
\[ \left| \varepsilon_{a,2,3} \right| = \frac{T_{\text{present}} - T_{\text{previous}}}{T_{\text{present}}} \times 100 \]
\[ = \frac{76.1516 - 66.5055}{76.1516} \times 100 \]
\[ = 12.67% \]
\(i=3 \text{ and } j=3\)

\[
T_{3,3} = \frac{T_{4,3} + T_{2,3} + T_{3,4} + T_{3,2}}{4} = \frac{100 + 122.095 + 220.704 + 76.1516}{4} = 129.738^\circ C
\]

\[
T_{3,3}^{\text{relaxed}} = \lambda T_{3,3}^{\text{new}} + (1 - \lambda) T_{3,3}^{\text{old}} = 1.4(129.738) + (1 - 1.4)66.4634 = 155.048^\circ C
\]

\[
E_{a,3,3} = \left| \frac{T_{3,3}^{\text{present}} - T_{3,3}^{\text{previous}}}{T_{3,3}^{\text{present}}} \right| \times 100 = \frac{155.048 - 66.4634}{155.048} \times 100 = 57.13\%
\]

\(i=3 \text{ and } j=4\)

\[
T_{3,4} = \frac{T_{4,4} + T_{2,4} + T_{3,5} + T_{3,3}}{4} = \frac{100 + 215.659 + 300 + 155.048}{4} = 192.677^\circ C
\]

\[
T_{3,4}^{\text{relaxed}} = \lambda T_{3,4}^{\text{new}} + (1 - \lambda) T_{3,4}^{\text{old}} = 1.4(192.677) + (1 - 1.4)220.704 = 181.466^\circ C
\]

\[
E_{a,3,4} = \left| \frac{T_{3,4}^{\text{present}} - T_{3,4}^{\text{previous}}}{T_{3,4}^{\text{present}}} \right| \times 100 = \frac{181.466 - 220.704}{181.466} \times 100 = 21.62\%
\]

The maximum absolute relative error at the end of iteration 2 is 81%.
It took nine iterations to get all of the temperature values within 1% error. The table below lists the temperature values at all nodes after each iteration.

<table>
<thead>
<tr>
<th>Node</th>
<th>Number of Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>$T_{1,1}$</td>
<td>43.7500</td>
</tr>
<tr>
<td>$T_{1,2}$</td>
<td>41.5625</td>
</tr>
<tr>
<td>$T_{1,3}$</td>
<td>40.7969</td>
</tr>
<tr>
<td>$T_{1,4}$</td>
<td>145.5289</td>
</tr>
<tr>
<td>$T_{2,1}$</td>
<td>32.8125</td>
</tr>
<tr>
<td>$T_{2,2}$</td>
<td>26.0313</td>
</tr>
<tr>
<td>$T_{2,3}$</td>
<td>23.3898</td>
</tr>
<tr>
<td>$T_{2,4}$</td>
<td>164.1216</td>
</tr>
<tr>
<td>$T_{3,1}$</td>
<td>63.9844</td>
</tr>
<tr>
<td>$T_{3,2}$</td>
<td>66.5055</td>
</tr>
<tr>
<td>$T_{3,3}$</td>
<td>66.4634</td>
</tr>
<tr>
<td>$T_{3,4}$</td>
<td>220.7047</td>
</tr>
<tr>
<td>Node</td>
<td>Number of Iterations</td>
</tr>
<tr>
<td>-------</td>
<td>----------------------</td>
</tr>
<tr>
<td></td>
<td>6</td>
</tr>
<tr>
<td>$T_{1,1}$</td>
<td>79.3934</td>
</tr>
<tr>
<td>$T_{1,2}$</td>
<td>92.3140</td>
</tr>
<tr>
<td>$T_{1,3}$</td>
<td>119.9649</td>
</tr>
<tr>
<td>$T_{1,4}$</td>
<td>173.4118</td>
</tr>
<tr>
<td>$T_{2,1}$</td>
<td>77.1177</td>
</tr>
<tr>
<td>$T_{2,2}$</td>
<td>102.4498</td>
</tr>
<tr>
<td>$T_{2,3}$</td>
<td>137.6794</td>
</tr>
<tr>
<td>$T_{2,4}$</td>
<td>198.2290</td>
</tr>
<tr>
<td>$T_{3,1}$</td>
<td>82.8338</td>
</tr>
<tr>
<td>$T_{3,2}$</td>
<td>103.6414</td>
</tr>
<tr>
<td>$T_{3,3}$</td>
<td>130.8010</td>
</tr>
<tr>
<td>$T_{3,4}$</td>
<td>182.2354</td>
</tr>
</tbody>
</table>
Alternative Boundary Conditions

