Discrete Fourier Transform (DFT)

Major: All Engineering Majors

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Numerical Methods for STEM undergraduates

Discrete Fourier Transform

Recalled the exponential form of Fourier series (see Eqs. 26, 28 in Ch. 11.01), one gets:

$$f(t) = \sum_{k=-\infty}^{\infty} \widetilde{C}_k e^{ikw_0 t} \qquad \text{(26, repeated)}$$

$$\widetilde{C}_k = \left(\frac{1}{T}\right)\left\{\int_0^T f(t) \times e^{-ikw_0 t} dt\right\} \qquad \text{(28, repeated)}$$

If time " t " is discretized at $t_1 = \Delta t, t_2 = 2\Delta t, t_3 = 3\Delta t, \ldots\ldots, t_n = n\Delta t,$ then Eq. (26) becomes:

$$f(t_n) = \sum_{k=0}^{N-1} \widetilde{C}_k e^{ikw_0 t_n} \qquad (1)$$

Discrete Fourier Transform cont.

To simplify the notation, define:

$$t_n = n \tag{2}$$

Then, Eq. (2) can be written as:

$$f(n) = \sum_{k=0}^{N-1} \tilde{C}_k e^{ikw_0 n} \tag{3}$$

Multiplying both sides of Eq. (3) by $e^{-ilw_0 n}$, and performing the summation on "n", one obtains (note: *l* = integer number)

$$\sum_{n=0}^{N-1} f(n) \times e^{-ilw_0 n} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} \tilde{C}_k e^{ikw_0 n} \times e^{-ilw_0 n} \tag{4}$$

$$= \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} \tilde{C}_k e^{i(k-l)\frac{2\pi}{N} n} \tag{5}$$

Discrete Fourier Transform cont.

Switching the order of summations on the right-hand-side of Eq.(5), one obtains:

$$\sum_{n=0}^{N-1} f(n) \times e^{-il\left(\frac{2\pi}{N}\right)n} = \sum_{k=0}^{N-1} \tilde{C}_k \sum_{n=0}^{N-1} e^{i(k-l)\left(\frac{2\pi}{N}\right)n} \qquad (6)$$

Define:

$$A = \sum_{n=0}^{N-1} e^{i(k-l)\left(\frac{2\pi}{N}\right)n} \qquad (7)$$

There are 2 possibilities for $(k-l)$ to be considered in Eq. (7)

Discrete Fourier Transform—Case 1

<u>Case(1)</u>: $(k-l)$ is a multiple integer of N, such as: $(k-l)=mN$; or $k=\pm mN$ where $m=0,\pm1,\pm2,......$

Thus, Eq. (7) becomes:

$$A=\sum_{n=0}^{N-1}e^{im2\pi n}=\sum_{n=0}^{N-1}\cos(mn2\pi)+i\sin(mn2\pi) \qquad (8)$$

Hence:

$$A=N \qquad (9)$$

Discrete Fourier Transform—Case 2

Case(2): $(k-l)$ is NOT a multiple integer of N. In this case, from Eq. (7) one has:

$$A=\sum_{n=0}^{N-1}\left\{e^{i(k-l)\left(\frac{2\pi}{N}\right)}\right\}^{n} \qquad (10)$$

Define:

$$a=e^{i(k-l)\frac{2\pi}{N}}=\cos\left\{(k-l)\frac{2\pi}{N})\right\}+i\sin\left\{(k-l)\frac{2\pi}{N})\right\} \qquad (11)$$

$a\neq 1$; because $(k-l)$ is “NOT” a multiple integer of N

Then, Eq. (10) can be expressed as:

$$A=\sum_{n=0}^{N-1}\{a\}^{n} \qquad (12)$$

Discrete Fourier Transform—Case 2

From mathematical handbooks, the right side of Eq. (12) represents the “geometric series”, and can be expressed as:

$$A = \sum_{n=0}^{N-1} \{a\}^n = N; \text{ if } a = 1 \quad (13)$$

$$= \frac{1-a^N}{1-a}; \text{ if } a \neq 1 \quad (14)$$

Because of Eq. (11), hence Eq. (14) should be used to compute A. Thus:

$$A = \frac{1-a^N}{1-a} = \frac{1-e^{i(k-l)2\pi}}{1-a} \quad \text{(See Eq. (10))} \quad (15)$$

