Romberg Rule of Integration

Mechanical Engineering Majors

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Basis of Romberg Rule

Integration

The process of measuring the area under a curve.

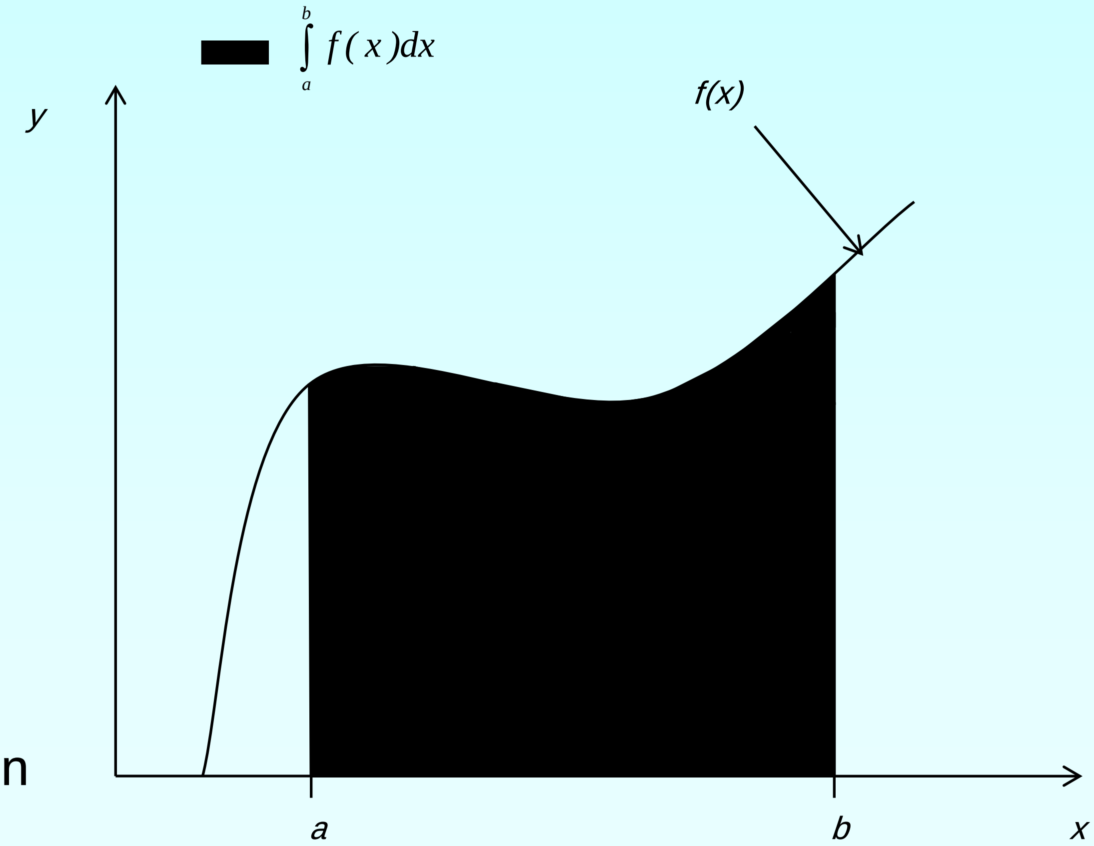
$$I = \int_a^b f(x)\,dx$$

Where:

f(x) is the integrand

a= lower limit of integration

b= upper limit of integration



What is The Romberg Rule?

Romberg Integration is an extrapolation formula of the Trapezoidal Rule for integration. It provides a better approximation of the integral by reducing the True Error.

Error in Multiple Segment Trapezoidal Rule

The true error in a multiple segment Trapezoidal Rule with n segments for an integral

$$I = \int_a^b f(x)dx$$

Is given by

$$E_t = \frac{(b-a)^3}{12n^2} \frac{\sum_{i=1}^{n} f''(\xi_i)}{n}$$

where for each *i*, ξ_i is a point somewhere in the domain , $[a+(i-1)h, a+ih]$.

Error in Multiple Segment Trapezoidal Rule

The term $\dfrac{\sum_{i=1}^{n} f''(\xi_i)}{n}$ can be viewed as an approximate average value of $f''(x)$ in $[a,b]$.

