Bisection Method

Chemical Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

Bisection Method

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Basis of Bisection Method

Theorem An equation $f(x)=0$, where $f(x)$ is a real continuous function, has at least one root between x_l and x_u if $f(x_l)\, f(x_u) < 0$.

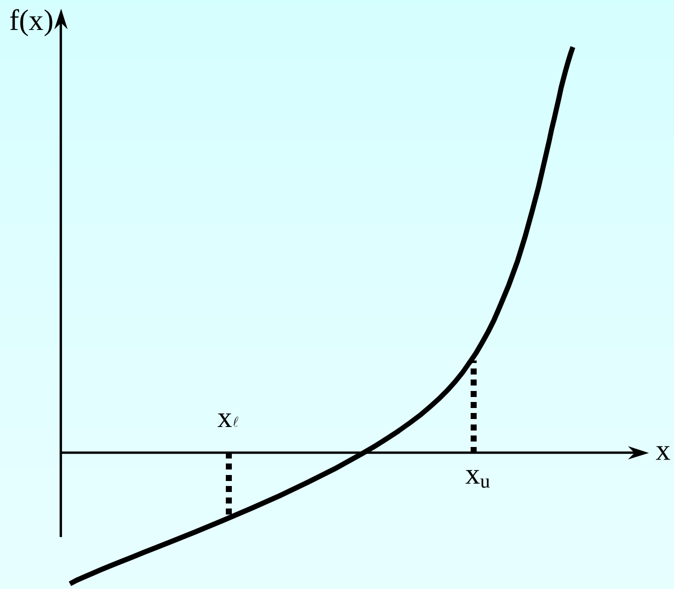


Figure 1 At least one root exists between the two points if the function is real, continuous, and changes sign.

Basis of Bisection Method

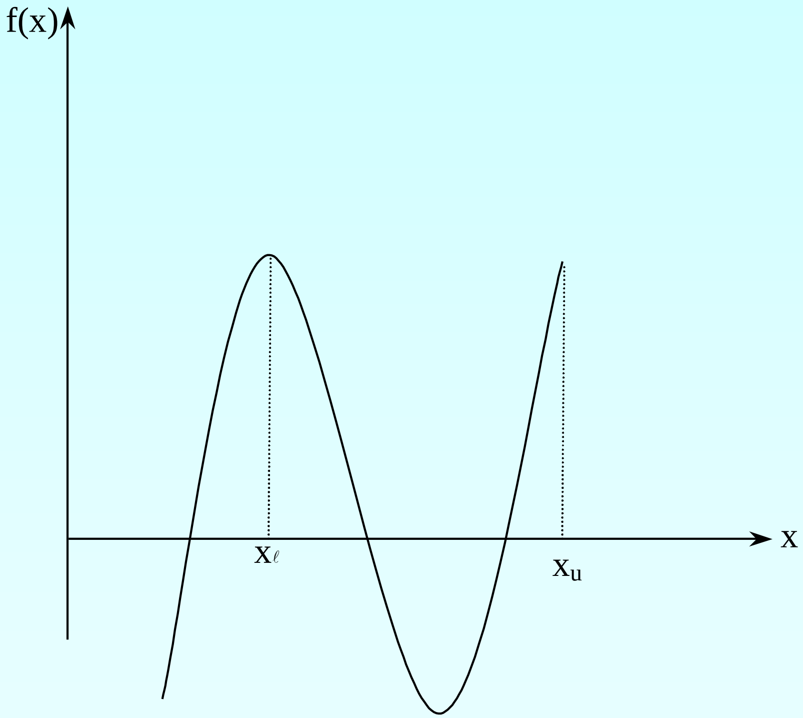


Figure 2 If function $f(x)$ does not change sign between two points, roots of the equation $f(x)=0$ may still exist between the two points.