$$e^{i(k-l)2\pi} \equiv \cos\{(k-l)2\pi\} + i\sin\{(k-l)2\pi\} = 1 \quad (16)$$

Discrete Fourier Transform—Case 2

Substituting Eq. (16) into Eq. (15), one gets

$$A = 0 \qquad (17)$$

Thus, combining the results of case 1 and case 2, we get

$$A = N + 0 = N \qquad (18)$$

Substituting Eq.(18) into Eq.(7), and then referring to Eq.(6), one gets:

$$\sum_{n=0}^{N-1} f(n) e^{-ilw_0 n} = \sum_{k=0}^{N-1} \tilde{C}_k \times N \qquad (18A)$$

Recall $k = l + mN$ (where l, m are integer numbers), and since k must be in the range $0 \rightarrow N-1$, m=0. Thus:

$k = l + mN$ becomes $k = l$

Discrete Fourier Transform—Case 2

Eq. (18A) can, therefore, be simplified to

$$\sum_{n=0}^{N-1} f(n)e^{-ilw_0 n} = \tilde{C}_l \times N \qquad (18B)$$

Thus:

$$\tilde{C}_k = \left(\frac{1}{N}\right)\sum_{n=0}^{N-1} f(n)e^{-ikw_0 n} = \left(\frac{1}{N}\right)\sum_{n=0}^{N-1} f(n)\left\{\cos(lw_0 n) - i\sin(lw_0 n)\right\} \qquad (19)$$

where $n \equiv t_n$ and

$$f(n) = \sum_{k=0}^{N-1} \tilde{C}_k\, e^{ikw_0 n} = \sum_{k=0}^{N-1} \tilde{C}_k \left\{\cos(kw_0 n) + i\sin(kw_0 n)\right\} \qquad (1, \text{ repeated})$$

Aliasing Phenomenon, Nyquist samples, Nyquist rate

When a function $f(t)$, which may represent the signals from some real-life phenomenon (shown in Figure 1), is sampled, it basically converts that function into a sequence $\tilde{f}(k)$ at discrete locations of t.

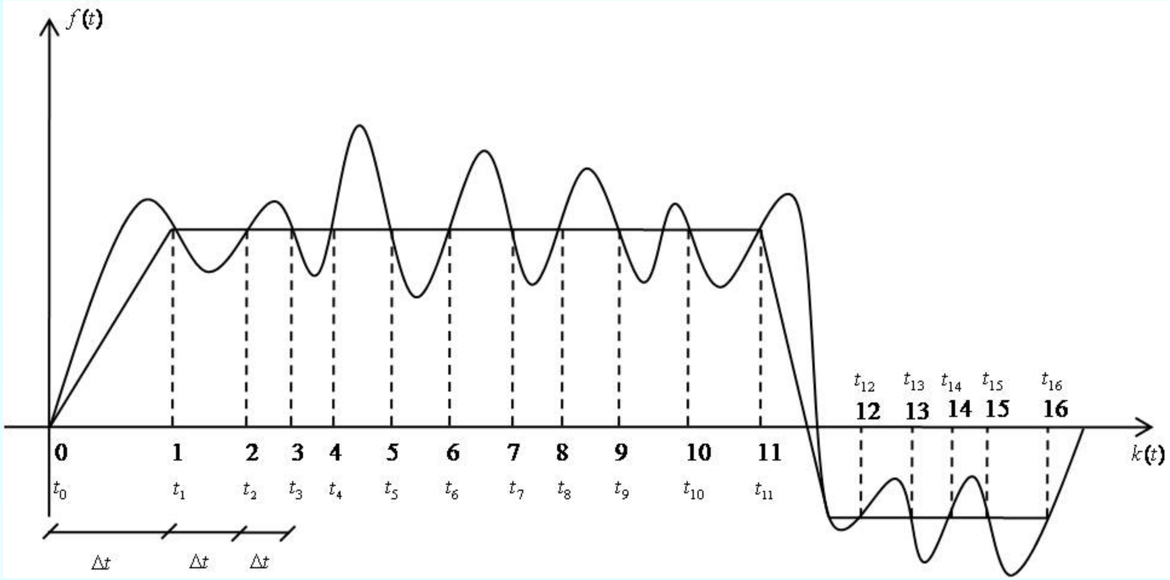

Figure 1 Function to be sampled and "Aliased" sample problem.

