Differentiation-Discrete Functions

Civil Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

Differentiation –Discrete Functions

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Forward Difference Approximation

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

For a finite 'Δx'

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Graphical Representation Of Forward Difference Approximation

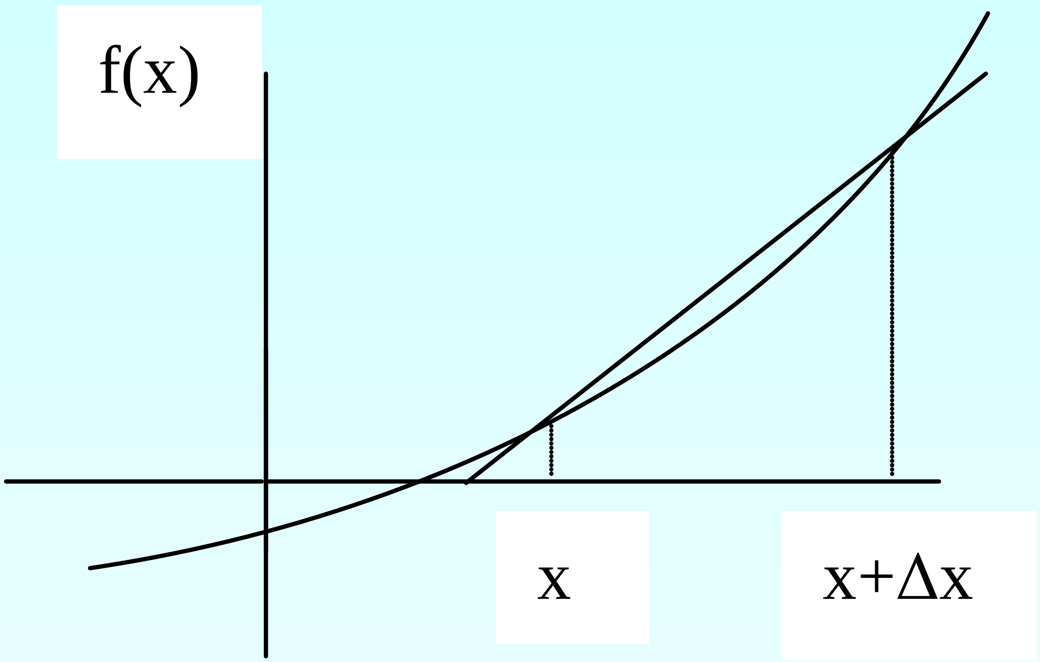

Figure 1 Graphical Representation of forward difference approximation of first derivative.

Example 1

To find the stress concentration around a hole in a plate under a uniform stress, a finite difference program has been written that calculates the radial and tangential displacements at different points in the plate. To find the stresses and hence the stress concentration factor, one needs to find the derivatives of these displacements. In Table 1 the radial displacements , u , are given along the y-axes. The radius of the hole is 1.0 cm.

a) At $x = 0$ if the radial strain, ε_r is given by $\varepsilon_r = \frac{\partial u}{\partial r}$, find the radial strain at $r = 1.1\text{cm}$ using forward divided difference method.

b) If the tangential strain at $r = 1.1\text{cm}, \theta = 90^0$ is given to you as $\varepsilon_\theta = 0.0029733$, find the hoop stress, σ_θ , at $r = 1.1cm, \theta = 90^0$ if $\sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta)$, where $E = 200\,\text{GPa}$ and $\nu = 0.3$.

Example 1 Cont.

Table 1 Radial displacement as a function of location.

$r(cm)$	$u(cm)$
1.0	−0.0010000
1.1	−0.0010689
1.2	−0.0011088
1.3	−0.0011326
1.4	−0.0011474
1.5	−0.0011574
1.6	−0.0011650
1.7	−0.0011718
1.8	−0.0011785
1.9	−0.0011857

Example 1 Cont.

Solution:

a) $$\varepsilon_r = \frac{\partial u}{\partial r}$$

$$\approx \frac{u_{i+1} - u_i}{\Delta r}$$

$$r_i = 1.1$$

$$r_{i+1} = 1.2$$

$$\Delta r = r_{i+1} - r_i$$

$$= 1.2 - 1.1$$

$$= 0.1$$

$$u_{i+1} = -0.0011088$$

$$u_i = -0.0010689$$

Example 1 Cont.

$$\begin{aligned}\varepsilon_r &\approx \frac{u_{i+1}-u_i}{\Delta r}\\ &\approx \frac{-0.0011088-(-0.0010689)}{0.1}\\ &\approx -0.00039900\end{aligned}$$

b) $$\sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta)$$

$$= \frac{2\times 10^{11}}{1-0.3^2}(-0.00039900 + 0.3\times 0.0029733)$$

$$= 108.35\times 10^6\ \mathrm{Pa}$$

Direct Fit Polynomials

In this method, given $'n+1'$ data points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ one can fit a n^{th} order polynomial given by

$$P_n(x) = a_0 + a_1 x + \ldots\ldots + a_{n-1} x^{n-1} + a_n x^n$$

To find the first derivative,

$$P_n'(x) = \frac{dP_n(x)}{dx} = a_1 + 2a_2 x + \ldots\ldots + (n-1)a_{n-1} x^{n-2} + n a_n x^{n-1}$$

Similarly other derivatives can be found.

