

Bisection Method

Civil Engineering Majors

Author(s): Autar Kaw, Jai Paul

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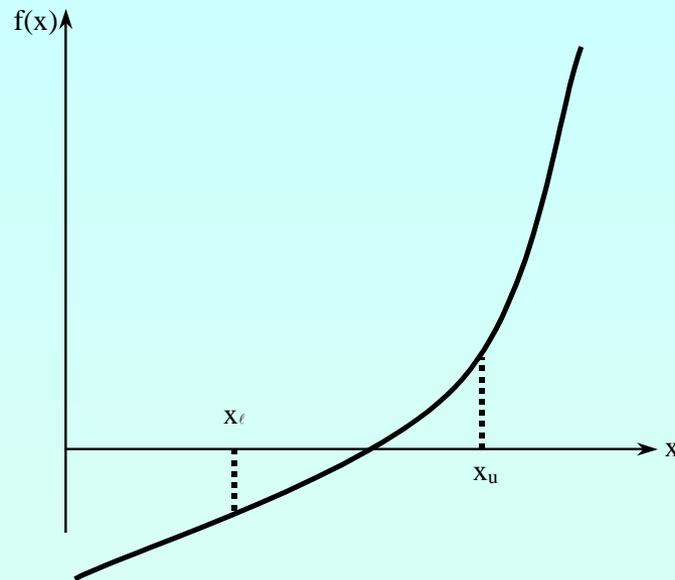
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Bisection Method

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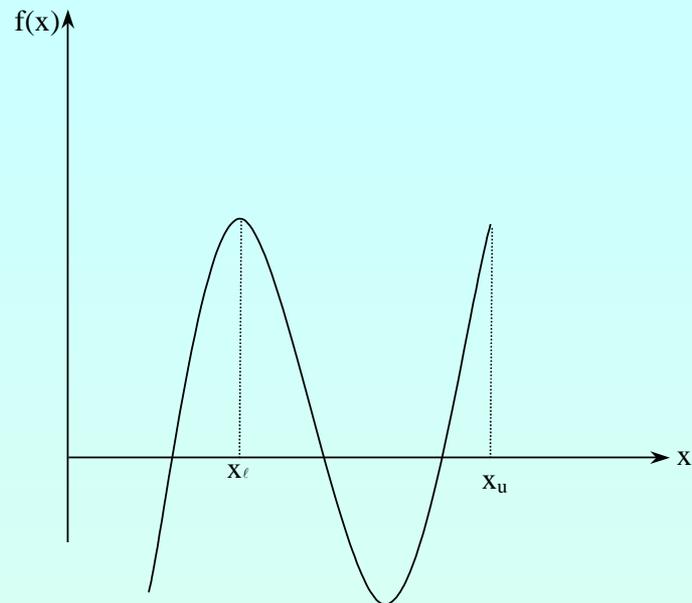
Basis of Bisection Method

Theorem: An equation $f(x)=0$, where $f(x)$ is a real continuous function, has at least one root between x_l and x_u if $f(x_l) f(x_u) < 0$.



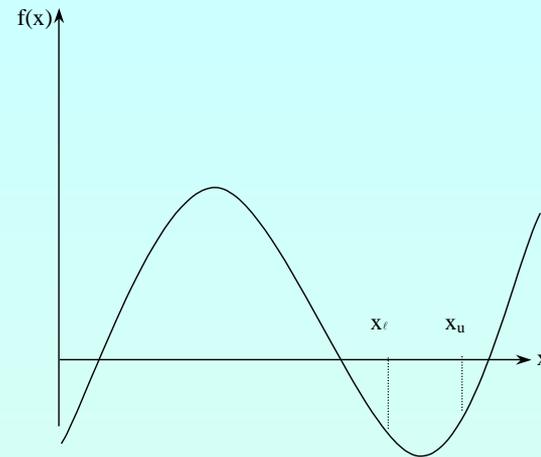
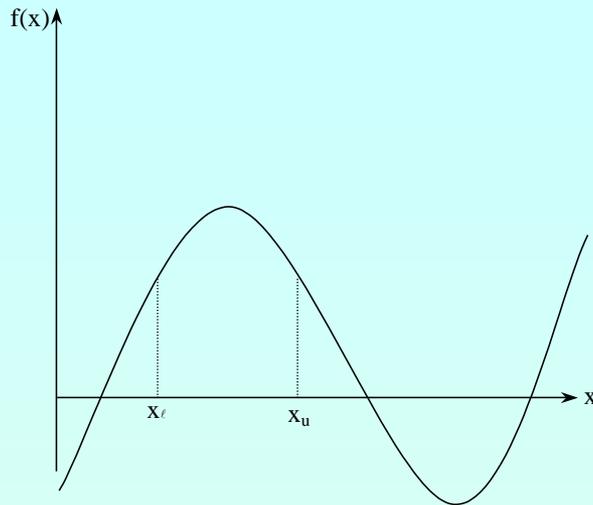
Theorem

If function $f(x)$ in $f(x)=0$ does not change sign between two points, roots may still exist between the two points.



