Chapter 11.02
Continuous Fourier Series

For a function with period \( T \), a continuous Fourier series can be expressed as [1-5]

\[
f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kw_0 t) + b_k \sin(kw_0 t)
\]  
(1)

The unknown Fourier coefficients \( a_0, a_k \) and \( b_k \) can be computed as

\[
a_0 = \left( \frac{1}{T} \right) \int_0^T f(t) dt
\]  
(2)

Thus, \( a_0 \) can be interpreted as the “average” function value between the period interval [0, \( T \)].

\[
a_k = \left( \frac{2}{T} \right) \int_0^T f(t) \cos(kw_0 t) dt
\]  
(3)

\[= a_{-k} \] (hence \( a_k \) is an “even” function)

\[
b_k = \left( \frac{2}{T} \right) \int_0^T f(t) \sin(kw_0 t) dt
\]  
(4)

\[= -b_{-k} \] (hence \( b_k \) is an “odd” function)

**Derivation of formulas for \( a_0, a_k \) and \( b_k \)**

Integrating both sides of Equation 1 with respect to time, one gets

\[
\int_0^T f(t) dt = \int_0^T a_0 dt + \int_0^T \sum_{k=1}^{\infty} a_k \cos(kw_0 t) dt + \int_0^T \sum_{k=1}^{\infty} b_k \sin(kw_0 t) dt
\]  
(5)

The second and third terms on the right hand side of the above equations are both zeros, due to the result stated in Equation (1) of Chapter 11.01.

Thus,

\[
\int_0^T f(t) dt = [a_0 t]_0^T
\]  
(6)

\[= a_0 T
\]

Hence,

\[
a_0 = \left( \frac{1}{T} \right) \int_0^T f(t) dt
\]  
(7)
Now, if both sides of Equation (1) are multiplied by \( \sin(mw_0t) \) and then integrated with respect to time, one obtains
\[
\int_0^T f(t) \times \sin(mw_0t) dt = \int_0^T a_0 \sin(mw_0t) dt + \int_0^T \sum_{k=1}^{\infty} a_k \cos(kw_0t) \sin(mw_0t) dt \\
+ \int_0^T \sum_{k=1}^{\infty} b_k \sin(kw_0t) \sin(mw_0t) dt
\]

Due to Equations (1) and (3) of Chapter 11.01, the first and second terms on the right hand side (RHS) of Equation (8) are zero.

Due to Equation (4) of Chapter 11.01, the third RHS term of Equation (8) is also zero, with the exception when \( k = m \), which will become (by referring to Equation (2) of Chapter 11.01)
\[
\int_0^T f(t) \sin(kw_0t) dt = 0 + 0 + \int_0^T b_k \sin^2(kw_0t) dt
\]
\[
= b_k \times \frac{T}{2}
\]

Thus,
\[
b_k = \left(\frac{2}{T}\right) \int_0^T f(t) \sin(kw_0t) dt
\]

Similar derivation can be used to obtain \( a_k \), as shown in Equation (3)

**A FORTRAN Program for finding Fourier Coefficients** \( a_0 \), \( a_k \), and \( b_k \)

Based upon the derived formulas for \( a_0 \), \( a_k \) and \( b_k \) (shown in Equations 2-4), a FORTRAN/MATLAB computer program has been developed. (The program is available at [http://numericalmethods.eng.usf.edu/simulations/mlt/11fft/f_coeff_final.m](http://numericalmethods.eng.usf.edu/simulations/mlt/11fft/f_coeff_final.m))

**Example 1**

Using the continuous Fourier series to approximate the following periodic function \( (T = 2\pi \) seconds) shown in Figure 1.
Figure 1 A Periodic Function (Between 0 and $2\pi$).

$$f(t) = \begin{cases} 
  t & \text{for } 0 < t \leq \pi \\
  \pi & \text{for } \pi \leq t < 2\pi 
\end{cases}$$

Specifically, find the Fourier coefficients $a_0, a_1, \ldots, a_8$ and $b_1, \ldots, b_8$.

**Solution**

The unknown Fourier coefficients $a_0, a_k$, and $b_k$ can be computed based on Equations (2–4); as following:

$$a_0 = \frac{1}{T} \int_0^{2\pi} f(t) \, dt$$

$$a_0 = \frac{1}{2(2\pi)} \times \left\{ \int_0^\pi t \, dt + \int_{\pi}^{2\pi} \pi \, dt \right\}$$

$$a_0 = 2.35619$$

$$a_k = \frac{2}{T} \int_0^{2\pi} f(t) \cos(kt) \, dt$$

$$a_k = \frac{2}{2\pi} \times \left\{ \int_0^{\pi} t \cos \left( \frac{2\pi}{T} \times t \right) \, dt + \int_{\pi}^{2\pi} \pi \cos \left( \frac{2\pi}{T} \times t \right) \, dt \right\}$$

$$a_k = \left( \frac{1}{\pi} \right) \times \left\{ \int_0^{\pi} t \cos(kt) \, dt + \int_{\pi}^{2\pi} \pi \cos(kt) \, dt \right\}$$

The “integration by part” formula can be utilized to compute the first integral on the right-hand-side of the above equation.