Basis of Bisection Method

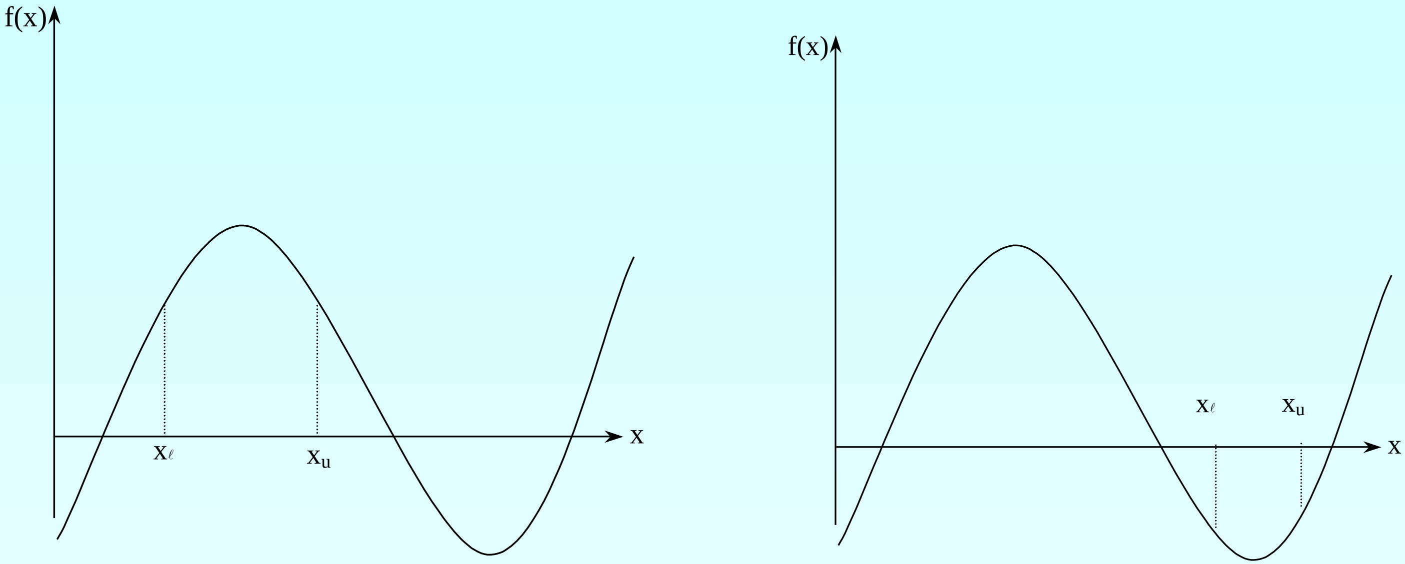


Figure 3 If the function $f(x)$ does not change sign between two points, there may not be any roots for the equation $f(x)=0$ between the two points.

Basis of Bisection Method

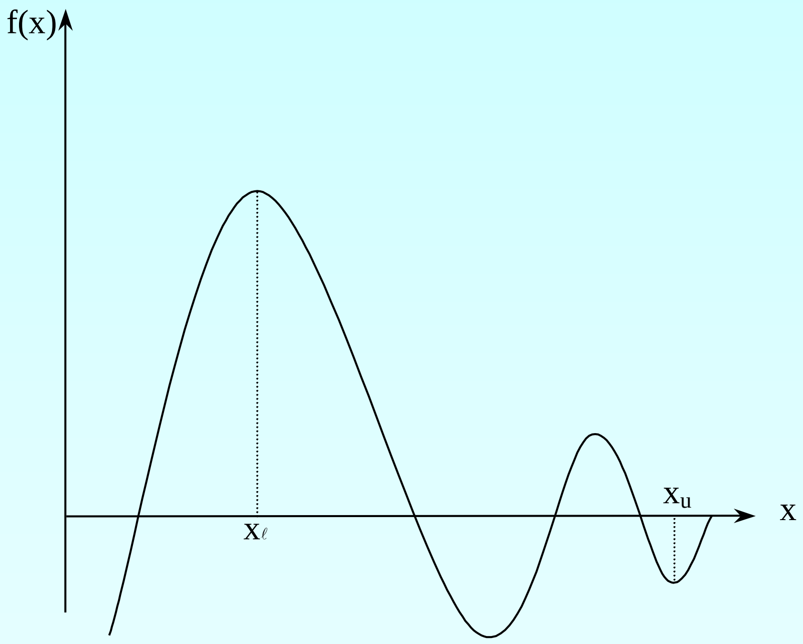


Figure 4 If the function $f(x)$ changes sign between two points, more than one root for the equation $f(x)=0$ may exist between the two points.

Algorithm for Bisection Method

Step 1

Choose x_ℓ and x_u as two guesses for the root such that $f(x_\ell)\, f(x_u) < 0$, or in other words, f(x) changes sign between x_ℓ and x_u. This was demonstrated in Figure 1.