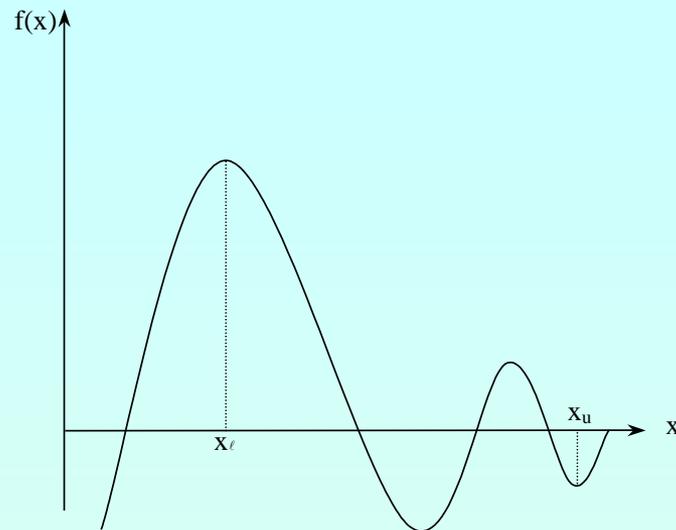
Theorem

If the function $f(x)$ in $f(x)=0$ does not change sign between two points, there may not be any roots between the two points.



Theorem

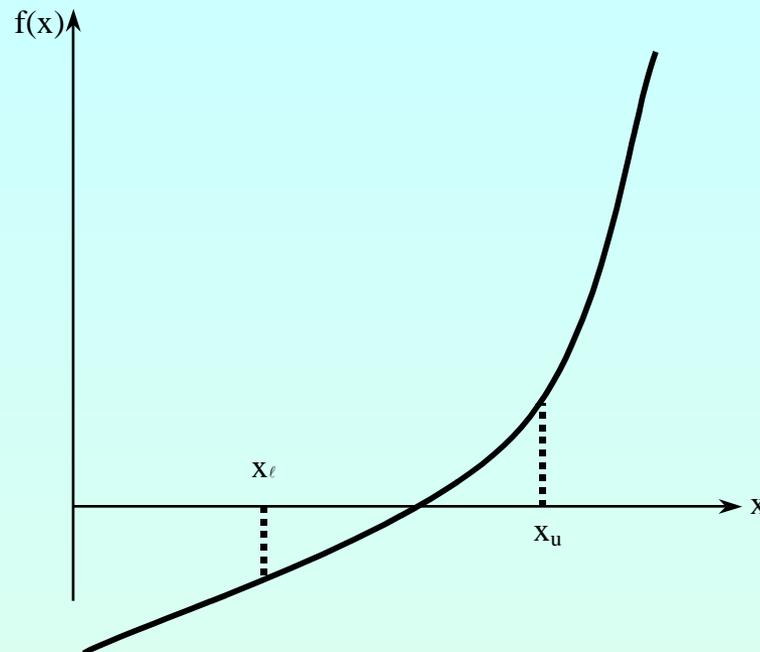
If the function $f(x)$ in $f(x)=0$ changes sign between two points, more than one root may exist between the two points.



Algorithm for Bisection Method

Step 1

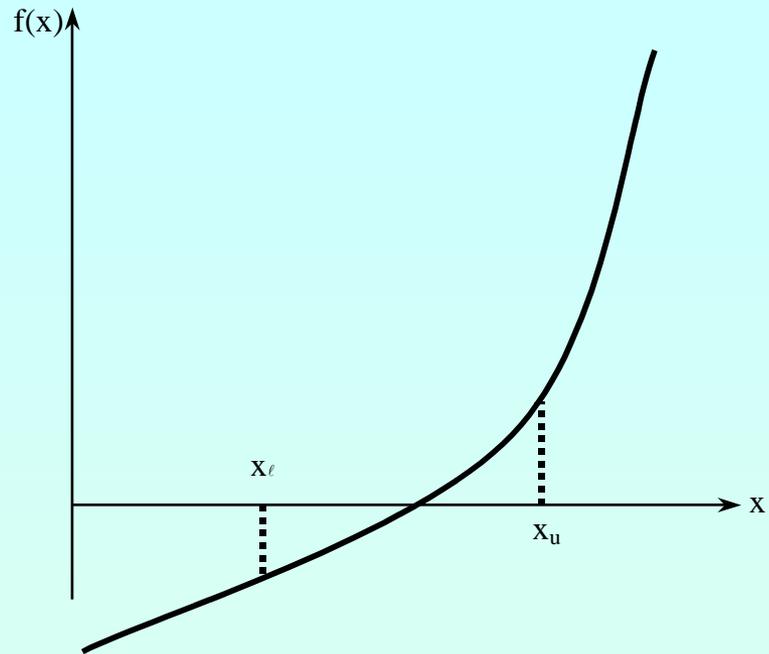
- Choose x_l and x_u as two guesses for the root such that $f(x_l) f(x_u) < 0$, or in other words, $f(x)$ changes sign between x_l and x_u .