For $k = 1, 2, \ldots, 8$, the Fourier coefficients $a_k$ can be computed as

$$a_1 = -0.6366257003116296$$

$$a_2 = -5.070352857678721 \times 10^{-6} \approx 0$$

$$a_3 = -0.07074100153210318$$
Similarly,
\[
b_k = \left( \frac{2}{T} \right) \int_0^{2\pi} f(t) \sin(kw_0 t) dt
\]
\[
b_k = \left( \frac{1}{\pi} \right) \times \left\{ \int_0^{\pi} t \sin(kt) dt + \int_0^{2\pi} \pi \sin(kt) dt \right\}
\]

For \( k = 1, 2, \ldots, 8 \), the Fourier coefficients \( b_k \) can be computed as
\[
b_1 = -0.9999986528958207
\]
\[
b_2 = -0.4999993232285269
\]
\[
b_3 = -0.333314439509194
\]
\[
b_4 = -0.24999804122384547
\]
\[
b_5 = -0.19999713794872364
\]
\[
b_6 = -0.1666635603759553
\]
\[
b_7 = -0.14285324664625462
\]
\[
b_8 = -0.12499577981019251
\]

Any periodic function \( f(t) \), such as the one shown in Figure 1 can be represented by the Fourier series as
\[
f(t) = a_0 + \sum_{k=1}^{\infty} \left\{ a_k \cos(kw_0 t) + b_k \sin(kw_0 t) \right\}
\]

where \( a_0 \), \( a_k \) and \( b_k \) have already been computed (for \( k = 1, 2, \ldots, 8 \)); and \( w_0 = 2\pi \)
\[
= \frac{2\pi}{T}
\]
\[
= \frac{2\pi}{2\pi}
\]
\[
= 1
\]

Thus, for \( k = 1 \), one obtains
\[
\tilde{f}_1(t) \approx a_0 + a_1 \cos(t) + b_1 \sin(t)
\]

For \( k = 1 \rightarrow 2 \), one obtains
\[
\tilde{f}_2(t) \approx a_0 + a_1 \cos(t) + b_1 \sin(t) + a_2 \cos(2t) + b_2 \sin(2t)
\]

For \( k = 1 \rightarrow 4 \), one obtains
\[
\tilde{f}_4(t) \approx a_0 + a_1 \cos(t) + b_1 \sin(t) + a_2 \cos(2t) + b_2 \sin(2t) + a_3 \cos(3t) + b_3 \sin(3t) + a_4 \cos(4t) + b_4 \sin(4t)
\]
Plots for $\tilde{f}_1(t)$, $\tilde{f}_2(t)$ and $\tilde{f}_4(t)$ are shown in Figure 2.

![Figure 2 Fourier Approximated Functions (for Example 1).](image)

It can be observed from Figure 2 that as more terms are included in the Fourier series, the approximated Fourier functions are more closely resemble the original periodic function as shown in Figure 1.

**Example 2**

The periodic triangular wave function $f(t)$ is defined as

$$f(t) = \begin{cases} 
-\frac{\pi}{2} & \text{for } -\pi < t < -\frac{\pi}{2} \\
-t & \text{for } -\frac{\pi}{2} < t < \frac{\pi}{2} \\
-\frac{\pi}{2} & \text{for } \frac{\pi}{2} < t < \pi 
\end{cases}$$

Find the Fourier coefficients $a_0, a_1, \ldots, a_8$ and $b_1, \ldots, b_8$ and approximate the periodic triangular wave function by the Fourier series.
Solution

The unknown Fourier Coefficients $a_0$, $a_k$ and $b_k$ can be computed based on Equations (2-4) as follows

$$a_0 = \left(\frac{1}{T}\right) \int_{-\pi}^{\pi} f(t)dt$$

$$a_0 = \frac{1}{2\pi} \times \left\{ \int_{-\pi}^{-\pi/2} \frac{\pi}{2} dt + \int_{-\pi/2}^{\pi/2} (-t)dt + \int_{\pi/2}^{\pi} \frac{\pi}{2} dt \right\}$$

$$a_0 = -0.78539753$$

$$a_k = \left(\frac{2}{T}\right) \int_{-\pi}^{\pi} f(t) \cos(kw_0t) dt$$

where

$$w_0 = \frac{2\pi}{T}$$

$$= \frac{2\pi}{2\pi}$$

$$= 1$$

Hence,

$$a_k = \left(\frac{2}{T}\right) \int_{-\pi}^{\pi} f(t) \cos(kt) dt$$
or

\[ a_k = \left( \frac{2}{2\pi} \right) \left\{ \int_{-\pi/2}^{\pi/2} \left( -\frac{\pi}{2} \right) \cos(kt)dt + \int_{-\pi/2}^{\pi/2} (-t)\cos(kt)dt + \int_{-\pi/2}^{\pi/2} \left( -\frac{\pi}{2} \right) \cos(kt)dt \right\} \]