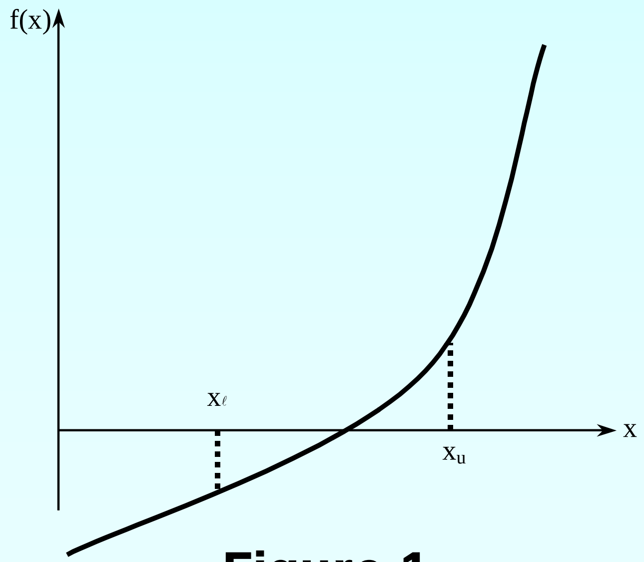


Figure 1

Step 2

Estimate the root, x_m of the equation f (x) = 0 as the mid point between x_ℓ and x_u as

$$x_m = \frac{x_\ell + x_u}{2}$$

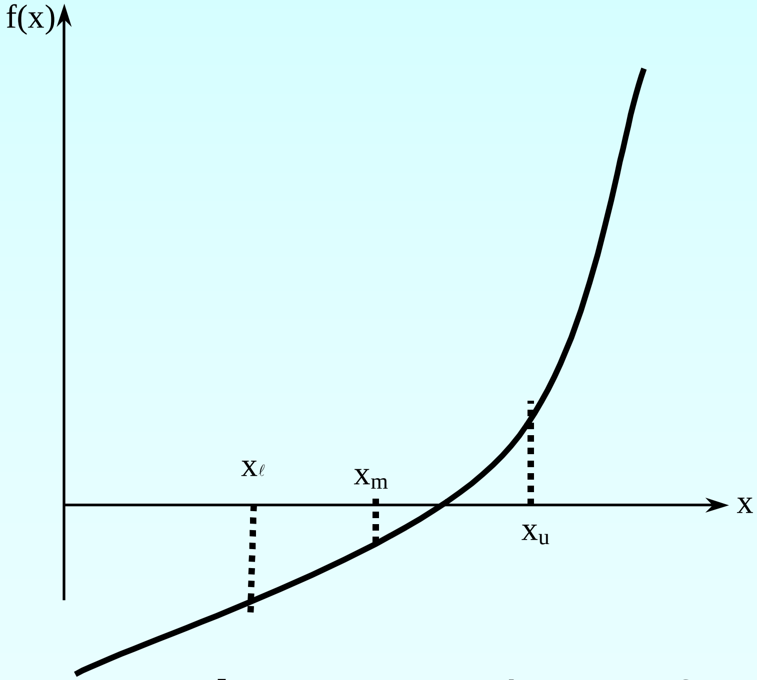


Figure 5 Estimate of x_m

Step 3

Now check the following

a) If $f(x_l)f(x_m) < 0$, then the root lies between x_ℓ and x_m; then $x_\ell = x_\ell$; $x_u = x_m$.

b) If $f(x_l)f(x_m) > 0$, then the root lies between x_m and x_u; then $x_\ell = x_m$; $x_u = x_u$.

c) If $f(x_l)f(x_m) = 0$; then the root is x_m. Stop the algorithm if this is true.