Step 2

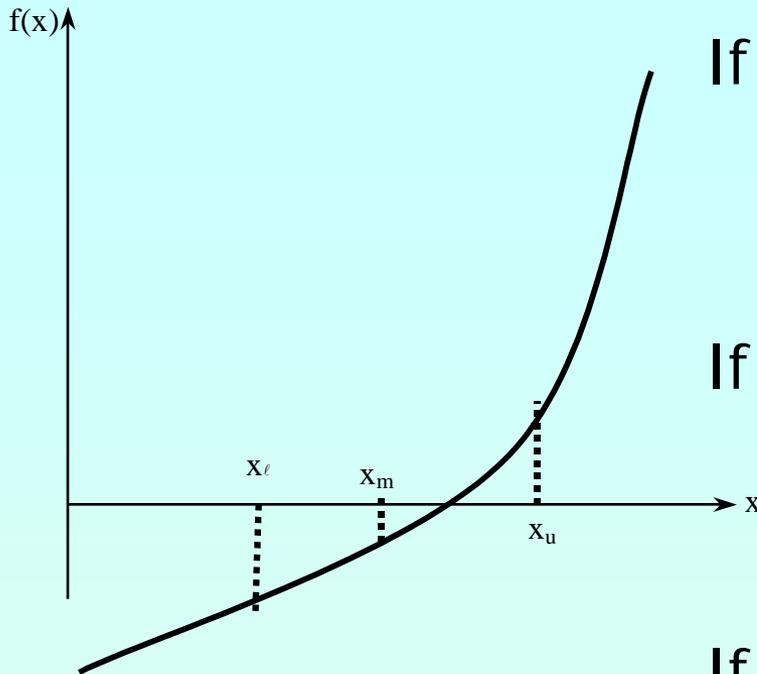
Estimate the root, x_m of the equation $f(x) = 0$ as the mid-point between x_ℓ and x_u as

$$x_m = \frac{x_\ell + x_u}{2}$$



Step 3

Now check the following



If $f(x_l) f(x_m) < 0$, then the root lies between x_l and x_m ; then $x_l = x_l$; $x_u = x_m$.

If $f(x_l) f(x_m) > 0$, then the root lies between x_m and x_u ; then $x_l = x_m$; $x_u = x_u$.

If $f(x_l) f(x_m) = 0$; then the root is x_m .
Stop the algorithm if this is true.

Step 4

New estimate

$$x_m = \frac{x_\ell + x_u}{2}$$

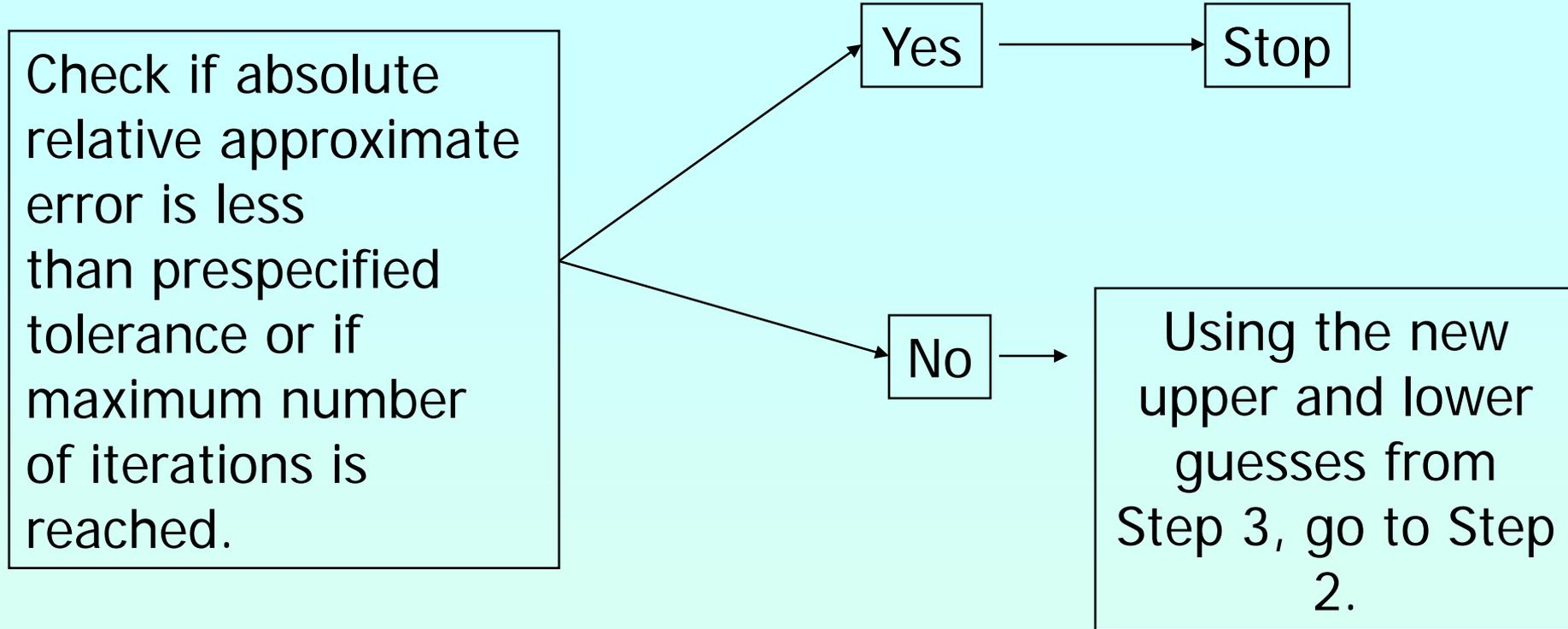
Absolute Relative Approximate Error

$$|\epsilon_a| = \left| \frac{x_m^{new} - x_m^{old}}{x_m^{new}} \right| \times 100$$

x_m^{old} = previous estimate of root

x_m^{new} = current estimate of root

Step 5



Example 1

You are making a bookshelf to carry books that range from 8 ½ " to 11" in height and would take 29" of space along length. The material is wood having Young's Modulus 3.667 Msi, thickness 3/8 " and width 12". You want to find the maximum vertical deflection of the bookshelf. The vertical deflection of the shelf is given by

$$v(x) = 0.42493 \times 10^{-4} x^3 - 0.13533 \times 10^{-8} x^5 - 0.66722 \times 10^{-6} x^4 - 0.018507 x$$

where x is the position along the length of the beam. Hence to find the maximum deflection we need to find where $f(x) = \frac{dv}{dx} = 0$ and conduct the second derivative test.