This leads us to say that the true error, E_t previously defined can be approximated as

$$E_t \approx \alpha \frac{1}{n^2}$$

Error in Multiple Segment Trapezoidal Rule

Table 1 shows the results obtained for the integral using multiple segment Trapezoidal rule for

$$x = \int_{8}^{30}\left(2000\,ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

n	Value	E_t	$\lvert\in_t\rvert\%$	$\lvert\in_a\rvert\%$
1	11868	807	7.296	---
2	11266	205	1.854	5.343
3	11153	91.4	0.8265	1.019
4	11113	51.5	0.4655	0.3594
5	11094	33.0	0.2981	0.1669
6	11084	22.9	0.2070	0.09082
7	11078	16.8	0.1521	0.05482
8	11074	12.9	0.1165	0.03560

Table 1: Multiple Segment Trapezoidal Rule Values

Error in Multiple Segment Trapezoidal Rule

The true error gets approximately quartered as the number of segments is doubled. This information is used to get a better approximation of the integral, and is the basis of Richardson's extrapolation.

Richardson's Extrapolation for Trapezoidal Rule

The true error, E_t in the *n*-segment Trapezoidal rule is estimated as

$$E_t \approx \frac{C}{n^2}$$

where *C* is an *approximate constant* of proportionality. Since

$$E_t = TV - I_n$$

Where TV = true value and I_n = approx. value

Richardson's Extrapolation for Trapezoidal Rule

From the previous development, it can be shown that

$$\frac{C}{(2n)^2} \approx TV - I_{2n}$$

when the segment size is doubled and that

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

which is Richardson's Extrapolation.

Example 1

A trunnion of diameter 12.363" has to be cooled from a room temperature of 80^0F before it is shrink fit into a steel hub (Figure 2).

The equation that gives the diametric contraction of the trunnion in dry-ice/alcohol (boiling temperature is -108^0F) is given by:

$$\Delta D = 12.363 \int_{80}^{-108} \left(-1.2278\times10^{-11}T^2 + 6.1946\times10^{-9}T + 6.015\times10^{-6}\right)dT$$

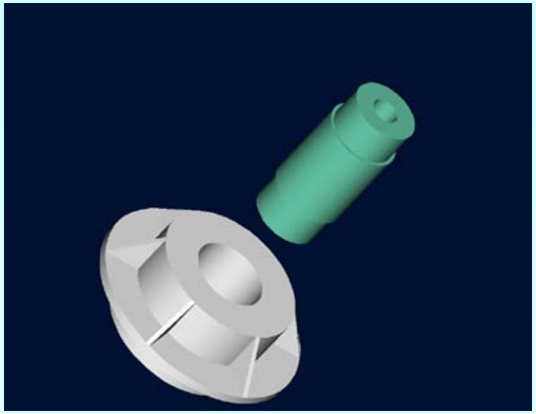


Figure 2. Trunnion to be slided through the hub after contracting.

a) Use Richardson's rule to find the contraction. Use the 2-segment and 4-segment Trapezoidal rule results given in Table 1.

b) Find the true error, E_t for part (a).

c) Find the absolute relative true error, $|\in_a|$ for part (a).

Solution

n	Trapezoidal Rule
1	−0.013536
2	−0.013630
4	−0.013679
8	−0.013687

a) $I_2 = -0.013630 in$

$$I_4 = -0.013679 in$$

Using Richardson's extrapolation formula for Trapezoidal rule

$TV \approx I_{2n} + \dfrac{I_{2n} - I_n}{3}$ and choosing *n=2*,

$$TV \approx I_4 + \frac{I_4 - I_2}{3} \approx -0.013679 + \frac{-0.013679 - (-0.01360)}{3} \approx -0.013670 in$$

Solution (cont.)

b) The exact value of the above integral is

$$\Delta D = 12.363 \int_{80}^{-108} \left(-1.2278\times10^{-11}T^2 + 6.1946\times10^{-9}T + 6.015\times10^{-6}\right)dT$$

$$= -0.013689 in$$

Hence

$$E_t = True\ Value - Approximate\ Value$$

$$= -0.013689 - \left(-0.013670\right)$$

$$= 5.5212\times10^{-7}\ \text{in}$$

Solution (cont.)

c) The absolute relative true error $\left|\in_t\right|$ would then be

$$\left|\in_t\right| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100$$

$$= \left|\frac{5.5212 \times 10^{-7}}{-0.013689}\right| \times 100$$

$$= 0.0040332\%$$

Table 2 shows the Richardson's extrapolation results using 1, 2, 4, 8 segments. Results are compared with those of Trapezoidal rule.

Solution (cont.)

