

Romberg Rule of Integration

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Basis of Romberg Rule

Integration

The process of measuring the area under a curve.

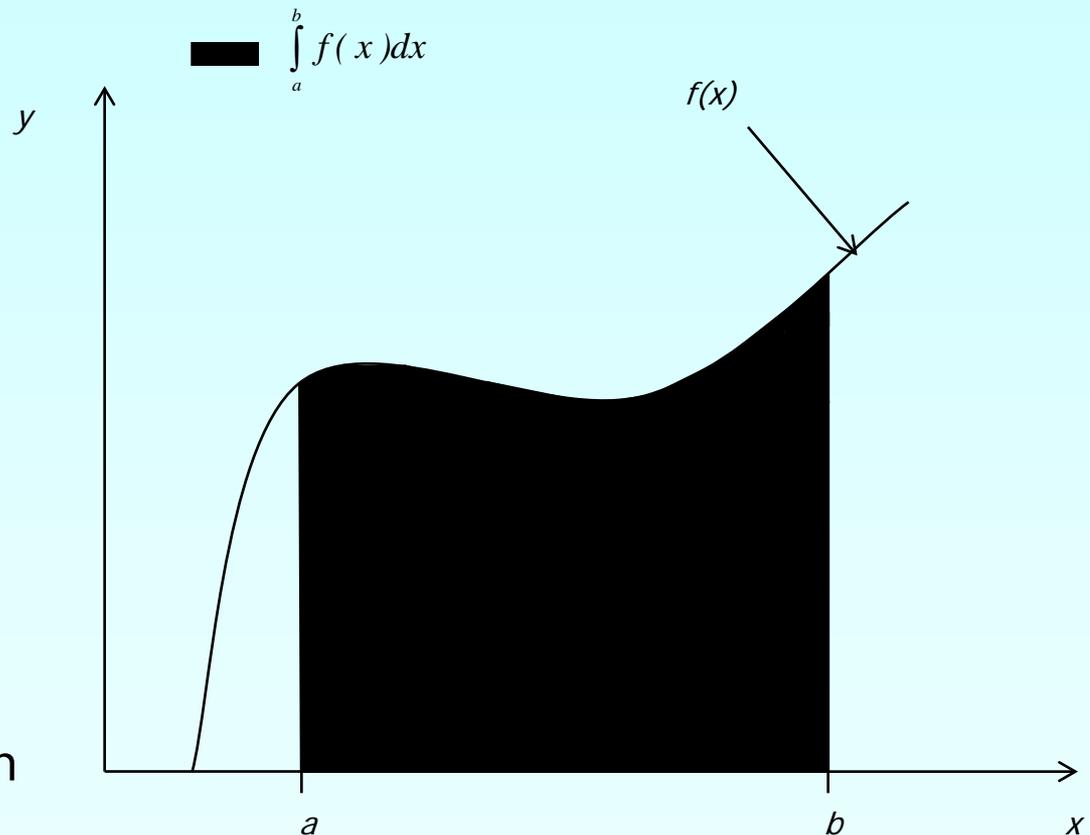
$$I = \int_a^b f(x) dx$$

Where:

$f(x)$ is the integrand

a = lower limit of integration

b = upper limit of integration



What is The Romberg Rule?

Romberg Integration is an extrapolation formula of the Trapezoidal Rule for integration. It provides a better approximation of the integral by reducing the True Error.

Error in Multiple Segment Trapezoidal Rule

The true error in a multiple segment Trapezoidal Rule with n segments for an integral

$$I = \int_a^b f(x) dx$$

Is given by

$$E_t = \frac{(b-a)^3}{12n^2} \frac{\sum_{i=1}^n f''(\xi_i)}{n}$$

where for each i , ξ_i is a point somewhere in the domain, $[a + (i-1)h, a + ih]$.

Error in Multiple Segment Trapezoidal Rule

The term $\frac{\sum_{i=1}^n f''(\xi_i)}{n}$ can be viewed as an approximate average value of $f''(x)$ in $[a,b]$.