Aliasing Phenomenon, Nyquist samples, Nyquist rate cont.

Thus, $\tilde{f}(k)$ represents the value of $f(t)\ at\ t = t_0 + k\Delta t,$ where t_0 is the location of the first sample $(at\ k = 0)$.

In Figure 1, the samples have been taken with a fairly large Δt. Thus, these sequence of discrete data will not be able to recover the original signal function $f(t)$.

For example, if all discrete values of $f(t)$ were connected by piecewise linear fashion, then a nearly horizontal straight line will occur between t_1 through t_8 and t_9 through t_{12} respectively (See Figure 1).

Aliasing Phenomenon, Nyquist samples, Nyquist rate cont.

These piecewise linear interpolation (or other interpolation schemes) will NOT produce a curve which closely resembles the original function $f(t)$. This is the case where the data has been "ALIASED".

“Windowing” phenomenon

Another potential difficulty in sampling the function is called “windowing” problem. As indicated in Figure 2, while Δt is small enough so that a piecewise linear interpolation for connecting these discrete values will adequately resemble the original function $f(t)$, however, only a portion of the function has been sampled (from t_0 through t_{17}) rather than the entire one. In other words, one has placed a “window” over the function.

“Windowing” phenomenon cont.

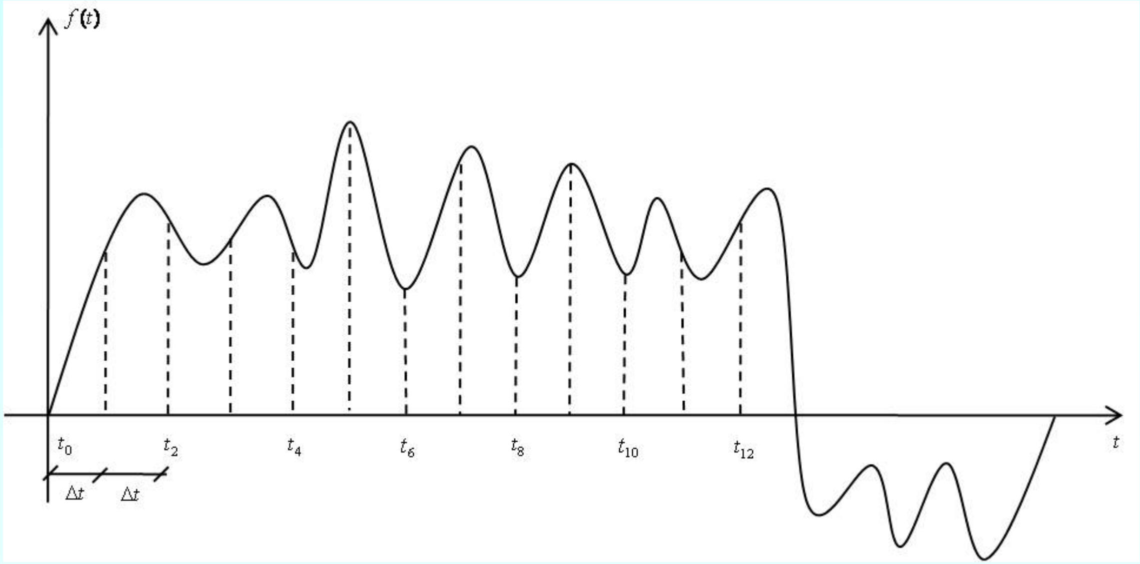

Figure 2. Function to be sampled and “windowing” sample problem.

“Windowing” phenomenon cont.