In Examples 1-3, the boundary conditions on the plate had a specified temperature on each edge. What if the conditions are different? For example; what if one of the edges of the plate is insulated? In this case, the boundary condition would be the derivative of the temperature (called the Neuman boundary condition). If the right edge of the plate is insulated, then the temperatures on the right edge nodes also become unknowns. The finite difference Equation (10) in this case for the right edge for the nodes \((m, j)\), \(j = 1,2,3,..,n-1\); \(i = 1,2,..,m\)

\[
T_{m+1,j} + T_{m-1,j} + T_{m,j-1} + T_{m,j+1} - 4T_{m,j} = 0
\]  

(13)

However, the node \((m+1, j)\) is not inside the plate. The derivative boundary condition needs to be used to account for these additional unknown nodal temperatures on the right edge. This is done by approximating the derivative at the edge node \((m, j)\) as

\[
\frac{\partial T}{\partial x}_{m,j} \approx \frac{T_{m+1,j} - T_{m-1,j}}{2(\Delta x)}
\]  

(14)

giving

\[
T_{m+1,j} = T_{m-1,j} + 2(\Delta x)\frac{\partial T}{\partial x}_{m,j}
\]  

(15)

substituting Equation (15) in Equation (13), gives

\[
2T_{m-1,j} + 2(\Delta x)\frac{\partial T}{\partial x}_{m,j} + T_{m,j-1} + T_{m,j+1} - 4T_{m,j} = 0
\]  

(16)

Now if the edge is insulated,

\[
\frac{\partial T}{\partial x}_{m,j} = 0
\]  

(17)

substituting Equation (17) in Equation (16), gives an equation to use at the Neuman Boundary condition

\[
2T_{m-1,j} + T_{m,j-1} + T_{m,j+1} - 4T_{m,j} = 0
\]  

(18)

Example 4

A plate \(2.4 m \times 3.0 m\) is subjected to the temperatures and insulated boundary conditions as shown in Fig. 12. Use a square grid length of \(0.6 m\). Assume the initial temperatures at all of the interior nodes to be \(0^\circ C\). Find the temperatures at the interior nodes using the direct method.
Re-writing Equations (5) and (6) we have

\[ m = \frac{L}{\Delta x} = \frac{2.4}{0.6} = 4 \]

\[ n = \frac{W}{\Delta y} = \frac{3}{0.6} = 5 \]

The unknown temperature nodes are shown in Figure 13.
All of the nodes on the boundary have an $i$ value of either zero or $m$. All of the nodes on the boundary have a $j$ value of either zero or $n$.

From the boundary conditions
\[
\begin{align*}
T_{0,j} &= 75; j = 1,2,3,4 \\
T_{i,0} &= 50; i = 1,2,3,4 \\
T_{i,5} &= 300; i = 1,2,3,4 \\
\frac{\partial T}{\partial x}_{i,j} &= 0; j = 1,2,3,4
\end{align*}
\]
(E4.1)

Now in order to find the temperatures at the interior nodes, we have to write Equation (10) for all of the combinations of $i$ and $j$. We express this using $i$ from 1 to $m-1$ and $j$ from 1 to $n-1$. For the right side boundary nodes, where $i = m = 4$, we have to write Equation (18) for $j = 1,2,3,4$. This would give $m \times n - 1$ simultaneous linear equations with $m \times n - 1$ unknowns.

\begin{align*}
\text{\textit{i=1 and j=1}} & \quad T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0} - 4T_{1,1} = 0 \\
& \quad T_{2,1} + 75 + T_{1,2} + 50 - 4T_{1,1} = 0 \\
& \quad - 4T_{1,1} + T_{1,2} + T_{2,1} = -125
\end{align*}
(E4.2)

\begin{align*}
\text{\textit{i=1 and j=2}} & \quad T_{2,2} + T_{0,2} + T_{1,3} + T_{1,1} - 4T_{1,2} = 0 \\
& \quad T_{2,2} + 75 + T_{1,3} + T_{1,1} - 4T_{1,2} = 0 \\
& \quad T_{1,1} - 4T_{1,2} + T_{1,3} + T_{2,2} = -75
\end{align*}
(E4.3)
\( i=1 \) and \( j=3 \)
\[
\begin{align*}
T_{2,3} + T_{0,3} + T_{1,4} + T_{1,2} - 4T_{1,3} &= 0 \\
T_{2,3} + 75 + T_{1,4} + T_{1,2} - 4T_{1,3} &= 0 \\
T_{1,2} - 4T_{1,3} + T_{1,4} + T_{2,3} &= -75
\end{align*}
\] (E4.4)