Step 4

Find the new estimate of the root

$$x_m = \frac{x_\ell + x_u}{2}$$

Find the absolute relative approximate error

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

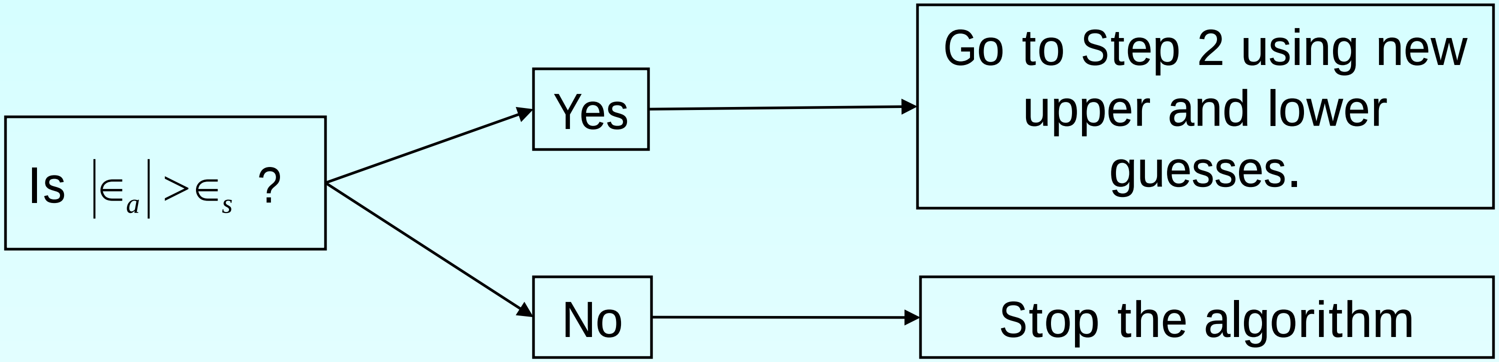
where

$$x_m^{old} = \text{previous estimate of root}$$

$$x_m^{new} = \text{current estimate of root}$$

Step 5

Compare the absolute relative approximate error $|\in_a|$ with the pre-specified error tolerance $\in_s$.



Note one should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user about it.

Example 1

You have a spherical storage tank containing oil. The tank has a diameter of 6 ft. You are asked to calculate the height, *h*, to which a dipstick 8 ft long would be wet with oil when immersed in the tank when it contains 4 ft^3 of oil.

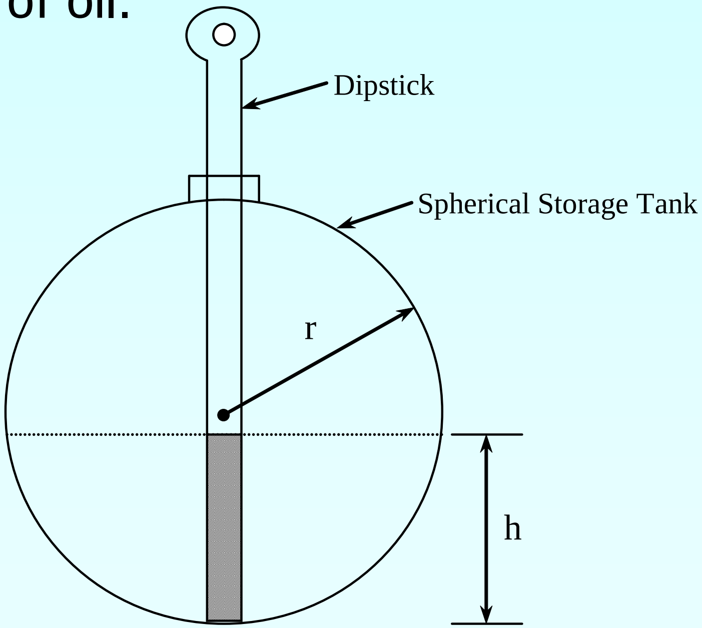


Figure 5 Spherical storage tank problem.

Example 1 Cont.

The equation that gives the height, *h*, of liquid in the spherical tank for the given volume and radius is given by

$$f(h) = h^3 - 9h^2 + 3.8197 = 0$$

Use the bisection method of finding roots of equations to find the height, *h*, to which the dipstick is wet with oil. Conduct three iterations to estimate the root of the above equation. Find the absolute relative approximate error at the end of each iteration and the number of significant digits at least correct at the end of each iteration.

Example 1 Cont.