Similarly,

\[ b_k = \left( \frac{2}{T} \right) \int_{-\pi}^{\pi} f(t) \sin(kw_0 t)dt = \left( \frac{2}{T} \right) \int_{-\pi}^{\pi} f(t) \sin(kt)dt \]

or,

\[ b_k = \left( \frac{2}{2\pi} \right) \left\{ \int_{-\pi/2}^{\pi/2} \left( -\frac{\pi}{2} \right) \sin(kt)dt + \int_{-\pi/2}^{\pi/2} (-t)\sin(kt)dt + \int_{-\pi/2}^{\pi/2} \left( -\frac{\pi}{2} \right) \sin(kt)dt \right\} \]

The “integration by part” formula can be utilized to compute the second integral on the right-hand-side of the above equations for \( a_k \) and \( b_k \).

For \( k = 1, 2, \ldots, 8 \), the Fourier coefficients \( a_k \) and \( b_k \) can be computed and summarized as following in Table 1

**Table 1** Fourier coefficients \( a_k \) and \( b_k \) for various \( k \) values.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( a_k )</th>
<th>( b_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.999997</td>
<td>-0.63661936</td>
</tr>
<tr>
<td>2</td>
<td>0.00</td>
<td>-0.49999932</td>
</tr>
<tr>
<td>3</td>
<td>-0.3333355</td>
<td>0.07073466</td>
</tr>
<tr>
<td>4</td>
<td>0.00</td>
<td>0.2499980</td>
</tr>
<tr>
<td>5</td>
<td>0.1999968</td>
<td>-0.02546389</td>
</tr>
<tr>
<td>6</td>
<td>0.00</td>
<td>-0.16666356</td>
</tr>
<tr>
<td>7</td>
<td>-0.14285873</td>
<td>0.0126991327</td>
</tr>
<tr>
<td>8</td>
<td>0.00</td>
<td>0.12499578</td>
</tr>
</tbody>
</table>

The periodic function (shown in Example 1) can be approximated by Fourier series as

\[ f(t) = a_0 + \sum_{k=1}^{\infty} \left\{ a_k \cos(kt) + b_k \sin(kt) \right\} \]

Thus, for \( k = 1 \), one obtains:

\[ \tilde{f}_1(t) = a_0 + a_1 \cos(t) + b_1 \sin(t) \]

For \( k = 1 \rightarrow 2 \), one obtains:

\[ \tilde{f}_2(t) = a_0 + a_1 \cos(t) + b_1 \sin(t) + a_2 \cos(2t) + b_2 \sin(2t) \]

Similarly, for \( k = 1 \rightarrow 4 \), one has:

\[ \tilde{f}_4(t) = a_0 + a_1 \cos(t) + b_1 \sin(t) + a_2 \cos(2t) + b_2 \sin(2t) + a_3 \cos(3t) + b_3 \sin(3t) + a_4 \cos(4t) + b_4 \sin(4t) \]

Plots for functions \( \tilde{f}_1(t) \), \( \tilde{f}_2(t) \) and \( \tilde{f}_4(t) \) are shown in Figure 4.
It can be observed from Figure 4 that as more terms are included in the Fourier series, the approximated Fourier functions closely resemble the original periodic function.

**Complex Form of the Fourier Series**

Using Euler’s identity, \( e^{ix} = \cos(x) + i\sin(x) \), and \( e^{-ix} = \cos(x) - i\sin(x) \), the sine and cosine can be expressed in the exponential form as

\[
\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} = \text{"odd" function, since } \sin(x) = -\sin(-x) \quad (10)
\]

\[
\cos(x) = \frac{e^{ix} + e^{-ix}}{2} = \text{"even" function, since } \cos(x) = \cos(-x) \quad (11)
\]