Example 1 Cont.

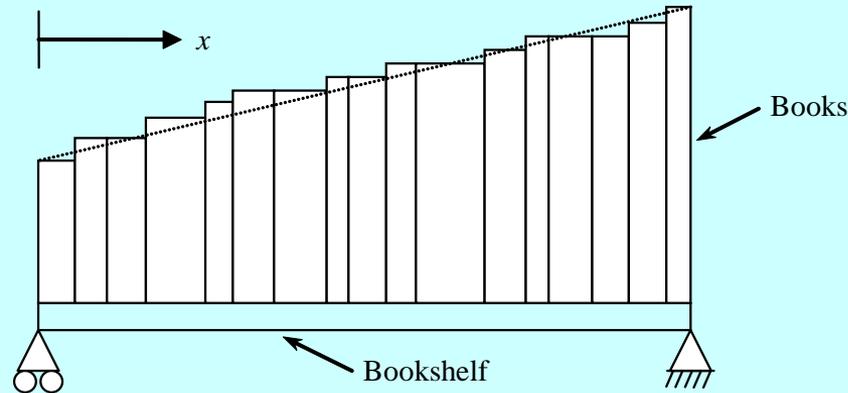


Figure 5 A loaded bookshelf.

The equation that gives the position x where the deflection is maximum is given by

$$f(x) = -0.67665 \times 10^{-8} x^4 - 0.26689 \times 10^{-5} x^3 + 0.12748 \times 10^{-3} x^2 - 0.018507 = 0$$

Use the bisection method of finding roots of equations to find the position x where the deflection is maximum. Conduct three iterations to estimate the root of the above equation. Find the absolute relative approximate error at the end of each iteration and the number of significant digits at least correct at the end of each iteration.

Example 1 Cont.

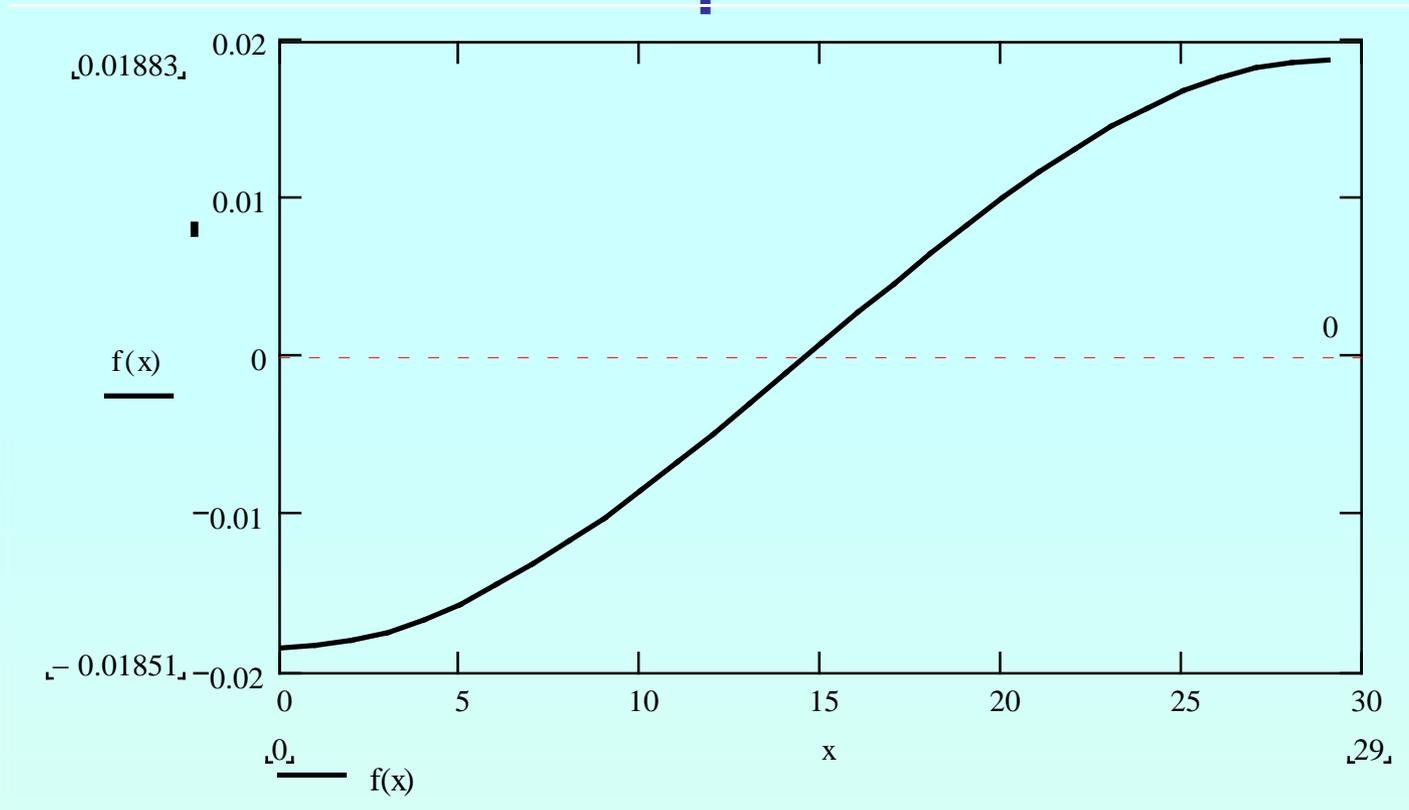


Figure 6 Graph of the function $f(x)$.

$$f(x) = -0.67665 \times 10^{-8} x^4 - 0.26689 \times 10^{-5} x^3 + 0.12748 \times 10^{-3} x^2 - 0.018507 = 0$$

Example 1 Cont.