\( i=1 \) and \( j=4 \)
\[
\begin{align*}
T_{2,4} + T_{0,4} + T_{1,5} + T_{1,3} - 4T_{1,4} &= 0 \\
T_{2,4} + 75 + 300 + T_{1,3} - 4T_{1,4} &= 0 \\
T_{1,3} - 4T_{1,4} + T_{2,4} &= -375
\end{align*}
\] (E4.5)

\( i=2 \) and \( j=1 \)
\[
\begin{align*}
T_{3,1} + T_{1,1} + T_{2,2} + T_{2,0} - 4T_{2,1} &= 0 \\
T_{3,1} + T_{1,1} + T_{2,2} + 50 - 4T_{2,1} &= 0 \\
T_{1,1} - 4T_{2,1} + T_{2,2} + T_{3,1} &= -50
\end{align*}
\] (E4.6)

\( i=2 \) and \( j=2 \)
\[
\begin{align*}
T_{3,2} + T_{1,2} + T_{2,3} + T_{2,1} - 4T_{2,2} &= 0 \\
T_{1,2} + T_{2,1} - 4T_{2,2} + T_{2,3} + T_{3,2} &= 0
\end{align*}
\] (E4.7)

\( i=2 \) and \( j=3 \)
\[
\begin{align*}
T_{3,3} + T_{1,3} + T_{2,4} + T_{2,2} - 4T_{2,3} &= 0 \\
T_{1,3} + T_{2,2} - 4T_{2,3} + T_{2,4} + T_{3,3} &= 0
\end{align*}
\] (E4.8)

\( i=2 \) and \( j=4 \)
\[
\begin{align*}
T_{3,4} + T_{1,4} + T_{2,5} + T_{2,3} - 4T_{2,4} &= 0 \\
T_{3,4} + T_{1,4} + 300 + T_{2,3} - 4T_{2,4} &= 0 \\
T_{1,4} + T_{2,3} - 4T_{2,4} + T_{3,4} &= -300
\end{align*}
\] (E4.9)

\( i=3 \) and \( j=1 \)
\[
\begin{align*}
T_{4,1} + T_{2,1} + T_{3,2} + T_{3,0} - 4T_{3,1} &= 0 \\
T_{4,1} + T_{2,1} + T_{3,2} + 50 - 4T_{3,1} &= 0 \\
T_{2,1} - 4T_{3,1} + T_{3,2} + T_{4,1} &= -50
\end{align*}
\] (E4.10)

\( i=3 \) and \( j=2 \)
\[
\begin{align*}
T_{4,2} + T_{2,2} + T_{3,3} + T_{3,1} - 4T_{3,2} &= 0 \\
T_{2,2} + T_{3,1} - 4T_{3,2} + T_{3,3} + T_{4,2} &= 0
\end{align*}
\] (E4.11)

\( i=3 \) and \( j=3 \)
\[
\begin{align*}
T_{4,3} + T_{2,3} + T_{3,4} + T_{3,2} - 4T_{3,3} &= 0 \\
T_{2,3} + T_{3,2} - 4T_{3,3} + T_{3,4} + T_{4,3} &= 0
\end{align*}
\] (E4.12)

\( i=3 \) and \( j=4 \)
\[
\begin{align*}
T_{4,4} + T_{2,4} + T_{3,5} + T_{3,3} - 4T_{3,4} &= 0
\end{align*}
\]
\[ T_{4,4} + T_{2,4} + 300 + T_{3,3} - 4T_{3,4} = 0 \]
\[ T_{2,4} + T_{3,3} - 4T_{3,4} + T_{4,4} = -300 \]  
(E4.13)

Now for \( i = 4 \) (for this problem \( m = 4 \)), all of these nodes are on the right hand side boundary which is insulated, so we use Equation (18) for \( j = 1, 2, 3 \) and 4. Substituting \( i \) for \( m \) variables gives

\( i = 4 \) and \( j = 1 \)

\[ 2T_{3,1} + T_{4,0} + T_{4,2} - 4T_{4,1} = 0 \]  
(E4.14)
\[ 2T_{3,1} + 50 + T_{4,2} - 4T_{4,1} = 0 \]
\[ 2T_{3,1} - 4T_{4,1} + T_{4,2} = -50 \]