This leads us to say that the true error, E_t previously defined can be approximated as

$$E_t \approx \alpha \frac{1}{n^2}$$

Error in Multiple Segment Trapezoidal Rule

Table 1 shows the results obtained for the integral using multiple segment Trapezoidal rule for

$$x = \int_8^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

n	Value	E_t	$ \epsilon_t \%$	$ \epsilon_a \%$
1	11868	807	7.296	---
2	11266	205	1.854	5.343
3	11153	91.4	0.8265	1.019
4	11113	51.5	0.4655	0.3594
5	11094	33.0	0.2981	0.1669
6	11084	22.9	0.2070	0.09082
7	11078	16.8	0.1521	0.05482
8	11074	12.9	0.1165	0.03560

Table 1: Multiple Segment Trapezoidal Rule Values

Error in Multiple Segment Trapezoidal Rule

The true error gets approximately quartered as the number of segments is doubled. This information is used to get a better approximation of the integral, and is the basis of Richardson's extrapolation.

Richardson's Extrapolation for Trapezoidal Rule

The true error, E_t in the n -segment Trapezoidal rule is estimated as

$$E_t \approx \frac{C}{n^2}$$

where C is an *approximate constant* of proportionality. Since

$$E_t = TV - I_n$$

Where TV = true value and I_n = approx. value

Richardson's Extrapolation for Trapezoidal Rule

From the previous development, it can be shown that

$$\frac{C}{(2n)^2} \approx TV - I_{2n}$$

when the segment size is doubled and that

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

which is Richardson's Extrapolation.

Example 1

In an attempt to understand the mechanism of the depolarization process in a fuel cell, an electro-kinetic model for mixed oxygen-methanol current on platinum was developed in the laboratory at FAMU. A very simplified model of the reaction developed suggests a functional relation in an integral form. To find the time required for 50% of the oxygen to be consumed, the time, T (s) is given by

$$T = -\int_{1.22 \times 10^{-6}}^{0.61 \times 10^{-6}} \left(\frac{6.73x + 4.3025 \times 10^{-7}}{2.316 \times 10^{-11} x} \right) dx$$

- Use Romberg's rule to find the time required for 50% of the oxygen to be consumed. Use the 2-segment and 4-segment Trapezoidal rule results given in Table 1.
- Find the true error, E_t for part (a).
- Find the absolute relative true error, $|\epsilon_a|$ for part (a).

Solution

Table 2 Values obtained using multiple-segment Trapezoidal rule.

a)

n	Value
1	191190
2	190420
3	190260
4	190200

$$I_2 = 190420s$$

$$I_4 = 190200s$$

Using Richardson's extrapolation formula for Trapezoidal rule

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3} \quad \text{and choosing } n=2,$$

$$TV \approx I_4 + \frac{I_4 - I_2}{3} = 190200 + \frac{190200 - (190420)}{3}$$

$$= 190130s$$

Solution (cont.)

b) The exact value of the above integral is

$$\begin{aligned} T &= -\int_{1.22 \times 10^{-6}}^{0.61 \times 10^{-6}} \left(\frac{6.73x + 4.3025 \times 10^{-7}}{2.316 \times 10^{-11} x} \right) dx \\ &= 1.90140 \times 10^5 \text{ s} \end{aligned}$$

Hence

$$\begin{aligned} E_t &= \textit{True Value} - \textit{Approximate Value} \\ &= 1.90140 \times 10^5 - 190130 \\ &= 8.3322 \end{aligned}$$

Solution (cont.)

c) The absolute relative true error $|\epsilon_t|$ would then be

$$|\epsilon_t| = \left| \frac{\text{True Error}}{\text{True Value}} \right| \times 100 = \left| \frac{8.3322}{1.90140 \times 10^5} \right| \times 100 = 0.0043823\%$$

Table 2 shows the Richardson's extrapolation results using 1, 2, 4, 8 segments. Results are compared with those of Trapezoidal rule.

Solution (cont.)

Table 2 The values obtained using Richardson's extrapolation formula for Trapezoidal rule for

$$T = -\int_{1.22 \times 10^{-6}}^{0.61 \times 10^{-6}} \left(\frac{6.73x + 4.3025 \times 10^{-7}}{2.316 \times 10^{-11} x} \right) dx$$

n	Trapezoidal Rule	$ \epsilon_t $ for Trapezoidal Rule	Richardson's Extrapolation	$ \epsilon_t $ for Richardson's Extrapolation
1	191190	0.55549	--	--
2	190420	0.14838	190163	0.014902
4	190210	0.037877	190127	0.0043823
8	190150	0.0095231	190133	0.00087599