Example 2-Direct Fit Polynomials

To find the stress concentration around a hole in a plate under a uniform stress, a finite difference program has been written that calculates the radial and tangential displacements at different points in the plate. To find the stresses and hence the stress concentration factor, one needs to find the derivatives of these displacements. In Table 2 the radial displacements, u, are given along y-axes. The radius of the hole is 1.0 cm.

a) At $x = 0$ if the radial strain, ε_r is given by $\varepsilon_r = \frac{\partial u}{\partial r}$, find the radial strain at $r = 1.1\text{cm}$. Use a third order polynomial interpolant for calculating the radial strain.

b) If the tangential strain at $r = 1.1\text{cm}, \theta = 90^0$ is given to you as $\varepsilon_\theta = 0.0029733$, find the hoop stress, σ_θ, at $r = 1.1\text{cm}, \theta = 90^0$ if $\sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta)$, where $E = 200\,\text{GPa}$ and $\nu = 0.3$.

Example 2-Direct Fit Polynomials cont.

Table 2 Radial displacement as a function of location.

$r(cm)$	$u(cm)$
1.0	−0.0010000
1.1	−0.0010689
1.2	−0.0011088
1.3	−0.0011326
1.4	−0.0011474
1.5	−0.0011574
1.6	−0.0011650
1.7	−0.0011718
1.8	−0.0011785
1.9	−0.0011857

Example 2-Direct Fit Polynomials cont.

Solution:

For the third order polynomial (also called cubic interpolation), we choose the displacement given by

$$u(r) = a_0 + a_1 r + a_2 r^2 + a_3 r^3$$

Since we want to find the radial strain at $r = 1.1\,\text{cm}$, and we are using a third order polynomial, we need to choose the four points closest to $r = 1.1$ and that also bracket $r = 1.1$ to evaluate it.

The four points are $r_0 = 1.0, r_1 = 1.1, r_2 = 1.2, \text{and } r_3 = 1.3$

$$r_o = 1.0,\ \ u(r_o) = -0.0010000$$

$$r_1 = 1.1,\ \ u(r_1) = -0.0010689$$

$$r_2 = 1.2,\ \ u(r_2) = -0.0011088$$

$$r_3 = 1.3,\ \ u(r_3) = -0.0011326$$

Example 2-Direct Fit Polynomials cont.

such that

$$u(1.0) = -0.0010000 = a_0 + a_1(1) + a_2(1)^2 + a_3(1)^3$$

$$u(1.1) = -0.0010689 = a_0 + a_1(1.1) + a_2(1.1)^2 + a_3(1.1)^3$$

$$u(1.2) = -0.0011088 = a_0 + a_1(1.2) + a_2(1.2)^2 + a_3(1.2)^3$$

$$u(1.3) = -0.0011326 = a_0 + a_1(1.3) + a_2(1.3)^2 + a_3(1.3)^3$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1.1 & 1.21 & 1.331 \\ 1 & 1.2 & 1.44 & 1.728 \\ 1 & 1.3 & 1.69 & 2.197 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -0.0010000 \\ -0.0010689 \\ -0.0011088 \\ -0.0011326 \end{bmatrix}$$

Example 2-Direct Fit Polynomials cont.

Solving the above four equations gives

$$a_0 = 0.0041220$$

$$a_1 = \text{-}0.0011517$$

$$a_2 = 0.0085450$$

$$a_3 = \text{-}0.0021500$$

Hence

$$u(r) = a_0 + a_1 r + a_2 r^2 + a_3 r^3$$

$$= 0.0041220 - 0.0011517r + 0.0085450r^2 - 0.0021500r^3, \quad 1 \le r \le 1.3$$

Example 2-Direct Fit Polynomials cont.

