Chapter 03.02
Solution of Cubic Equations

After reading this chapter, you should be able to:

1. *find the exact solution of a general cubic equation.*

How to Find the Exact Solution of a General Cubic Equation

In this chapter, we are going to find the exact solution of a general cubic equation

$$ax^3 + bx^2 + cx + d = 0 \tag{1}$$

To find the roots of Equation (1), we first get rid of the quadratic term $\left(x^2\right)$ by making the substitution

$$x = y - \frac{b}{3a} \tag{2}$$

to obtain

$$a\left(y - \frac{b}{3a}\right)^3 + b\left(y - \frac{b}{3a}\right)^2 + c\left(y - \frac{b}{3a}\right) + d = 0 \tag{3}$$

Expanding Equation (3) and simplifying, we obtain the following equation

$$ay^3 + \left(c - \frac{b^2}{3a}\right)y + \left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right) = 0 \tag{4}$$

Equation (4) is called the depressed cubic since the quadratic term is absent. Having the equation in this form makes it easier to solve for the roots of the cubic equation (Click here to know the history behind solving cubic equations exactly).

First, convert the depressed cubic Equation (4) into the form

$$y^3 + \frac{1}{a}\left(c - \frac{b^2}{3a}\right)y + \frac{1}{a}\left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right) = 0$$

$$y^3 + ey + f = 0 \tag{5}$$

where

$$e = \frac{1}{a}\left(c - \frac{b^2}{3a}\right)$$

$$f = \frac{1}{a}\left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right)$$

Now, reduce the above equation using Vieta's substitution

$$y = z + \frac{s}{z} \tag{6}$$

For the time being, the constant s is undefined. Substituting into the depressed cubic Equation (5), we get

$$\left(z + \frac{s}{z}\right)^3 + e\left(z + \frac{s}{z}\right) + f = 0 \tag{7}$$

Expanding out and multiplying both sides by z^3, we get

$$z^6 + (3s + e)z^4 + fz^3 + s(3s + e)z^2 + s^3 = 0 \tag{8}$$

Now, let $s = -\frac{e}{3}$ (s is no longer undefined) to simplify the equation into a tri-quadratic equation.

$$z^6 + fz^3 - \frac{e^3}{27} = 0 \tag{9}$$

By making one more substitution, $w = z^3$, we now have a general quadratic equation which can be solved using the quadratic formula.

$$w^2 + fw - \frac{e^3}{27} = 0 \tag{10}$$

Once you obtain the solution to this quadratic equation, back substitute using the previous substitutions to obtain the roots to the general cubic equation.

$$w \to z \to y \to x$$

where we assumed

$$w = z^3 \tag{11}$$

$$y = z + \frac{s}{z}$$

$$s = -\frac{e}{3} \tag{12}$$

$$x = y - \frac{b}{3a}$$

Note: You will get two roots for w as Equation (10) is a quadratic equation. Using Equation (11) would then give you three roots for each of the two roots of w, hence giving you six root values for z. But the six root values of z would give you six values of y (Equation (6)); but three values of y will be identical to the other three. So one gets only three values of y, and hence three values of x. (Equation (2))

Example 1

Find the roots of the following cubic equation.

$$x^3 - 9x^2 + 36x - 80 = 0$$

Solution

For the general form given by Equation (1)

$$ax^3 + bx^2 + cx + d = 0$$

we have

$$a = 1,\ b = -9,\ c = 36,\ d = -80$$

in

$$x^3 - 9x^2 + 36x - 80 = 0 \tag{E1-1}$$

Equation (E1-1) is reduced to

$$y^3 + ey + f = 0$$

where

$$e = \frac{1}{a}\left(c - \frac{b^2}{3a}\right)$$
$$= \frac{1}{1}\left(36 - \frac{(-9)^2}{3(1)}\right)$$
$$= 9$$

and

$$f = \frac{1}{a}\left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right)$$
$$= \frac{1}{1}\left(-80 + \frac{2(-9)^3}{27(1)^2} - \frac{(-9)(36)}{3(1)}\right)$$
$$= -26$$

giving

$$y^3 + 9y - 26 = 0 \tag{E1-2}$$

For the general form given by Equation (5)

$$y^3 + ey + f = 0$$

we have

$$e = 9,\ f = -26$$

in Equation (E1-2).