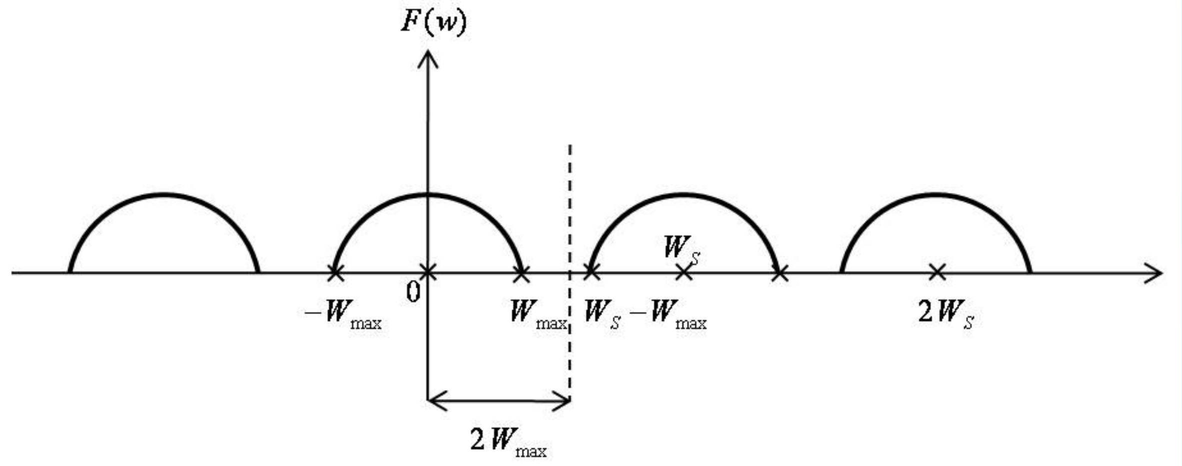

Figure 3. Frequency of sampling rate (w_S) versus maximum frequency content (w_{max}).

In order to satisfy $F(w)=0 \;\; for \; |w| \geq w_{max}$ the frequency (w) should be between points A and B of Figure 3.

"Windowing" phenomenon cont.

Hence:

$$w_{\max} \le w \le w_s - w_{\max}$$

which implies:

$$w_s \ge 2w_{\max}$$

Physically, the above equation states that one must have at least 2 samples per cycle of the highest frequency component present (Nyquist samples, Nyquist rate).

“Windowing” phenomenon cont.

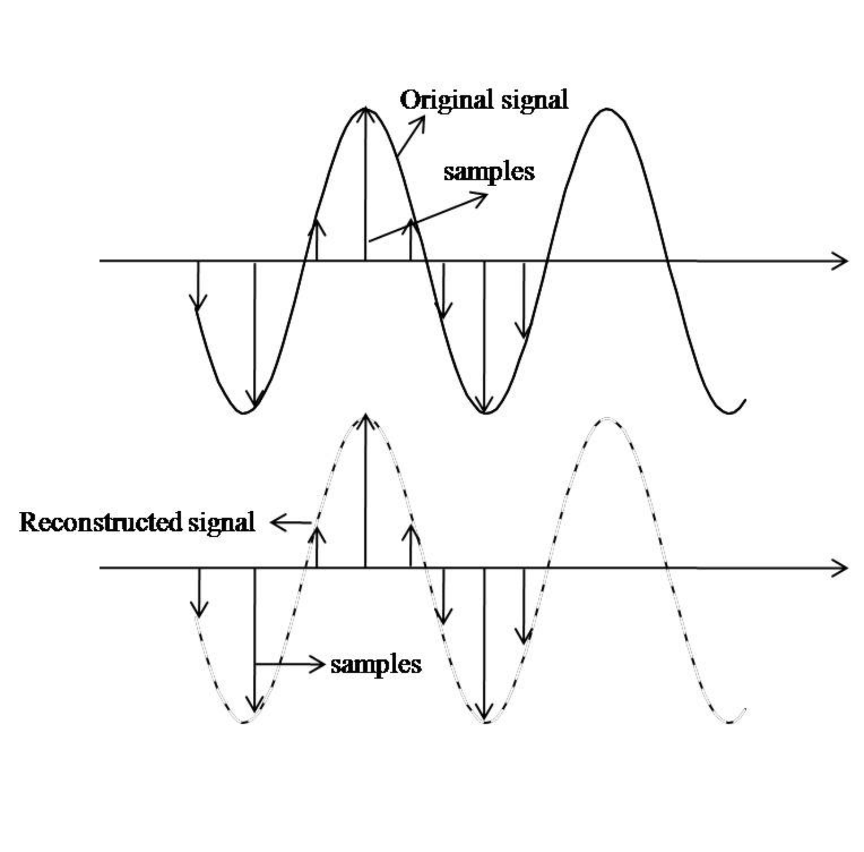

Figure 4. Correctly reconstructed signal.

