

Chapter 07.00F

Physical Problem for Integration Industrial Engineering

Problem Statement

An industrial engineer works as a quality control engineer for a company making toilet paper. The company advertises that every roll of toilet paper has at least 250 sheets. It is the industrial (quality) engineer's responsibility to validate this claim by sampling rolls off of the assembly line and determining the probability (that is, confidence) that the company can make the claim. Note that validating the claim is important to avoid frivolous lawsuits associated with false advertising.

Background: Let us assume that the number of sheets in a roll of toilet paper, y , is governed by the normal probability distribution (see figure on next page), that is,

$$y \sim N(\mu, \sigma^2)$$

where N is a normal random variable described by its mean, μ , and standard deviation, σ . The probability distribution function of the normal random variable y is

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)[(y-\mu)/\sigma]^2}$$

and is shown in the Figure 1.

Note that the previously described normal probability density function was first derived by Abraham de Moivre in 1733 in his book called *The Doctrine of Chance*. Initially, the discovery went unnoticed until it was re-derived by Laplace and Gauss about 50 years. Hence, it is often called the Gaussian distribution. The normal probability distribution, which is a continuous distribution, is derived from the binomial probability distribution, which is a discrete distribution. The discrete binomial distribution is given below

$$f(y) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

where n is the number of samples and k is the number of successes with a probability of p . When n is large the binomial distribution approaches the normal distribution. In fact, when n is greater than or equal to 6, the normal distribution is an excellent approximation to the binomial distribution (Figure 2).

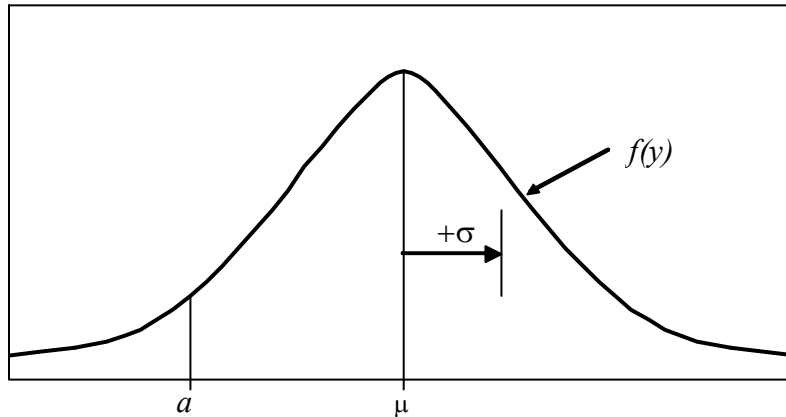


Figure 1 Probability distribution function, $f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)[(y-\mu)/\sigma]^2}$

The actual proof that the binomial distribution approaches the normal distribution involves a special case of the central limit theorem and is beyond the scope of this problem. Furthermore, the derivation also involves the inclusion of a continuity correction factor.

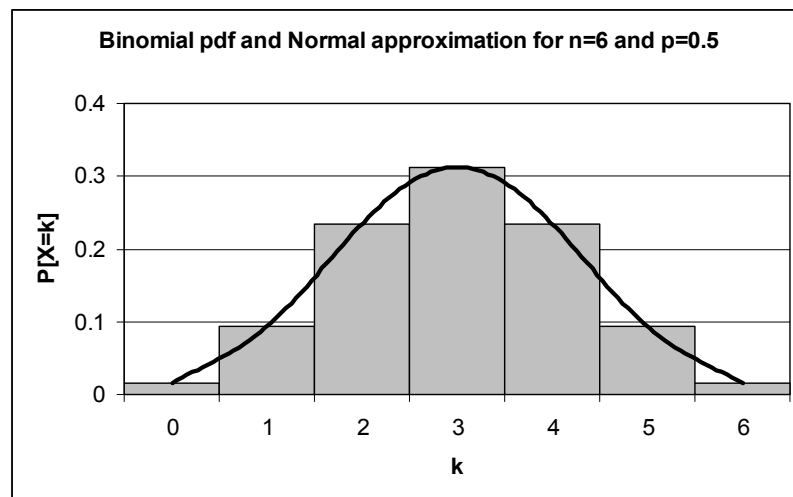


Figure 2 Binomial pdf and normal approximation for $n = 6$ and $p = 0.5$

Since the quality engineer cannot sample every roll of toilet paper and count the number of sheets, the mean and standard deviation are estimated based on a sample size of n rolls of toilet paper. The mean, \bar{y} , and standard deviation, s , of the sample (that is, sample mean and sample standard deviation) are estimates of the actual mean, μ , and standard deviation, σ , and are given by

$$\bar{y} = \frac{\sum_{i=1}^n y_i}{n} \approx \mu$$

$$s = \sqrt{\frac{\sum_{i=1}^n (y_i - \bar{y})^2}{n-1}} \approx \sigma$$

where n is the number of samples (that is, rolls of toilet paper). The probability that the number of sheets in a roll is greater than or equal to some specified number, a , is the area under the probability density function (that is, integral) from a to ∞ , and is given by

$$P(y \geq a) = \int_a^{\infty} f(y) dy = \int_a^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)[(y-\mu)/\sigma]^2} dy$$

Note that the result of the previous equation is a number from 0 to 1 that correlates to a 0% to 100% probability, respectively.

Solution

A co-operative education student working for the industrial (quality) engineer samples 10 rolls of toilet paper on a monthly basis and determines that the number of sheets in each roll is

Roll	Number of sheets
1	253
2	250
3	251
4	252
5	253
6	253
7	252
8	254
9	252
10	252

The industrial (quality) engineer calculates the following mean and standard deviation of the sample of 10 rolls of toilet paper.

$$\bar{y} = \frac{253 + 250 + 251 + 252 + 253 + 253 + 252 + 254 + 252 + 252}{10} = 252.2 \approx \mu$$

$$s = \sqrt{\frac{(253 - 252.2)^2 + (250 - 252.2)^2 + (251 - 252.2)^2 + (252 - 252.2)^2 + (253 - 252.2)^2 + (253 - 252.2)^2 + (252 - 252.2)^2 + (254 - 252.2)^2 + (252 - 252.2)^2 + (252 - 252.2)^2}{10-1}} = 1.135 \approx \sigma$$

Finally, the probability that the number of sheets in a roll is greater $a = 250$ sheets is

$$P(y \geq 250) = \int_{250}^{\infty} \frac{1}{1.135\sqrt{2\pi}} e^{-(1/2)[(y-252.2)/1.135]^2} dy$$

$$P(y \geq 250) = \int_{250}^{\infty} 0.3515 e^{-0.3881(y-252.2)^2} dy$$

$$P(y \geq 250) = ??$$

Solution to the previous equation yields the probability that the number of sheets in a roll is greater than or equal to 250.

QUESTIONS

1. Although all the samples have more than 250 sheets, why do we not get 100% probability that all the rolls will have more than 250 sheets.
2. Change one of the data points to 248 sheets and rework the problem.
3. To find the integral numerically, one may substitute infinity by 5. How will you justify this substitution quantitatively.

Topic	INTEGRATION
Sub Topic	Physical Problem
Summary	Find the probability, based on a sampling of the assembly line that the number of sheets in a roll of toilet paper is greater than a specified number.
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