From Equation (12)

$$s = -\frac{e}{3}$$
$$= -\frac{9}{3}$$
$$= -3$$

From Equation (10)

$$w^2 + fw - \frac{e^3}{27} = 0$$
$$w^2 - 26w - \frac{9^3}{27} = 0$$
$$w^2 - 26w - 27 = 0$$

where

$$w = z^3$$

and

$$y = z + \frac{s}{z}$$

$$= z - \frac{3}{z}$$

$$w = \frac{-(-26) \pm \sqrt{(-26)^2 - 4(1)(-27)}}{2(1)}$$

$$= 27, -1$$

The solution is

$$w_1 = 27$$

$$w_2 = -1$$

Since

$$w = z^3$$

$$z^3 = w$$

For $w = w_1$

$$z^3 = w_1$$

$$= 27$$

$$= 27e^{i0}$$

Since

$$w = z^3$$

$$re^{i\theta} = \left(ue^{i\alpha}\right)^3 = u^3 e^{3i\alpha}$$

$$r(\cos\theta + i\sin\theta) = u^3(\cos 3\alpha + i\sin 3\alpha)$$

resulting in

$$r = u^3$$

$$\cos\theta = \cos 3\alpha$$

$$\sin\theta = \sin 3\alpha$$

Since $\sin\theta$ and $\cos\theta$ are periodic of 2π,

$$3\alpha = \theta + 2\pi k$$

$$\alpha = \frac{\theta + 2\pi k}{3}$$

k will take the value of 0, 1 and 2 before repeating the same values of α.

So,

$$\alpha = \frac{\theta + 2\pi k}{3}, k = 0, 1, 2$$

$$\alpha_1 = \frac{\theta}{3}$$

$$\alpha_2 = \frac{(\theta + 2\pi)}{3}$$

$$\alpha_3 = \frac{(\theta + 4\pi)}{3}$$

So roots of $w = z^3$ are

$$z_1 = r^{\frac{1}{3}}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right)$$

$$z_2 = r^{\frac{1}{3}}\left(\cos\frac{\theta + 2\pi}{3} + i\sin\frac{\theta + 2\pi}{3}\right)$$

$$z_3 = r^{\frac{1}{3}}\left(\cos\frac{\theta + 4\pi}{3} + i\sin\frac{\theta + 4\pi}{3}\right)$$

gives

$$z_1 = (27)^{1/3}\left(\cos\frac{0}{3} + i\sin\frac{0}{3}\right)$$

$$= 3$$

$$z_2 = (27)^{1/3}\left(\cos\frac{0 + 2\pi}{3} + i\sin\frac{0 + 2\pi}{3}\right)$$

$$= 3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$= 3\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= -\frac{3}{2} + i\frac{3\sqrt{3}}{2}$$

$$z_3 = (27)^{1/3}\left(\cos\frac{0 + 4\pi}{3} + i\sin\frac{0 + 4\pi}{3}\right)$$

$$= 3\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

$$= 3\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= -\frac{3}{2} - i\frac{3\sqrt{3}}{2}$$

Since

$$y = z - \frac{3}{z}$$

$$y_1 = z_1 - \frac{3}{z_1}$$

$$= 3 - \frac{3}{3}$$

$$= 2$$

$$\begin{aligned}
y_2 &= z_2 - \frac{3}{z_2} \\
&= \left(-\frac{3}{2} + i\frac{3\sqrt{3}}{2}\right) - \frac{3}{\left(-\frac{3}{2} + i\frac{3\sqrt{3}}{2}\right)} \\
&= -\frac{5 + 3i\sqrt{3}}{-1 + i\sqrt{3}} \\
&= -\frac{5 + 3i\sqrt{3}}{-1 + i\sqrt{3}} \times \frac{-1 - i\sqrt{3}}{-1 - i\sqrt{3}} \\
&= -1 + i2\sqrt{3}
\end{aligned}$$

$$\begin{aligned}
y_3 &= z_3 - \frac{3}{z_3} \\
&= \left(-\frac{3}{2} - i\frac{3\sqrt{3}}{2}\right) - \frac{3}{\left(-\frac{3}{2} - i\frac{3\sqrt{3}}{2}\right)} \\
&= \frac{5 - i3\sqrt{3}}{1 + i\sqrt{3}} \\
&= \frac{5 - i3\sqrt{3}}{1 + i\sqrt{3}} \times \frac{1 - i\sqrt{3}}{1 - i\sqrt{3}} \\
&= -1 - i2\sqrt{3}
\end{aligned}$$