Thus, the Fourier series (expressed in Equation 1) can be converted into the following form

\[
f(t) = a_0 + \sum_{k=1}^{\infty} a_k \left( \frac{e^{ik\omega_d t} + e^{-ik\omega_d t}}{2} \right) + b_k \left( \frac{e^{ik\omega_d t} - e^{-ik\omega_d t}}{2i} \right) \quad (12)
\]

or

\[
f(t) = a_0 + \sum_{k=1}^{\infty} e^{ik\omega_d t} \left( \frac{a_k}{2} + \frac{b_k}{2i} \right) + e^{-ik\omega_d t} \left( \frac{a_k}{2} - \frac{b_k}{2i} \right) \quad (13)
\]

or, since \( i^2 = -1 \), one obtains

\[
f(t) = a_0 + \sum_{k=1}^{\infty} e^{ik\omega_d t} \left( \frac{a_k - ib_k}{2} \right) + e^{-ik\omega_d t} \left( \frac{a_k + ib_k}{2} \right) \quad (13)
\]

Define the following constants

\[
\tilde{C}_0 \equiv a_0 \quad (14)
\]
\[ \tilde{C}_k \equiv \frac{a_k - ib_k}{2} \]  \hspace{1cm} (15)

Hence:
\[ \tilde{C}_{-k} \equiv \frac{a_{-k} - ib_{-k}}{2} \]  \hspace{1cm} (16)

Using the even and odd properties shown in Equations (3) and (4) respectively, Equation (16) becomes
\[ \tilde{C}_{-k} \equiv \frac{a_k + ib_k}{2} \]  \hspace{1cm} (17)

Substituting Equations (14), (15), (17) into Equation (13), one gets
\[
\begin{align*}
\sum_{\infty}^{\infty} \sum_{\infty}^{\infty} & = \sum_{k=1}^{\infty} \sum_{k=1}^{\infty} \tilde{C}_k e^{ikw_0 t} + \sum_{k=1}^{\infty} \tilde{C}_{-k} e^{-ikw_0 t} \\
& = \sum_{k=0}^{\infty} \tilde{C}_k e^{ikw_0 t} + \sum_{k=1}^{\infty} \tilde{C}_k e^{ikw_0 t} \\
& = \sum_{k=0}^{\infty} \tilde{C}_k e^{ikw_0 t} + \sum_{k=-\infty}^{-1} \tilde{C}_k e^{ikw_0 t} \\
& = \sum_{k=0}^{\infty} \tilde{C}_k e^{ikw_0 t} \\
& = \sum_{k=-\infty}^{\infty} \tilde{C}_k e^{ikw_0 t} \\
& = \sum_{k=0}^{\infty} \tilde{C}_k e^{ikw_0 t} \hspace{1cm} (18)
\end{align*}
\]

The coefficient \( \tilde{C}_k \) can be computed, by substituting Equations (3) and (4) into Equation (15) to obtain
\[
\begin{align*}
\tilde{C}_k & = \left( \frac{1}{2} \right) \left( \frac{2}{T} \right) \left\{ \int_{0}^{T} f(t) \cos(kw_0 t) dt - i \int_{0}^{T} f(t) \sin(kw_0 t) dt \right\} \\
& = \left( \frac{1}{T} \right) \left\{ \int_{0}^{T} f(t) \times \left[ \cos(kw_0 t) - i \sin(kw_0 t) \right] dt \right\} \\
& = \left( \frac{1}{T} \right) \left\{ \int_{0}^{T} f(t) \times \left[ \frac{e^{ikw_0 t} + e^{-ikw_0 t}}{2} - i \times \frac{e^{ikw_0 t} - e^{-ikw_0 t}}{2i} \right] dt \right\} \\
& = \left( \frac{1}{T} \right) \left\{ \int_{0}^{T} f(t) \times e^{-ikw_0 t} dt \right\} \hspace{1cm} (19)
\end{align*}
\]

Substituting Equations (10, 11) into the above equation, one gets
\[
\begin{align*}
\tilde{C}_k & = \left( \frac{1}{T} \right) \left\{ \int_{0}^{T} f(t) \times \left[ \frac{e^{ikw_0 t} + e^{-ikw_0 t}}{2} - i \times \frac{e^{ikw_0 t} - e^{-ikw_0 t}}{2i} \right] dt \right\} \\
& = \left( \frac{1}{T} \right) \left\{ \int_{0}^{T} f(t) \times e^{-ikw_0 t} dt \right\} \hspace{1cm} (20)
\end{align*}
\]

Thus, Equations (18) and (20) are the equivalent complex version of Equations (1)-(4).

**References**


<table>
<thead>
<tr>
<th>FAST FOURIER TRANSFORM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Topic</td>
</tr>
<tr>
<td>Summary</td>
</tr>
<tr>
<td>Major</td>
</tr>
<tr>
<td>Authors</td>
</tr>
<tr>
<td>Date</td>
</tr>
<tr>
<td>Web Site</td>
</tr>
</tbody>
</table>