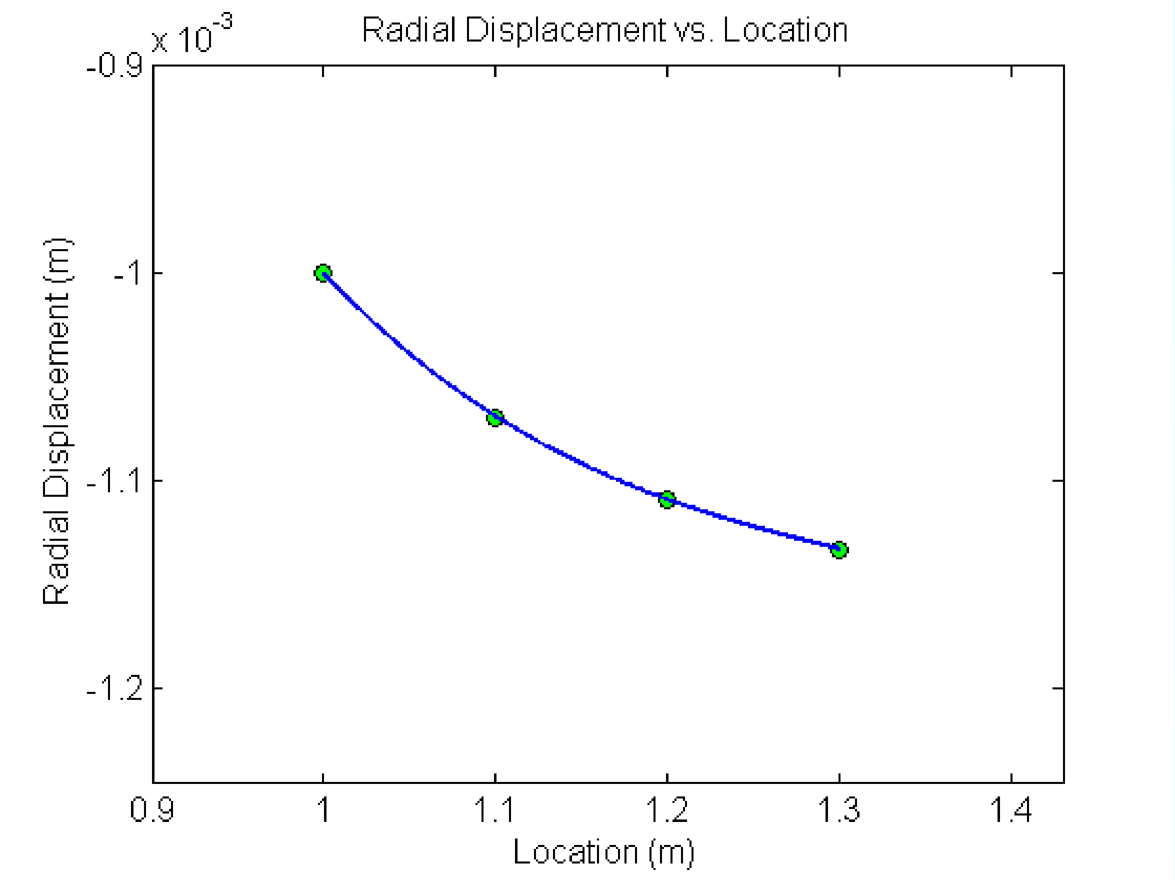

Figure 2 Graph of Radial Displacement vs. Location

Example 2-Direct Fit Polynomials cont.

The derivative of radial displacement at $r = 1.1\text{cm}$ is given by

$$u'(1.1) = \frac{d}{dr}u(r)\Big|_{r=1.1}$$

Given that

$$u(r) = 0.0041220 - 0.0011517r + 0.0085450r^2 - 0.0021500r^3, 1 \le r \le 1.3$$

$$\begin{aligned} u'(r) &= \frac{d}{dr}u(r) \\ &= \frac{d}{dr}\left(0.0041220 - 0.0011517r + 0.0085450r^2 - 0.0021500r^3\right) \\ &= -0.0011517 + 0.017090r - 0.0064500r^2, \quad 1 \le r \le 1.3 \end{aligned}$$

$$\begin{aligned} u'(1.1) &= -0.0011517 + 0.017090(1.1) - 0.0064500(1.1)^2 \\ &= -0.00052250\text{cm/cm} \end{aligned}$$

$$\varepsilon_r = -0.00052250\text{cm/cm}$$

Example 2-Direct Fit Polynomials cont.

b) $$\sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta)$$

$$= \frac{2\times10^{11}}{1-0.3^2}(-0.00052250 + 0.3\times0.0029733)$$

$$= 81.207\times10^6\,\mathrm{Pa}$$

Lagrange Polynomial

In this method, given $(x_1, y_1), \ldots, (x_n, y_n)$, one can fit a $(n-1)^{th}$ order Lagrangian polynomial given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where ' n ' in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $(n+1)$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.