Since

$$x = y + 3$$

$$\begin{aligned}
x_1 &= y_1 + 3 \\
&= 2 + 3 \\
&= 5
\end{aligned}$$

$$\begin{aligned}
x_2 &= y_2 + 3 \\
&= \left(-1 + i2\sqrt{3}\right) + 3 \\
&= 2 + i2\sqrt{3}
\end{aligned}$$

$$\begin{aligned}
x_3 &= y_3 + 3 \\
&= \left(-1 - i2\sqrt{3}\right) + 3 \\
&= 2 - i2\sqrt{3}
\end{aligned}$$

The roots of the original cubic equation

$$x^3 - 9x^2 + 36x - 80 = 0$$

are $x_1, x_2,$ and x_3, that is,

$$5,\ 2 + i2\sqrt{3},\ 2 - i2\sqrt{3}$$

Verifying

$$(x-5)\left(x-\left(2+i2\sqrt{3}\right)\right)\left(x-\left(2-i2\sqrt{3}\right)\right)=0$$

gives

$$x^3-9x^2+36x-80=0$$

Using

$$w_2=-1$$

would yield the same values of the three roots of the equation. Try it.

Example 2

Find the roots of the following cubic equation

$$x^3-0.03x^2+2.4\times10^{-6}=0$$

Solution

For the general form

$$ax^3+bx^2+cx+d=0$$

$$a=1, b=-0.03, c=0, d=2.4\times10^{-6}$$

Depress the cubic equation by letting (Equation (2))

$$\begin{aligned}x&=y-\frac{b}{3a}\\&=y-\frac{(-0.03)}{3(1)}\\&=y+0.01\end{aligned}$$

Substituting the above equation into the cubic equation and simplifying, we get

$$y^3-\left(3\times10^{-4}\right)y+\left(4\times10^{-7}\right)=0$$

That gives $e=-3\times10^{-4}$ and $f=4\times10^{-7}$ for Equation (5), that is, $y^3+ey+f=0$.

Now, solve the depressed cubic equation by using Vieta's substitution as

$$y=z+\frac{s}{z}$$

to obtain

$$z^6+\left(3s-3\times10^{-4}\right)z^4+\left(4\times10^{-7}\right)z^3+s\left(3s-3\times10^{-4}\right)z^2+s^3=0$$

Letting

$$s=-\frac{e}{3}=-\frac{-3\times10^{-4}}{3}=10^{-4}$$

we get the following tri-quadratic equation

$$z^6+\left(4\times10^{-7}\right)z^3+1\times10^{-12}=0$$

Using the following conversion, $w=z^3$, we get a general quadratic equation

$$w^2+\left(4\times10^{-7}\right)w+\left(1\times10^{-12}\right)=0$$

Using the quadratic equation, the solutions for w are

$$w=\frac{-4\times10^{-7}\pm\sqrt{\left(4\times10^{-7}\right)^2-4(1)\left(1\times10^{-12}\right)}}{2(1)}$$

giving

$$w_1 = -2\times10^{-7} + i\left(9.79795897113\times10^{-7}\right)$$
$$w_2 = -2\times10^{-7} - i\left(9.79795897113\times10^{-7}\right)$$

Each solution of $w = z^3$ yields three values of z. The three values of z from w_1 are in rectangular form.

Since

$$w = z^3$$

Then

$$z = w^{\frac{1}{3}}$$

Let

$$w = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

then

$$z = u(\cos\alpha + i\sin\alpha) = ue^{i\alpha}$$

This gives

$$w = z^3$$
$$re^{i\theta} = \left(ue^{i\alpha}\right)^3 = u^3 e^{3i\alpha}$$
$$r(\cos\theta + i\sin\theta) = u^3(\cos 3\alpha + i\sin 3\alpha)$$

resulting in

$$r = u^3$$
$$\cos\theta = \cos 3\alpha$$
$$\sin\theta = \sin 3\alpha$$

Since $\sin\theta$ and $\cos\theta$ are periodic of 2π,

$$3\alpha = \theta + 2\pi k$$
$$\alpha = \frac{\theta + 2\pi k}{3}$$

k will take the value of 0, 1 and 2 before repeating the same values of α.