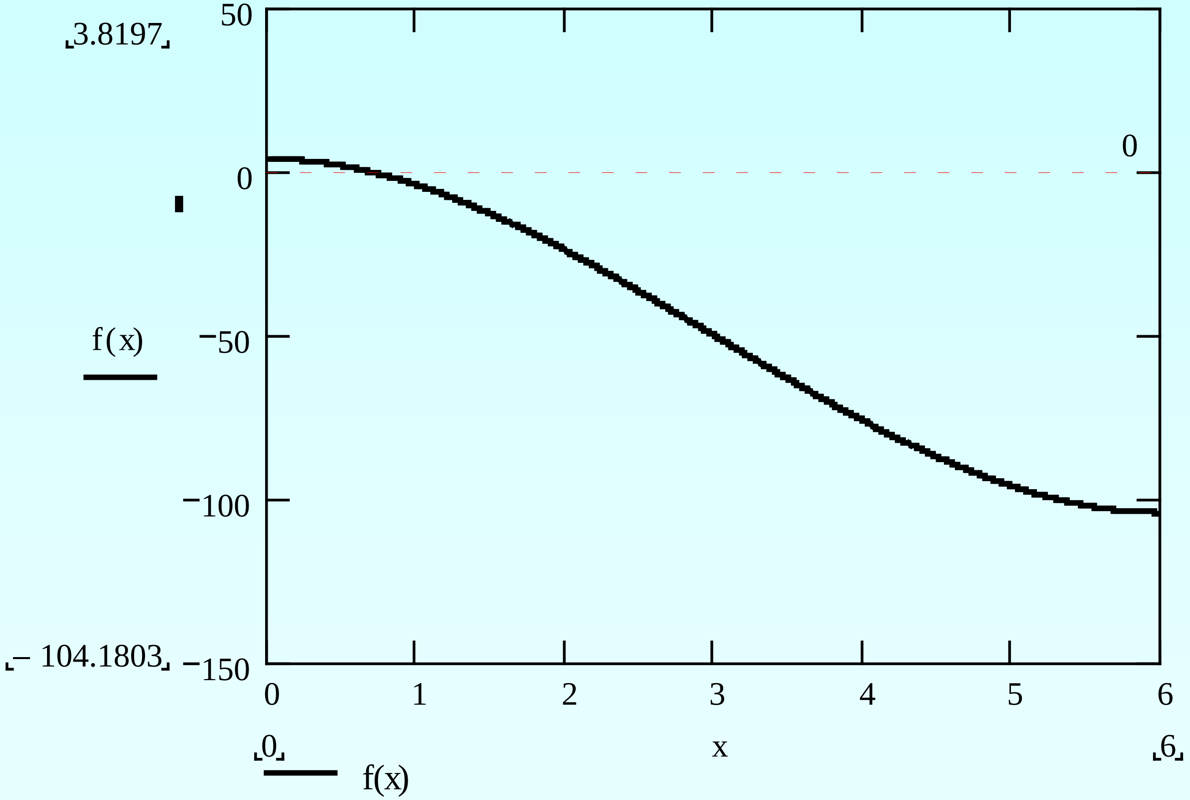


Figure 6 Graph of the function f(h).

$$f(h) = h^3 - 9h^2 + 3.8197 = 0$$

Example 1 Cont.

Solution

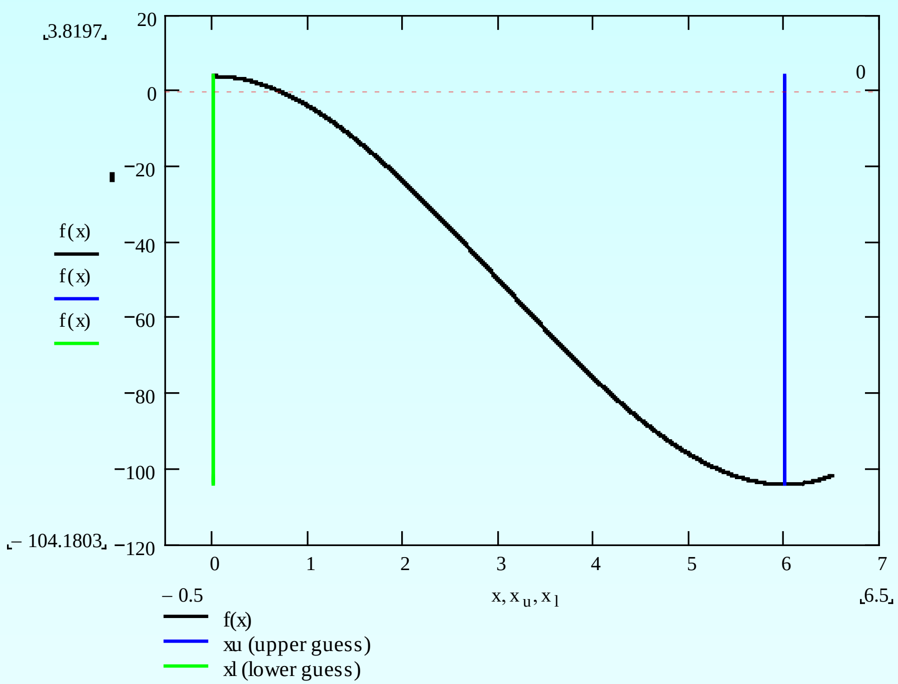


Figure 7 Graph showing sign change between limits.