Solution

From the physics of the problem, the maximum deflection would be between $x = 0$ and $x = L$, where

$L =$ length of the bookshelf

that is

$$0 \leq x \leq L$$

$$0 \leq x \leq 29$$

Let us assume

$$x_l = 0, x_u = 29$$

Example 1 Cont.

Check if the function changes sign between x_l and x_u .

$$f(x_l) = f(0)$$

$$\begin{aligned} &= -0.67665 \times 10^{-8} (0)^4 - 0.26689 \times 10^{-5} (0)^3 + 0.12748 \times 10^{-3} (0)^2 - 0.018507 \\ &= -0.018507 \end{aligned}$$

$$f(x_u) = f(29)$$

$$\begin{aligned} &= -0.67665 \times 10^{-8} (29)^4 - 0.26689 \times 10^{-5} (29)^3 + 0.12748 \times 10^{-3} (29)^2 - 0.018507 \\ &= 0.018826 \end{aligned}$$

Hence

$$f(x_l)f(x_u) = f(0)f(29) = (-0.018507)(0.018826) < 0$$

So there is at least one root between x_l and x_u that is between 0 and 29.

Example 1 Cont.

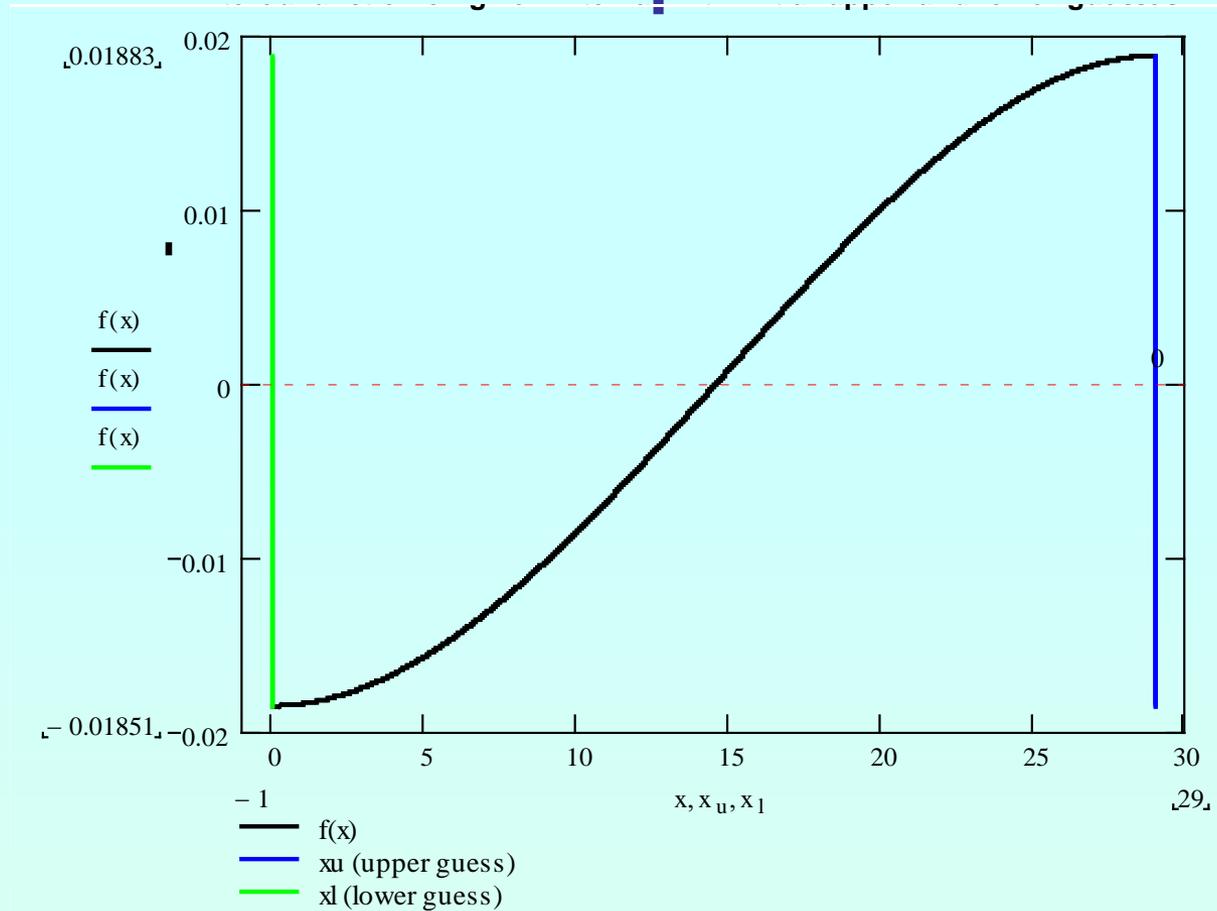


Figure 7 Checking the validity of the bracket.

Example 1 Cont.

