Chapter 10.00B

Physical Problem for Partial Differential Equations Mechanical Engineering

Problem Statement

Imagine that you are using a poker to poke the fire. What would the temperature be on the end of poker in your hand if you keep the poker in the fire for a while?

System Model

According to Fourier's law of conduction heat flow along the length dimension of the poker is given by

$$q(x) = -k\frac{\partial T}{\partial x} \tag{1}$$

where

$q(x) =$ Heat flow per unit time per unit area

$k =$ Thermal conductivity of the material

$T(x,t) =$ Temperature of the rod at a particular location on the rod at a particular time

Heat conducted in or out

$$\begin{aligned} q_x &= q(x+dx) - q(x) \\ &= \left(q(x) + \frac{dq}{dx}\right) - q(x) \\ &= \frac{dq}{dx} \\ &= \frac{d}{dx}(q) \end{aligned} \tag{2}$$

Here in one dimensional equation, heat conducted is a function of x on only, so the total differential operator and the partial differential operator are the same, i.e. $\frac{d}{dx}(q) = \frac{\partial}{\partial x}(q)$.

Substituting in equation (2) we have

$$q_x = \frac{\partial}{\partial x}(q) \tag{3}$$

substituting equation (3) in equation (1), we have

$$q_x = \frac{\partial}{\partial x}\left(-k\frac{\partial T}{\partial x}\right) \tag{4}$$

Let heat generation rate be a function of x.

$$q_{generated} = g(x) \tag{5}$$

Rate of heat stored in the body is given by

$$q_{stiored} = \rho C_p \frac{\partial T}{\partial t} \tag{6}$$

where

$\rho =$ Density of the material

$C_p =$ Specific heat of the material of the body

$t =$ Time

According to the law of conservation of mass:

Rate of total heat energy generated = rate of heat stored inside body + rate of heat conducted by the body

$$q_{generated} = q_{stored} + q_x \tag{7}$$

Substituting Equations (4-6) in Equation (7), we have

$$g(x,y,z) = \rho C_p \frac{\partial T}{\partial t} - \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)$$

Re-writing the above equation we have the three-dimensional heat conduction equation

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + g(x) = \rho C_p \frac{\partial T}{\partial t} \tag{8}$$

In this problem, there is no heat generation inside the rod, so $g(x) = 0$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) = \rho C_p \frac{\partial T}{\partial t}$$

$$k\frac{\partial^2 T}{\partial x^2} = \rho C_p \frac{\partial T}{\partial t}$$

$$\alpha\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \tag{9}$$

in which α is given by

$$\alpha = \frac{k}{\rho C}$$

Equation (9) represents the mathematical model of the transient heat conduction inside a rod with no heat generation.

Boundary Conditions

$$T(0,t) = T_{Source} \tag{10}$$

$$T(L,t) = T_{\text{sink}} \tag{11}$$

Questions

1. Solve the mathematical model represented by Equation 9 with the boundary conditions to find the temperature distribution along the rod.

PARTIAL DIFFERENTIAL EQUATIONS	
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