Let us assume

$$h_l = 0, \; h_u = 6$$

Check if the function changes sign between h_ℓ and h_u .

$$f(h_l) = (0)^3 - 9(0)^2 + 3.8197 = 3.8197$$

$$f(h_u) = (6)^3 - 9(6)^2 + 3.8197 = -104.18$$

$$f(h_l)f(h_u) = f(0)f(6)$$
$$= (3.8197)(-104.18) < 0$$

There is at least one root between h_l and h_u.

Example 1 Cont.

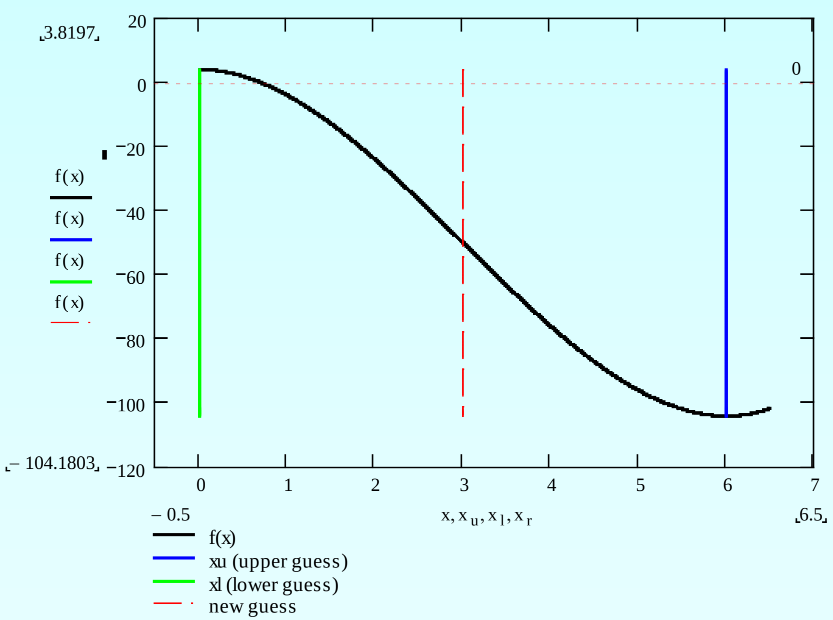


Figure 8 Graph of the estimated root after Iteration 1.

<u>Iteration 1</u>

The estimate of the root is

$$h_m = \frac{h_l + h_u}{2} = \frac{0+6}{2} = 3$$

$$f(h_m) = f(3) = (3)^3 - 9(3)^2 + 3.8197 = -50.180$$

$$f(h_l)f(h_m) = f(0)f(3) = (3.1897)(-50.180) < 0$$

The root is bracketed between h_l and h_m.

The lower and upper limits of the new bracket are

$$h_l = 0, \; h_u = 3$$

The absolute relative approximate error $|\in_a|$ cannot be calculated, as we do not have a previous approximation.

Example 1 Cont.

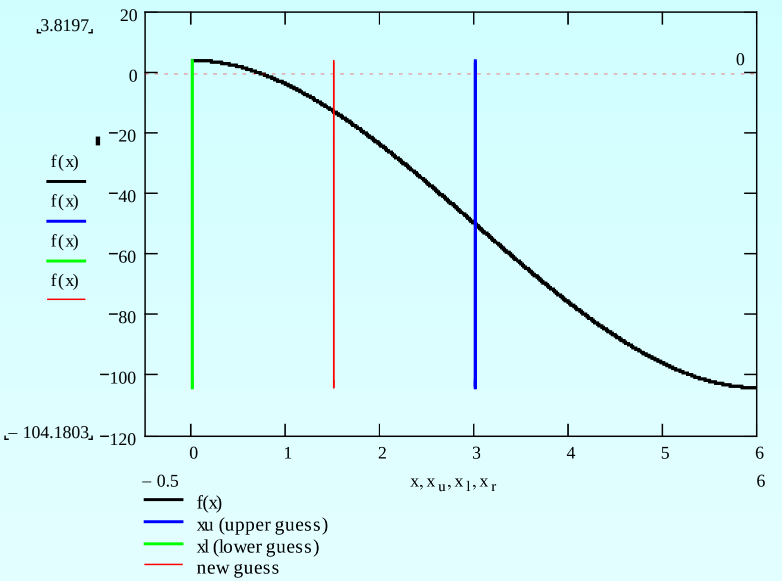


Figure 9 Graph of the estimated root after Iteration 2.