\( i = 4 \) and \( j = 2 \)

\[ 2T_{3,2} + T_{4,1} + T_{4,3} - 4T_{4,2} = 0 \]  
(E4.15)
\[ 2T_{3,2} + T_{4,1} - 4T_{4,2} + T_{4,3} = 0 \]

\( i = 4 \) and \( j = 3 \)

\[ 2T_{3,3} + T_{4,2} + T_{4,4} - 4T_{4,3} = 0 \]  
(E4.16)
\[ 2T_{3,3} + T_{4,2} - 4T_{4,3} + T_{4,4} = 0 \]

\( i = 4 \) and \( j = 4 \)

\[ 2T_{3,4} + T_{4,3} + T_{4,5} - 4T_{4,4} = 0 \]  
(E4.17)
\[ 2T_{3,4} + T_{4,3} + 300 - 4T_{4,4} = 0 \]
\[ 2T_{3,4} + T_{4,3} - 4T_{4,4} = -300 \]
Equations (E4.2) to (E4.17) represent a set of sixteen simultaneous linear equations, and solving them gives the temperature at sixteen interior nodes. The solution is

\[
\begin{bmatrix}
T_{1,1} \\
T_{1,2} \\
T_{1,3} \\
T_{1,4} \\
T_{2,1} \\
T_{2,2} \\
T_{2,3} \\
T_{2,4} \\
T_{3,1} \\
T_{3,2} \\
T_{3,3} \\
T_{3,4} \\
T_{4,1} \\
T_{4,2} \\
T_{4,3} \\
T_{4,4}
\end{bmatrix}
= \begin{bmatrix}
76.8254 \\
99.4444 \\
128.617 \\
180.410 \\
82.8571 \\
117.335 \\
159.614 \\
218.021 \\
87.2678 \\
127.426 \\
174.483 \\
232.060 \\
88.7882 \\
130.617 \\
178.830 \\
232.738
\end{bmatrix} \degree C
\]
APPENDIX A

Analytical Solution of Example 1

The differential equation for Example 1 is
\[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0. \]

The temperature boundary conditions are given on the four sides of the plate (Dirichlet boundary conditions). This problem is too complex to solve analytically. To make this simple, we split the problem into two problems and using the principle of superposition. We then superimpose the solutions of the two simple problems to get the final solution. How the total problem is split is shown in Figure A.1.

**Figure A.1:** Splitting of non-homogeneous problem into two homogeneous problems
From Figure A.1, the total solution of the problem is obtained by the summation of the solutions of Problem 1 and Problem 2.

**Solution to Problem 1**

Let the solution to problem 1 be $T_1$.

Then the differential equation is

$$\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2} = 0 \quad 0 < x < L \quad ; \quad 0 < y < W$$ \hspace{1cm} (A.1)

with boundary conditions

$$T_1(0,y) = 0 \hspace{1cm} (A.2)$$
$$T_1(2.4,y) = 0 \hspace{1cm} (A.3)$$
$$T_1(x,0) = 50 \hspace{1cm} (A.4)$$
$$T_1(x,3.0) = 300 \hspace{1cm} (A.5)$$

Let $T_1$ be a function of $X(x)$ and $Y(y)$

$$T_1(x,y) = X(x).Y(y) \hspace{1cm} (A.6)$$

Substituting Equation (A.6) in Equation (A.1), we have

$$X''Y + Y''X = 0$$
$$\frac{X''}{X} = -\frac{Y''}{Y}$$
$$\frac{X'}{X} = -\frac{Y'}{Y} = -\beta^2$$ \hspace{1cm} (A.7)

**Spatial Y solution**

Now from Equation (A.7) we can write

$$\frac{Y''}{Y} = \beta^2$$
$$Y'' - \beta^2 Y = 0$$ \hspace{1cm} (A.8)

Equation (A.8) is a homogeneous second order differential equation. These type of equations have the solution of the form $Y(y) = e^{my}$. Substituting $Y(y) = e^{my}$ in Equation (A.8) we get,

$$m^2e^{my} - \beta^2e^{my} = 0$$
$$e^{my}(m^2 - \beta^2) = 0$$
$$m^2 - \beta^2 = 0$$