"Windowing" phenomenon cont.

In Figure 4, a sinusoidal signal is sampled at the rate of 6 samples per 1 cycle (or $w_s = 6w_0$). Since this sampling rate does satisfy the sampling theorem requirement of $\left(w_s \geq 2w_{\max}\right)$, the reconstructed signal does correctly represent the original signal.

"Windowing" phenomenon cont.

In Figure 5 a sinusoidal signal is sampled at the rate of 6 samples per 4 cycles $\left(or\ w_s = \frac{6}{4} w_0\right)$

Since this sampling rate does NOT satisfy the requirement $\left(w_s \geq 2w_{\max}\right)$, the reconstructed signal was wrongly represent the original signal!

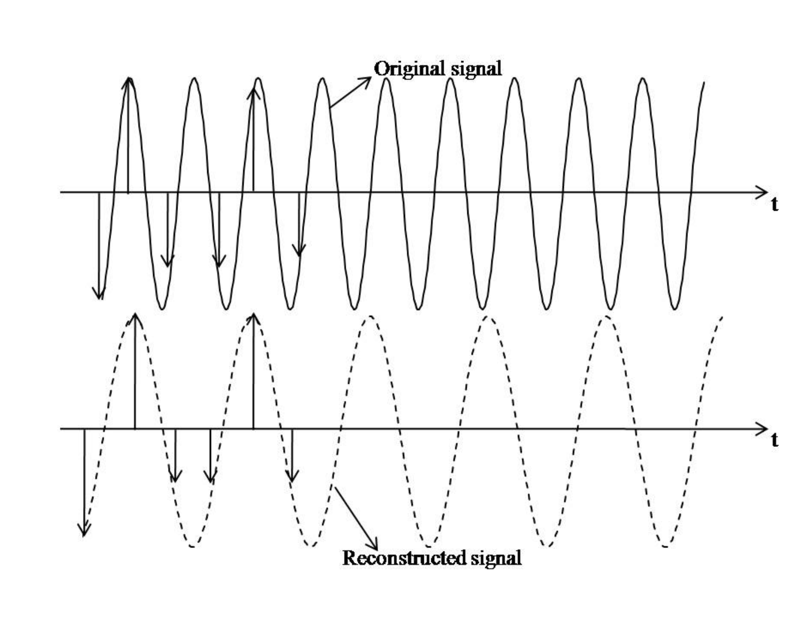

Figure 5. Wrongly reconstructed signal.

Discrete Fourier Transform cont.

Equations (19) and (1) can be rewritten as

$$\tilde{C}_n = \sum_{k=0}^{N-1} f(k) e^{-ik\left(w_0 = \frac{2\pi}{N}\right)n} \quad (20)$$

$$f(k) = \left(\frac{1}{N}\right) \sum_{n=0}^{N-1} \tilde{C}_n e^{ik\left(w_0 = \frac{2\pi}{N}\right)n} \quad (21)$$

To avoid computation with "complex numbers", Equation (20) can be expressed as

$$\tilde{C}_n^R + i\tilde{C}_n^I = \sum_{k=0}^{N-1} \left\{ f^R(k) + i \quad f^I(k) \right\} \times \left\{ \cos(\theta) - i \sin(\theta) \right\} \quad (20A)$$

where

$$\theta = k\left(w_0 = \frac{2\pi}{N}\right)n$$

Discrete Fourier Transform cont.

$$\tilde{C}_n^R + i\tilde{C}_n^I = \sum_{k=0}^{N-1} \left\{ f^R(k) \times \cos(\theta) + f^I(k)\sin(\theta) \right\} + i\left\{ f^I(k)\cos(\theta) - f^R(k)\sin(\theta) \right\} \tag{20B}$$

The above “complex number” equation is equivalent to the following 2 “real number” equations:

$$\tilde{C}_n^R = \sum_{k=0}^{N-1} \left\{ f^R(k)\cos(\theta) + f^I(k)\sin(\theta) \right\} \tag{20C}$$

$$\tilde{C}_n^I = \sum_{k=0}^{N-1} \left\{ f^I(k)\cos(\theta) - f^R(k)\sin(\theta) \right\} \tag{20D}$$