Lagrange Polynomial Cont.

Then to find the first derivative, one can differentiate $f_n(x)$ once, and so on for other derivatives.

For example, the second order Lagrange polynomial passing through $(x_0, y_0),(x_1, y_1),(x_2, y_2)$ is

$$f_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Differentiating equation (2) gives

Lagrange Polynomial Cont.

$$f_2'(x) = \frac{2x-(x_1+x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{2x-(x_0+x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{2x-(x_0+x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Differentiating again would give the second derivative as

$$f_2''(x) = \frac{2}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{2}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{2}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Example 3

To find the stress concentration around a hole in a plate under a uniform stress, a finite difference program has been written that calculates the radial and tangential displacements at different points in the plate. To find the stresses and hence the stress concentration factor, one needs to find the derivatives of these displacements. In Table 3 the radial displacements u, are given along the y-axes. The radius of the hole is 1.0 cm

a) At $x = 0$, if the radial strain, ε_r , is given by $\varepsilon_r = \frac{\partial u}{\partial r}$, find the radial strain at $r = 1.1\text{cm}$. Use a second order Lagrange polynomial interpolant for calculating the radial strain.

b) If the tangential strain at $r = 1.1\text{cm}, \theta = 90^0$ is given to you as $\varepsilon_\theta = 0.0029733$, find the hoop stress, σ_θ , at $r = 1.1\text{cm}, \theta = 90^0$ if $\sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta)$, where $E = 200\,\text{GPa}$ and $\nu = 0.3$.

Example 3 Cont.

Table 3 Radial displacement as a function of location.

$r(cm)$	$u(cm)$
1.0	−0.0010000
1.1	−0.0010689
1.2	−0.0011088
1.3	−0.0011326
1.4	−0.0011474
1.5	−0.0011574
1.6	−0.0011650
1.7	−0.0011718
1.8	−0.0011785
1.9	−0.0011857

Example 3 Cont.

Solution:

For second order Lagrangian interpolation, we choose the radial displacement given by

$$u(r) = \left(\frac{r-r_1}{r_0-r_1}\right)\left(\frac{r-r_2}{r_0-r_2}\right)u(r_0) + \left(\frac{r-r_0}{r_1-r_0}\right)\left(\frac{r-r_2}{r_1-r_2}\right)u(r_1) + \left(\frac{r-r_0}{r_2-r_0}\right)\left(\frac{r-r_1}{r_2-r_1}\right)u(r_2)$$

Since we want to find the rate of change in the radial displacement at $r = 1.1\,\text{cm}$, and we are using second order Lagrangian interpolation, we need to choose the three points closest to $r = 1.1\,\text{cm}$ that also bracket $r = 1.1\,\text{cm}$ to evaluate it.

The three points are $r_0 = 1.0,\ r_1 = 1.1,\ \text{and}\ r_2 = 1.2$.

$$r_0 = 1.0,\ \ u(r_0) = -0.0010000$$

$$r_1 = 1.1,\ \ u(r_1) = -0.0010689$$

$$r_2 = 1.2,\ \ u(r_2) = -0.0011088$$

Example 3 Cont.

The change in the radial displacement at $r = 1.1\,\text{cm}$ is given by

$$\frac{du(1.1)}{dr} = \frac{d}{dr}u(r)\Big|_{r=1.1}$$

Hence

$$u'(r) = \frac{2r-(r_1+r_2)}{(r_0-r_1)(r_0-r_2)}u(r_0) + \frac{2r-(r_0+r_2)}{(r_1-r_0)(r_1-r_2)}u(r_1) + \frac{2r-(r_0+r_1)}{(r_2-r_0)(r_2-r_1)}u(r_2)$$

$$u'(1.1) = \frac{2(1.1)-(1.1+1.2)}{(1.0-1.1)(1.0-1.2)}(-0.0010000) + \frac{2(1.1)-(1.0+1.2)}{(1.1-1.0)(1.1-1.2)}(-0.0010689)$$

$$+\frac{2(1.1)-(1.0+1.1)}{(1.2-1.0)(1.2-1.1)}(-0.0011088)$$

$$= -5(-0.0010000) + 0(-0.0010689) + 5(-0.0011088)$$

$$= -0.00054400\,\text{cm/cm}$$

Example 3 Cont.

b) $$\sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta)$$

$$= \frac{2\times 10^{11}}{1-0.3^2}(-0.00054400 + 0.3\times 0.0029733)$$

$$= 76.481\times 10^6\,\text{Pa}$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/discrete_02dif.html

THE END

http://numericalmethods.eng.usf.edu