$m_1, m_2 = \beta, -\beta$

From the values of $m_1$ and $m_2$, the solution of $Y(y)$ is written as

$$Y(y) = A \cosh(\beta y) + B \sinh(\beta y) \hspace{1cm} (A.9)$$

**Spatial X solution**

Now from Equation (A.7) we can write

$$\frac{X''}{X} = -\beta^2$$
$$X'' + \beta^2 X = 0$$ \hspace{1cm} (A.10)
Equation (A.10) is a homogeneous second order differential equation. These types of equations have the solution of the form $X(x) = e^{mx}$. Substituting $X(x) = e^{mx}$ in Equation (A.10), we get
\[ m^2 e^{mx} + \beta^2 e^{mx} = 0 \]
\[ e^{mx} (m^2 + \beta^2) = 0 \]
\[ m^2 + \beta^2 = 0 \]
\[ m_1, m_2 = i\beta, -i\beta \]

From the values of $m_1$ and $m_2$, the solution of $X(x)$ is written as
\[ X(x) = C \cos(\beta x) + D \sin(\beta x) \quad \text{(A.11)} \]

Substituting Equation (A.9) and Equation (A.11) in Equation (A.6) gives
\[ T_1(x, y) = \left[ C \cos(\beta y) + D \sin(\beta y) \right] \left[ A \cosh(\beta y) + B \sinh(\beta y) \right] \quad \text{(A.12)} \]

To find the value of the constants we must use the boundary conditions. Applying boundary condition represented by Equation (A.2), we have
\[ 0 = C \left[ A \cosh(\beta y) + B \sinh(\beta y) \right] \]
\[ C = 0 \]

Substituting $C = 0$ in Equation (A.12), we have
\[ T_1(x, y) = D \sin(\beta x) \left[ A \cosh(\beta y) + B \sinh(\beta y) \right] \]
\[ = \sin(\beta x) \left[ A \cosh(\beta y) + B \sinh(\beta y) \right] \quad \text{(A.13)} \]

Applying the boundary condition represented by Equation (A.13), we have
\[ 0 = \sin(2.4\beta) \left[ A \cosh(\beta y) + B \sinh(\beta y) \right] \]
\[ 0 = \sin(2.4\beta) \]
\[ 2.4\beta = n\pi \]
\[ \beta = \frac{n\pi}{2.4} \quad \text{(A.14)} \]

Substituting Equation (A.14) in Equation (A.13)
\[ T_1(x, y) = \sin\left( \frac{n\pi}{2.4} x \right) \left[ A \cosh\left( \frac{n\pi}{2.4} y \right) + B \sinh\left( \frac{n\pi}{2.4} y \right) \right] \]

Since the general solution can have any value of $n$,
\[ T_1(x, y) = \sum_{n=1}^{\infty} \sin\left( \frac{n\pi}{2.4} x \right) \left[ A_n \cosh\left( \frac{n\pi}{2.4} y \right) + B_n \sinh\left( \frac{n\pi}{2.4} y \right) \right] \quad \text{(A.15)} \]

Applying boundary condition represented by Equation (A.4), we have
\[ 50 = \sum_{n=1}^{\infty} A_n \sin\left( \frac{n\pi}{2.4} x \right) \quad \text{(A.16)} \]

A half range sine series is given by
\[ f(x) = \sum_{n=1}^{\infty} A_n \sin\left( \frac{n\pi}{L} x \right) \]

where
\[ A_n = \frac{2}{L} \int_{0}^{L} f(x) \sin\left( \frac{n\pi}{L} x \right) dx \]
Comparing Equation (A.16) with half range sine series, Equation (A.16) is a half-range expression of 50 in sine series with $L = 2.4$. Therefore

$$A_n = \frac{2}{2.4} \int_0^{2.4} 50 \sin\left(\frac{n\pi}{2.4}x\right) dx$$

$$= \frac{1}{1.2} \int_0^{2.4} 50 \sin\left(\frac{n\pi}{2.4}x\right) dx$$

$$= \frac{50}{n\pi} \left[ -\cos\left(\frac{n\pi}{2.4}x\right) \right]_0^{2.4}$$

$$= \frac{50 \times 2.4}{1.2 \times n\pi} [-\cos(n\pi) + 1]$$

$$= \frac{100}{n\pi} [-\cos(n\pi) + 1]$$

$$= \frac{100}{n\pi} [1 - (-1)^n] \quad (A.17)$$

Applying boundary condition represented by Equation (A.5), we have

$$300 = \sum_{n=1}^\infty \sin\left(\frac{n\pi}{2.4}x\right) \left[ A_n \cosh\left(\frac{n\pi}{2.4} 3.0\right) + B_n \sinh\left(\frac{n\pi}{2.4} 3.0\right) \right] \quad (A.18)$$