So,

$$\alpha = \frac{\theta + 2\pi k}{3}, k = 0, 1, 2$$
$$\alpha_1 = \frac{\theta}{3}$$
$$\alpha_2 = \frac{(\theta + 2\pi)}{3}$$
$$\alpha_3 = \frac{(\theta + 4\pi)}{3}$$

So the roots of $w = z^3$ are

$$z_1 = r^{\frac{1}{3}}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right)$$
$$z_2 = r^{\frac{1}{3}}\left(\cos\frac{\theta + 2\pi}{3} + i\sin\frac{\theta + 2\pi}{3}\right)$$

$$z_3 = r^{\frac{1}{3}}\left(\cos\frac{\theta+4\pi}{3} + i\sin\frac{\theta+4\pi}{3}\right)$$

So for

$$w_1 = -2\times10^{-7} + i\left(9.79795897113\times10^{-7}\right)$$

$$r = \sqrt{\left(-2\times10^{-7}\right)^2 + \left(9.79795897113\times10^{-7}\right)^2}$$

$$= 1\times10^{-6}$$

$$\theta = \tan^{-1}\frac{9.79795897113\times10^{-7}}{-2\times10^{-7}}$$

$= 1.772154248$ (2nd quadrant because y (the numerator) is positive and x (the denominator) is negative)

$$z_1 = \left(1\times10^{-6}\right)^{\frac{1}{3}}\left(\cos\frac{1.772154248}{3} + i\sin\frac{1.772154248}{3}\right)$$

$$= 0.008305409517 + i0.005569575635$$

$$z_2 = \left(1\times10^{-6}\right)^{\frac{1}{3}}\left(\cos\frac{1.772154248+2\pi}{3} + i\sin\frac{1.772154248+2\pi}{3}\right)$$

$$= -0.008976098746 + i0.004407907815$$

$$z_3 = \left(1\times10^{-6}\right)^{\frac{1}{3}}\left(\cos\frac{1.772154248+4\pi}{3} + i\sin\frac{1.772154248+4\pi}{3}\right)$$

$$= 0.0006706892313 - i0.009977483448$$

Compiling

$$z_1 = 0.008305409518 + i0.005569575634$$

$$z_2 = -0.008976098746 + i0.004407907814$$

$$z_3 = 6.70689228525\times10^{-4} - i0.009977483448$$

Similarly, the three values of z from w_2 in rectangular form are

$$z_4 = 0.008305409518 - i0.005569575634$$

$$z_5 = -0.008976098746 - i0.004407907814$$

$$z_6 = 6.70689228525\times10^{-4} + i0.009977483448$$

Using Vieta's substitution (Equation (6)),

$$y = z + \frac{s}{z}$$

$$y = z + \frac{\left(1\times10^{-4}\right)}{z}$$

we back substitute to find three values for y.

For example, choosing

$$z_1 = 0.008305409518 + i0.005569575634$$

gives

$$y_1 = 0.008305409518 + i0.005569575634 + \frac{1\times10^{-4}}{0.008305409518 + i0.005569575634}$$

$$= 0.008305409518 + i0.005569575634$$
$$+ \frac{1\times 10^{-4}}{0.008305409078 + i0.00556957634} \times \frac{0.008305409518 - i0.00556957634}{0.008305409518 - i0.00556957634}$$
$$= 0.008305409518 + i0.005569575634$$
$$+ \frac{1\times 10^{-4}}{1\times 10^{-4}}(0.008305409518 - i0.00556957634)$$
$$= 0.016610819036$$

The values of z_1, z_2 and z_3 give

$$y_1 = 0.016610819036$$
$$y_2 = -0.01795219749$$
$$y_3 = 0.001341378457$$

respectively. The three other z values of z_4, z_5 and z_6 give the same values as y_1, y_2 and y_3, respectively.

Now, using the substitution of

$$x = y + 0.01$$

the three roots of the given cubic equation are

$$x_1 = 0.016610819036 + 0.01$$
$$= 0.026610819036$$
$$x_2 = -0.01795219749 + 0.01$$
$$= -0.00795219749$$
$$x_3 = 0.001341378457 + 0.01$$
$$= 0.011341378457$$

NONLINEAR EQUATIONS	
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