Iteration 1

The estimate of the root is $x_m = \frac{x_l + x_u}{2} = \frac{0 + 29}{2} = 14.5$

$$f(x_m) = f(14.5)$$

$$\begin{aligned} &= -0.67665 \times 10^{-8} (14.5)^4 - 0.26689 \times 10^{-5} (14.5)^3 + 0.12748 \times 10^{-3} (14.5)^2 - 0.018507 \\ &= -1.4007 \times 10^{-4} \end{aligned}$$

$$f(x_m)f(x_u) = f(14.5)f(29) = (-1.4007 \times 10^{-4})(0.018826) < 0$$

The root is bracketed between x_m and x_u .

The lower and upper limits of the new bracket are

$$x_l = 14.5, \quad x_u = 29$$

The absolute relative approximate error $|\epsilon_a|$ cannot be calculated as we do not have a previous approximation.

Example 1 Cont.

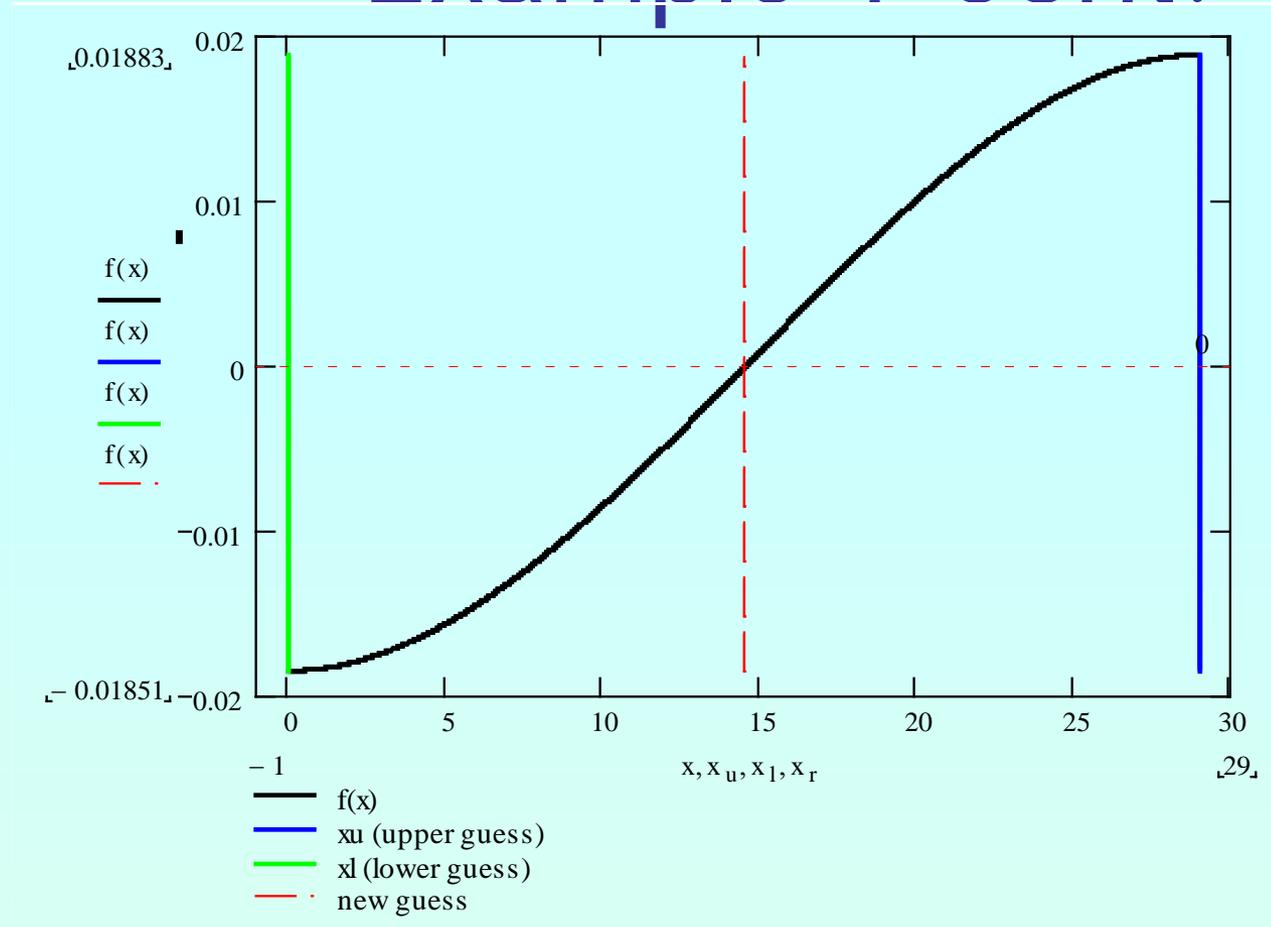


Figure 8 Graph of the estimate of the root after Iteration 1.

Example 1 Cont.

Iteration 2

The estimate of the root is $x_m = \frac{x_l + x_u}{2} = \frac{14.5 + 29}{2} = 21.75$

$$f(x_m) = f(21.75)$$

$$\begin{aligned} &= -0.67665 \times 10^{-8} (21.75)^4 - 0.26689 \times 10^{-5} (21.75)^3 + 0.12748 \times 10^{-3} (21.75)^2 - 0.018507 \\ &= 0.012824 \end{aligned}$$

$$f(x_m)f(x_u) = f(14.5)f(21.75) = (-1.4007 \times 10^{-4})(0.012824) < 0$$

The root is bracketed between x_l and x_m .

The lower and upper limits of the new bracket are

$$x_l = 14.5, \quad x_u = 21.75$$

Example 1 Cont.