Solving Equation (A.18) for $B_n$ gives

$$B_n = \frac{1}{\sin\left(\frac{3n\pi}{2.4}\right)} \left\{ \frac{2}{2.4} \int_0^{2.4} 300 \sin\left(\frac{n\pi}{2.4}x\right) dx - A_n \cos\left(\frac{3n\pi}{2.4}\right) \right\}$$

$$= \frac{1}{\sin\left(\frac{3n\pi}{2.4}\right)} \left\{ \frac{600}{2.4} \left[ -\cos\left(\frac{n\pi}{2.4}x\right) \right]_0^{2.4} - A_n \cos\left(\frac{3n\pi}{2.4}\right) \right\}$$

$$= \frac{1}{\sin\left(\frac{3n\pi}{2.4}\right)} \left\{ \frac{600}{n\pi} \left[-\cos(n\pi) + 1\right] - A_n \cos\left(\frac{3n\pi}{2.4}\right) \right\}$$

$$= \frac{1}{\sin\left(\frac{3n\pi}{2.4}\right)} \left\{ \frac{600}{n\pi} [1 - (-1)^n] - A_n \cos\left(\frac{3n\pi}{2.4}\right) \right\} \quad (A.19)$$

From Equations (A.15), (A.17) and (A.19), the solution $T_1$ is given as
\[ T_1(x, y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2.4} x\right) \left[ A_n \cosh\left(\frac{n\pi}{2.4} y\right) + B_n \sinh\left(\frac{n\pi}{2.4} y\right) \right] \] (A.20)

where

\[ A_n = \frac{100}{n\pi} \left[ 1 - (-1)^n \right] \]
\[ B_n = \frac{1}{\sin\left(\frac{3n\pi}{2.4}\right)} \left\{ 600 \frac{1 - (-1)^n}{n\pi} - A_n \cos\left(\frac{3n\pi}{2.4}\right) \right\} \]

**Solution Problem 2**

Let the solution to Problem 2 be \( T_2 \). Problem 2 can be solved similarly as Problem 1. The solution to Problem 2 is

\[ T_2(x, y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{3} y\right) \left[ C_n \cosh\left(\frac{n\pi}{3} x\right) + D_n \sinh\left(\frac{n\pi}{3} x\right) \right] \] (A.21)

where

\[ C_n = \frac{150}{n\pi} \left[ 1 - (-1)^n \right] \]
\[ D_n = \frac{1}{\sin\left(\frac{2.4n\pi}{3}\right)} \left\{ 200 \frac{1 - (-1)^n}{n\pi} - C_n \cos\left(\frac{2.4n\pi}{3}\right) \right\} \]

**Overall Solution**

The overall solution to the problem is

\[ T(x, y) = T_1(x, y) + T_2(x, y) \]
\[ T(x, y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2.4} x\right) \left[ A_n \cosh\left(\frac{n\pi}{2.4} y\right) + B_n \sinh\left(\frac{n\pi}{2.4} y\right) \right] + \]
\[ \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{3} y\right) \left[ C_n \cosh\left(\frac{n\pi}{3} x\right) + D_n \sinh\left(\frac{n\pi}{3} x\right) \right] \]

where

\[ A_n = \frac{100}{n\pi} \left[ 1 - (-1)^n \right], \]
\[ B_n = \frac{1}{\sin\left(\frac{3n\pi}{2.4}\right)} \left\{ 600 \frac{1 - (-1)^n}{n\pi} - A_n \cos\left(\frac{3n\pi}{2.4}\right) \right\}, \]
\[ C_n = \frac{150}{n\pi} \left[ 1 - (-1)^n \right], \]
\[ D_n = \frac{1}{\sin\left(\frac{2.4n\pi}{3}\right)} \left\{ 200 \frac{1 - (-1)^n}{n\pi} - C_n \cos\left(\frac{2.4n\pi}{3}\right) \right\}. \]
<table>
<thead>
<tr>
<th>PARTIAL DIFFERENTIAL EQUATIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Topic</strong></td>
</tr>
<tr>
<td><strong>Summary</strong></td>
</tr>
<tr>
<td><strong>Major</strong></td>
</tr>
<tr>
<td><strong>Authors</strong></td>
</tr>
<tr>
<td><strong>Date</strong></td>
</tr>
<tr>
<td><strong>Web Site</strong></td>
</tr>
</tbody>
</table>