Romberg Integration

Romberg integration is same as Richardson's extrapolation formula as given previously. However, Romberg used a recursive algorithm for the extrapolation. Recall

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

This can alternately be written as

$$(I_{2n})_R = I_{2n} + \frac{I_{2n} - I_n}{3} = I_{2n} + \frac{I_{2n} - I_n}{4^{2-1} - 1}$$

Romberg Integration

Note that the variable TV is replaced by $(I_{2n})_R$ as the value obtained using Richardson's extrapolation formula. Note also that the sign \cong is replaced by $=$ sign. Hence the estimate of the true value now is

$$TV \cong (I_{2n})_R + Ch^4$$

Where Ch^4 is an approximation of the true error.

Romberg Integration

Determine another integral value with further halving the step size (doubling the number of segments),

$$(I_{4n})_R = I_{4n} + \frac{I_{4n} - I_{2n}}{3}$$

It follows from the two previous expressions that the true value TV can be written as

$$\begin{aligned} TV &\cong (I_{4n})_R + \frac{(I_{4n})_R - (I_{2n})_R}{15} \\ &= I_{4n} + \frac{(I_{4n})_R - (I_{2n})_R}{4^{3-1} - 1} \end{aligned}$$

Romberg Integration

A general expression for Romberg integration can be written as

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1}, k \geq 2$$

The index k represents the order of extrapolation. $k=1$ represents the values obtained from the regular Trapezoidal rule, $k=2$ represents values obtained using the true estimate as $O(h^2)$. The index j represents the more and less accurate estimate of the integral.

Example 2

In an attempt to understand the mechanism of the depolarization process in a fuel cell, an electro-kinetic model for mixed oxygen-methanol current on platinum was developed in the laboratory at FAMU.

A very simplified model of the reaction developed suggests a functional relation in an integral form. To find the time required for 50% of the oxygen to be consumed, the time, T (s) is given by

$$T = -\int_{1.22 \times 10^{-6}}^{0.61 \times 10^{-6}} \left(\frac{6.73x + 4.3025 \times 10^{-7}}{2.316 \times 10^{-11} x} \right) dx$$

Use Romberg's rule to find the time required for 50% of the oxygen to be consumed. Use the 1, 2, 4, and 8-segment Trapezoidal rule results as given in the Table 1.

Solution

From Table 1, the needed values from original Trapezoidal rule are

$$I_{1,1} = 191190s \quad I_{1,2} = 190420s$$

$$I_{1,3} = 190200s \quad I_{1,4} = 190150s$$

where the above four values correspond to using 1, 2, 4 and 8 segment Trapezoidal rule, respectively.

Solution (cont.)

To get the first order extrapolation values,

$$\begin{aligned}I_{2,1} &= I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3} \\ &= 190420 + \frac{190420 - (191190)}{3} \\ &= 190160s\end{aligned}$$

Similarly,

$$\begin{aligned}I_{2,2} &= I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3} \\ &= 190200 + \frac{190200 - (190420)}{3} \\ &= 190130s\end{aligned}$$
$$\begin{aligned}I_{2,3} &= I_{1,4} + \frac{I_{1,4} - I_{1,3}}{3} \\ &= 190150 + \frac{190150 - (190200)}{3} \\ &= 190130s\end{aligned}$$

Solution (cont.)

For the second order extrapolation values,

$$\begin{aligned} I_{3,1} &= I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15} \\ &= 190130 + \frac{190130 - (190160)}{15} \\ &= 190120s \end{aligned}$$

Similarly,

$$\begin{aligned} I_{3,2} &= I_{2,3} + \frac{I_{2,3} - I_{2,2}}{15} \\ &= 190130 + \frac{190130 - (190130)}{15} \\ &= 190130s \end{aligned}$$

Solution (cont.)

For the third order extrapolation values,

$$\begin{aligned} I_{4,1} &= I_{3,2} + \frac{I_{3,2} - I_{3,1}}{63} \\ &= 190130 + \frac{190130 - (190120)}{63} \\ &= 190130_s \end{aligned}$$

Table 3 shows these increased correct values in a tree graph.

Solution (cont.)

Table 3: Improved estimates of the integral value using Romberg Integration

		1 st Order	2 nd Order	3 rd Order
1-segment	191190	190160	190120	190130
2-segment	190420			
4-segment	190200	190130		
8-segment	190150	190130		

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/romberg_method.html

THE END

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