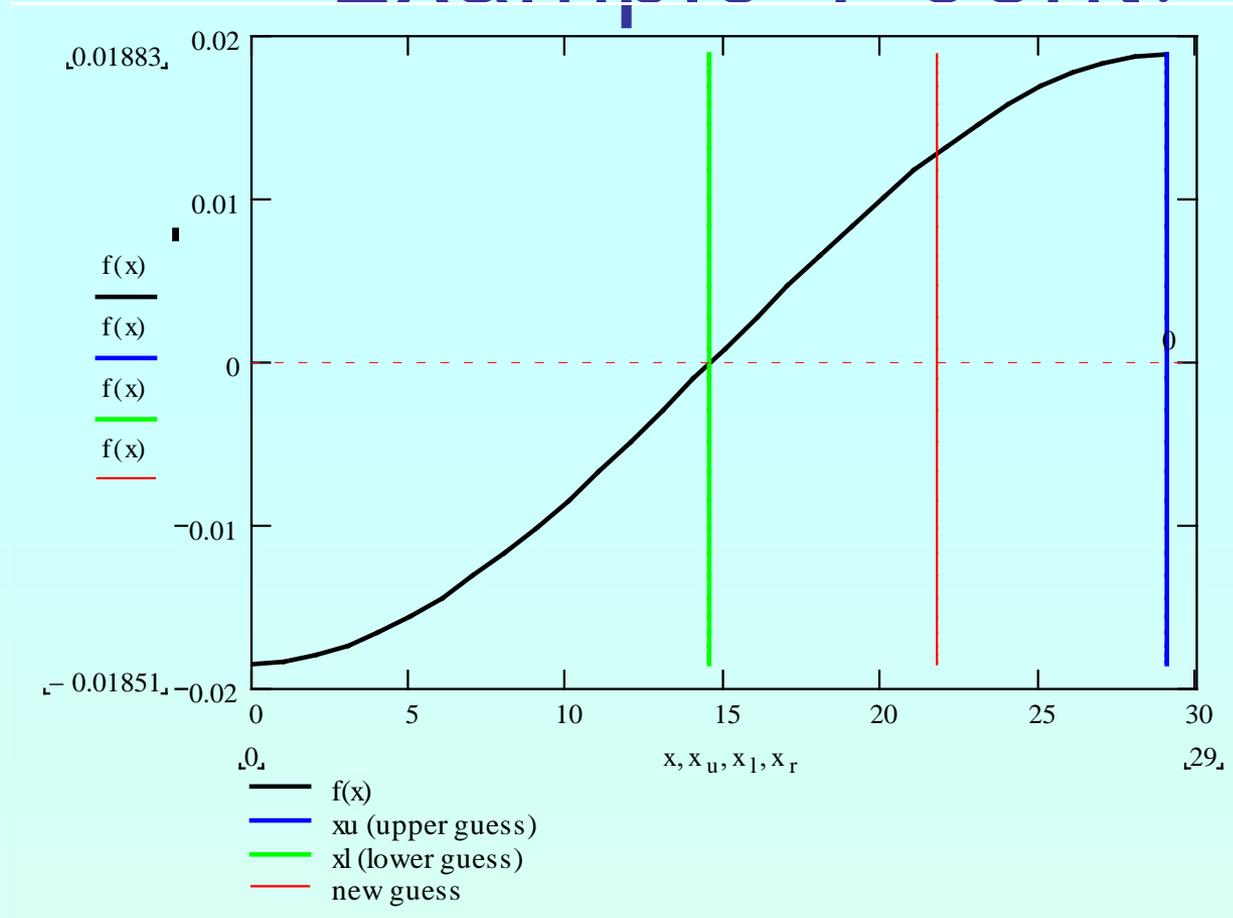


Figure 9 Graph of the estimate of the root after Iteration 2.

Example 1 Cont.

The absolute relative approximate error at the end of Iteration 2 is

$$\begin{aligned} |\epsilon_a| &= \left| \frac{x_m^{new} - x_m^{old}}{x_m^{new}} \right| \times 100 \\ &= \left| \frac{21.75 - 14.5}{21.75} \right| \times 100 \\ &= 33.333\% \end{aligned}$$

None of the significant digits are at least correct in the estimated root

$$x_m = 21.75$$

as the absolute relative approximate error is greater than 5%.

Example 1 Cont.

Iteration 3

The estimate of the root is $x_m = \frac{x_l + x_u}{2} = \frac{14.5 + 21.75}{2} = 18.125$

$$f(x_m) = f(18.125)$$

$$\begin{aligned} &= -0.67665 \times 10^{-8} (18.125)^4 - 0.26689 \times 10^{-5} (18.125)^3 + 0.12748 \times 10^{-3} (18.125)^2 - 0.018507 \\ &= 6.7502 \times 10^{-3} \end{aligned}$$

$$f(x_l)f(x_m) = f(14.5)f(18.125) = (-1.4007 \times 10^{-4})(6.7502 \times 10^{-3}) < 0$$

The root is bracketed between x_l and x_m .

The lower and upper limits of the new bracket are

$$x_l = 14.5, \quad x_u = 18.125$$

Example 1 Cont.

