Chapter 11.01
Introduction to Fourier Series

In general, curve fitting interpolation through a set of data points can be done by a linear combination of polynomial functions, with based functions $1, x, x^2, \ldots\ldots, x^m$. In this chapter, however, trigonometric functions such as $1, \cos(x), \cos(2x), \ldots\ldots \cos(nx)$, and $\sin(x), \sin(2x), \ldots\ldots, \sin(nx)$ will be used as based functions. In the former, the unknown coefficients of based functions can be found by solving the associated linear simultaneous equations (where the number of unknown coefficients will be matched with the same number of equations, provided by a set of given data points). In the latter, however, the unknown coefficients can be efficiently solved (by exploiting special properties of trigonometric functions) without requiring solving the expensive simultaneous linear equations (more details will be explained in Equation 6 of Chapter 11.05).

Introduction

The following relationships can be readily established, and will be used in subsequent sections for derivation of useful formulas for the unknown Fourier coefficients, in both time and frequency domains.

$$\int_0^T \sin(kw_0 t)dt = \int_0^T \cos(kw_0 t)dt \qquad (1)$$
$$= 0$$

$$\int_0^T \sin^2(kw_0 t)dt = \int_0^T \cos^2(kw_0 t)dt \qquad (2)$$
$$= \frac{T}{2}$$

$$\int_0^T \cos(kw_0 t)\sin(gw_0 t)dt = 0 \qquad (3)$$

$$\int_0^T \sin(kw_0 t)\sin(gw_0 t)dt = 0 \qquad (4)$$

$$\int_0^T \cos(kw_0 t)\cos(gw_0 t)dt = 0 \qquad (5)$$

where

$$w_0 = 2\pi f \tag{6}$$

$$f = \frac{1}{T} \tag{7}$$

where f and T represents the frequency (in cycles/time) and period (in seconds) respectively. Also, k and g are integers.

A periodic function $f(t)$ with a period T should satisfy the following equation

$$f(t+T) = f(t) \tag{8}$$

Example 1

Prove that

$$\int_0^{\pi} \sin(kw_0 t) = 0$$

for

$$w_0 = 2\pi f$$

$$f = \frac{1}{T}$$

and k is an integer.

Solution

Let

$$A = \int_0^T \sin(kw_0 t)dt \tag{9}$$

$$= -\left(\frac{1}{kw_0}\right)\left[\cos(kw_0 t)\right]_0^T$$

$$A = \left(\frac{-1}{kw_0}\right)\left[\cos(kw_0 T) - \cos(0)\right] \tag{10}$$

$$= \left(\frac{-1}{kw_0}\right)\left[\cos(k2\pi) - 1\right]$$

$$= 0$$

Example 2

Prove that

$$\int_0^{\pi} \sin^2(kw_0 t) = \frac{T}{2}$$

for

$$w_0 = 2\pi f$$

$$f = \frac{1}{T}$$

and k is an integer.

Solution

Let

$$B = \int_0^T \sin^2(kw_0 t)dt \tag{11}$$

Recall

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2} \tag{12}$$

Thus,

$$B = \int_o^T \left[\frac{1}{2} - \frac{1}{2}\cos(2kw_0 t)\right]dt \tag{13}$$

$$= \left[\left(\frac{1}{2}\right)t - \left(\frac{1}{2}\right)\left(\frac{1}{2kw_0}\right)\sin(2kw_0 t)\right]_0^T$$

$$B = \left[\frac{T}{2} - \frac{1}{4kw_0}\sin(2kw_0 T)\right] - [0] \tag{14}$$

$$= \frac{T}{2} - \left(\frac{1}{4kw_0}\right)\sin(2k * 2\pi)$$

$$= \frac{T}{2}$$

Example 3

Prove that

$$\int_0^\pi \sin(gw_0 t)\cos(kw_0 t) = 0$$

for

$$w_0 = 2\pi f$$

$$f = \frac{1}{T}$$

and k and g are integers.

Solution

Let

$$C = \int_0^T \sin(gw_0 t)\cos(kw_0 t)dt \tag{15}$$

Recall that

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha) \tag{16}$$

Hence,

$$C=\int_0^T \left[\sin\left[(g+k)w_0t\right]-\sin(kw_0t)\cos(gw_0t)\right]dt \tag{17}$$

$$=\int_0^T \sin\left[(g+k)w_0t\right]dt-\int_0^T \sin(kw_0t)\cos(gw_0t)dt \tag{18}$$

From Equation (1),

$$\int_0^T [\sin(g+k)w_0t]dt=0$$

then

$$C=0-\int_0^T \sin(kw_0t)\cos(gw_0t)dt \tag{19}$$

Adding Equations (15), (19),

$$2C=\int_0^T \sin(gw_0t)\cos(kw_0t)dt-\int_0^T \sin(kw_0t)\cos(gw_0t)dt$$

$$=\int_0^T \sin\left[(gw_0t)-(kw_0t)\right]dt=\int_0^T \sin\left[(g-k)w_0t\right]dt \tag{20}$$

$2C=0$, since the right side of the above equation is zero (see Equation 1). Thus,

$$C=\int_o^T \sin(gw_0t)\cos(kw_0t)dt=0 \tag{21}$$

$$=0$$

Example 4

Prove that

$$\int_0^T \sin(kw_0t)\sin(gw_0t)dt=0$$

for

$$w_0=2\pi f$$

$$f=\frac{1}{T}$$

$$k,g=\text{integers}$$

Solution

$$\text{Let } D=\int_0^T \sin(kw_0t)\sin(gw_0t)dt \tag{22}$$

Since

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$$

or

$$\sin(\alpha)\sin(\beta)=\cos(\alpha)\cos(\beta)-\cos(\alpha+\beta)$$

Thus,

$$D=\int_0^T \cos(kw_0t)\cos(gw_0t)dt-\int_0^T \cos[(k+g)w_0t]dt \tag{23}$$

From Equation (1)

$$\int_0^T \cos[(k+g)w_0t]dt=0$$

then

$$D=\int_0^T \cos(kw_0t)\cos(gw_0t)dt-0 \tag{24}$$

Adding Equations (23), (26)

$$2D=\int_0^T \sin(kw_0t)\sin(gw_0t)+\int_0^T \cos(kw_0t)\cos(gw_0t)dt$$

$$=\int_0^T \cos[kw_0t-gw_0t]dt \tag{25}$$

$$=\int_0^T \cos[(k-g)w_0t]dt$$

2D = 0, since the right side of the above equation is zero (see Equation 1). Thus,

$$D\equiv\int_0^T \sin(kw_0t)\sin(gw_0t)dt=0 \tag{26}$$

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