<u>Iteration 2</u>

The estimate of the root is

$$h_m = \frac{h_l + h_u}{2} = \frac{0+3}{2} = 1.5$$

$$f(h_m) = f(1.5) = (1.5)^3 - 9(1.5)^2 + 3.8197 = -13.055$$

$$f(h_l)f(h_m) = f(0)f(1.5) = (3.1897)(-13.055) < 0$$

The root is bracketed between h_l and h_m .

The lower and upper limits of the new bracket are

$$h_l = 0,\ h_u = 1.5$$

Example 1 Cont.

The absolute relative error $\left|\in_a\right|$ at the end of Iteration 2 is

$$\left|\in\right| = \left|\frac{h_m^{new} - h_m^{old}}{h_m^{new}}\right| \times 100$$

$$= \left|\frac{1.5-3}{1.5}\right| \times 100$$

$$= 100\%$$

None of the significant digits are at least correct in the estimated root

$$h_m = 1.5$$

as the absolute relative approximate error is greater than 5%.

Example 1 Cont.

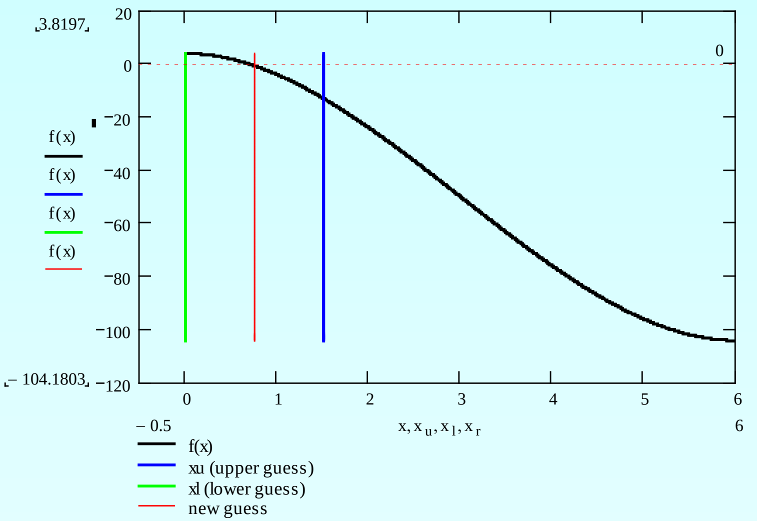


Figure 10 Graph of the estimated root after Iteration 3.

<u>Iteration 3</u>

The estimate of the root is

$$h_m = \frac{h_l + h_u}{2} = \frac{0 + 1.5}{2} = 0.75$$

$$f(h_m) = f(0.75) = (0.75)^3 - 9(0.75)^2 + 3.8197 = -0.82093$$

$$f(h_l)f(h_m) = f(0)f(0.75) = (3.1897)(-0.82093) < 0$$

The root is bracketed between h_l and h_m .

The lower and upper limits of the new bracket are

$$h_l = 0, \; h_u = 0.75$$

Example 1 Cont.

The absolute relative error $\left|\in_a\right|$ at the end of Iteration 3 is

$$\left|\in\right| = \left|\frac{h_m^{new} - h_m^{old}}{h_m^{new}}\right| \times 100$$

$$= \left|\frac{0.75 - 1.5}{0.75}\right| \times 100$$

$$= 100\%$$

Still none of the significant digits are at least correct in the estimated root of the equation as the absolute relative approximate error is greater than 5%.
The height of the liquid is estimated as 0.75 ft at the end of the third iteration.
Seven more iterations were conducted and these iterations are shown in Table 1.

Example 1 Cont.

Table 1 Root of $f(x)=0$ as function of number of iterations for bisection method.

Iteration	h_l	h_u	h_m	$\lvert\in_a\rvert\%$	$f(h_m)$
1	0.00	6	3	--------	−50.18
2	0.00	3	1.5	100	−13.055
3	0.00	1.5	0.75	100	−0.82093
4	0.00	0.75	0.375	100	2.6068
5	0.375	0.75	0.5625	33.333	1.1500
6	0.5625	0.75	0.65625	14.286	0.22635
7	0.65625	0.75	0.70313	6.6667	−0.28215
8	0.65625	0.70313	0.67969	3.4483	−0.024077
9	0.65625	0.67969	0.66797	1.7544	0.10210
10	0.66797	0.67969	0.67383	0.86957	0.039249

Example 1 Cont.