Table 2 The values obtained using Richardson's extrapolation formula for Trapezoidal rule for

$$\Delta D = 12.363 \int_{80}^{-108} \left(-1.2278 \times 10^{-11} T^2 + 6.1946 \times 10^{-9} T + 6.015 \times 10^{-6}\right) dT$$

n	**Trapezoidal Rule**	$\lvert \in_t \rvert$ **for Trapezoidal Rule**	**Richardson's Extrapolation**	$\lvert \in_t \rvert$ **for Richardson's Extrapolation**
1	−0.013536	1.1177	--	--
2	−0.013630	0.43100	−0.013661	0.20294
4	−0.013679	0.076750	−0.013695	0.045429
8	−0.013687	0.019187	−0.013690	0.0040332

Romberg Integration

Romberg integration is same as Richardson's extrapolation formula as given previously. However, Romberg used a recursive algorithm for the extrapolation. Recall

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

This can alternately be written as

$$\left(I_{2n}\right)_R = I_{2n} + \frac{I_{2n} - I_n}{3} = I_{2n} + \frac{I_{2n} - I_n}{4^{2-1} - 1}$$

Romberg Integration

Note that the variable TV is replaced by $\left(I_{2n}\right)_R$ as the value obtained using Richardson's extrapolation formula.
Note also that the sign $\approx$ is replaced by = sign.
Hence the estimate of the true value now is

$$TV \approx \left(I_{2n}\right)_R + Ch^4$$

Where Ch^4 is an approximation of the true error.

Romberg Integration

Determine another integral value with further halving the step size (doubling the number of segments),

$$(I_{4n})_R = I_{4n} + \frac{I_{4n} - I_{2n}}{3}$$

It follows from the two previous expressions that the true value TV can be written as

$$TV \approx (I_{4n})_R + \frac{(I_{4n})_R - (I_{2n})_R}{15}$$

$$= I_{4n} + \frac{(I_{4n})_R - (I_{2n})_R}{4^{3-1} - 1}$$

Romberg Integration

A general expression for Romberg integration can be written as

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1}, k \geq 2$$

The index *k* represents the order of extrapolation. k=1 represents the values obtained from the regular Trapezoidal rule, k=2 represents values obtained using the true estimate as O(h^2). The index *j* represents the more and less accurate estimate of the integral.

Example 2

A trunnion of diameter 12.363" has to be cooled from a room temperature of 80°F before it is shrink fit into a steel hub (Figure 2). The equation that gives the diametric contraction of the trunnion in dry-ice/alcohol (boiling temperature is −108°F) is given by:

$$\Delta D = 12.363 \int_{80}^{-108} \left(-1.2278 \times 10^{-11} T^2 + 6.1946 \times 10^{-9} T + 6.015 \times 10^{-6} \right) dT$$

Use Romberg's rule to find the contraction. Use the 1, 2, 4, and 8-segment Trapezoidal rule results as given in the Table 1.

Solution

From Table 1, the needed values from original Trapezoidal rule are

$$I_{1,1} = -0.013536in \qquad I_{1,2} = -0.013630in$$

$$I_{1,3} = -0.013679in \qquad I_{1,4} = -0.013687in$$

where the above four values correspond to using 1, 2, 4 and 8 segment Trapezoidal rule, respectively.

Solution (cont.)

To get the first order extrapolation values,

$$I_{2,1} = I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3}$$

$$= -0.013630 + \frac{-0.013630 - (-0.013536)}{3}$$

$$= -0.013661 in$$

Similarly,

$$I_{2,2} = I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3}$$

$$= -0.013679 + \frac{-0.013679 - (-0.013630)}{3}$$

$$= -0.013695 in$$

$$I_{2,3} = I_{1,4} + \frac{I_{1,4} - I_{1,3}}{3}$$

$$= -0.013687 + \frac{-0.013687 - (-0.013679)}{3}$$

$$= -0.013695 in$$

Solution (cont.)

For the second order extrapolation values,

$$I_{3,1} = I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15}$$

$$= -0.013695 + \frac{-0.013695 - (-0.013661)}{15}$$

$$= -0.013698 in$$

Similarly,

$$I_{3,2} = I_{2,3} + \frac{I_{2,3} - I_{2,2}}{15}$$

$$= -0.013695 + \frac{-0.013695 - (-0.013695)}{15}$$

$$= -0.013690 in$$

Solution (cont.)

For the third order extrapolation values,

$$I_{4,1} = I_{3,2} + \frac{I_{3,2} - I_{3,1}}{63}$$

$$= -0.013690 + \frac{-0.013690 - (-0.013698)}{63}$$

$$= -0.013689 in$$

Table 3 shows these increased correct values in a tree graph.

Solution (cont.)

Table 3: Improved estimates of the integral value using Romberg Integration

		1st Order	2nd Order	3rd Order
1-segment	−0.013536	− 0.013661	− 0.013698	− 0.013689
2-segment	− 0.013630	− 0.013695	− 0.013690	
4-segment	−0.013679	− 0.013695		
8-segment	− 0.013687			

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/romberg_method.html

THE END

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