At the end of the 10th iteration,

$$\left|\in_a\right| = 0.86957\%$$

Hence the number of significant digits at least correct is given by the largest value of *m* for which

$$\left|\in_a\right| \le 0.5 \times 10^{2-m}$$

$$0.86957 \le 0.5 \times 10^{2-m}$$

$$1.7391 \le 10^{2-m}$$

$$\log(1.7391) \le 2 - m$$

$$m \le 2 - \log(1.7391) = 1.759$$

So

$$m = 1$$

The number of significant digits at least correct in the estimated root 0.67383 is 2.

Advantages

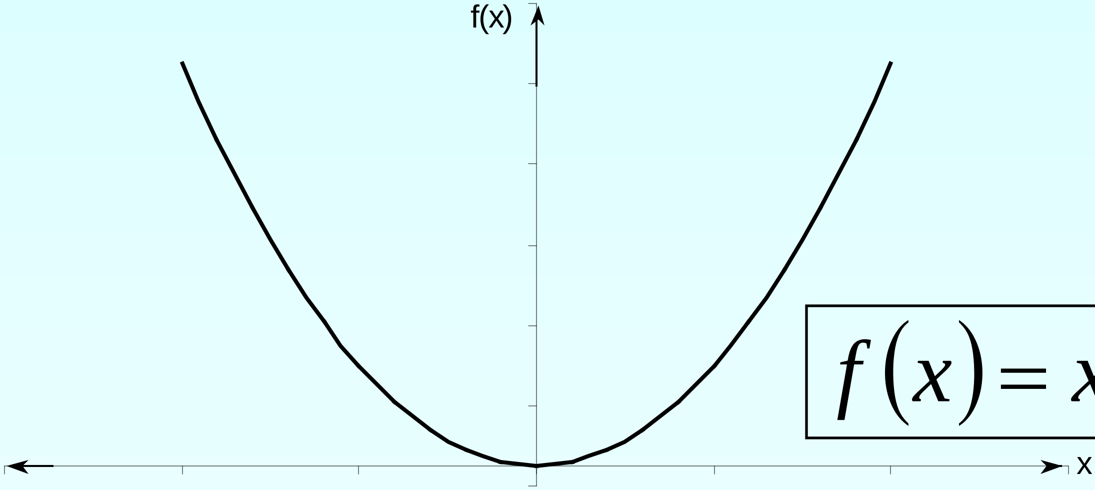
- Always convergent
- The root bracket gets halved with each iteration - guaranteed.

Drawbacks

- Slow convergence
- If one of the initial guesses is close to the root, the convergence is slower

Drawbacks (continued)

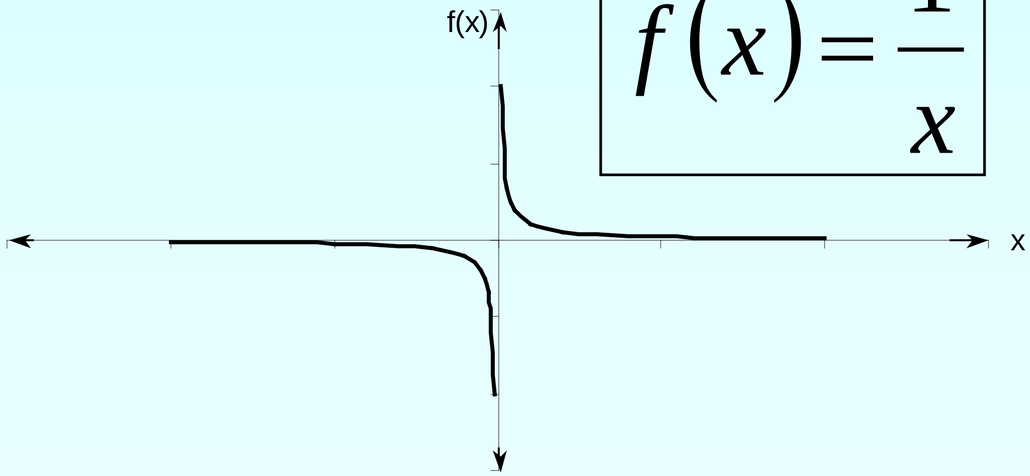
- If a function f(x) is such that it just touches the x-axis it will be unable to find the lower and upper guesses.



$$f(x) = x^2$$

Drawbacks (continued)

- Function changes sign but root does not exist



$$f(x) = \frac{1}{x}$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/bisection_method.html

THE END

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