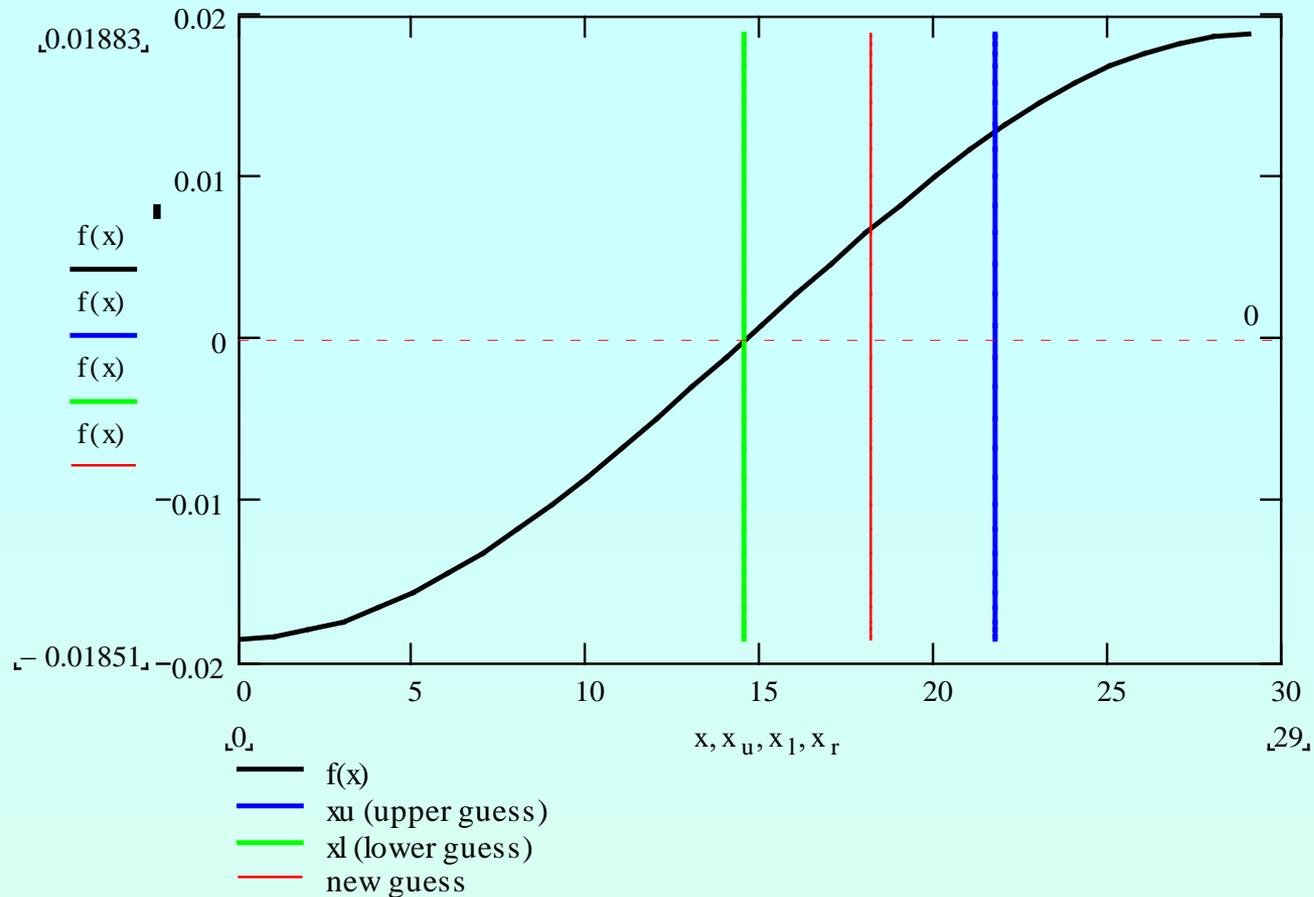


Figure 10 Graph of the estimate of the root after Iteration 3.

Example 1 Cont.

The absolute relative approximate error at the end of Iteration 3 is

$$\begin{aligned} |\epsilon_a| &= \left| \frac{x_m^{new} - x_m^{old}}{x_m^{new}} \right| \times 100 \\ &= \left| \frac{18.125 - 21.75}{18.125} \right| \times 100 \\ &= 20\% \end{aligned}$$

Still none of the significant digits are at least correct in the estimated root as the absolute relative approximate error is greater than 5%.

Seven more iterations were conducted and these iterations are shown in Table 1.

Example 1 Cont.

Table 1 Root of $f(x)=0$ as function of number of iterations for bisection method.

Iteration	x_ℓ	x_u	x_m	$ \epsilon_a \%$	$f(x_m)$
1	0	29	14.5	-----	-1.3992×10^{-4}
2	14.5	29	21.75	33.333	0.012824
3	14.5	21.75	18.125	20	6.7502×10^{-3}
4	14.5	18.125	16.313	11.111	3.3509×10^{-3}
5	14.5	16.313	15.406	5.8824	1.6099×10^{-3}
6	14.5	15.406	14.953	3.0303	7.3521×10^{-4}
7	14.5	14.953	14.727	1.5385	2.9753×10^{-4}
8	14.5	14.727	14.613	0.77519	7.8708×10^{-5}
9	14.5	14.613	14.557	0.38911	-3.0688×10^{-5}
10	14.5566	14.613	14.585	0.19417	2.4009×10^{-5}

Example 1 Cont.

At the end of the 10th iteration,

$$|\epsilon_a| = 0.19417\%$$

Hence the number of significant digits at least correct is given by the largest value of m for which

$$|\epsilon_a| \leq 0.5 \times 10^{2-m}$$

$$0.19417 \leq 0.5 \times 10^{2-m}$$

$$0.38835 \leq 10^{2-m}$$

$$\log(0.38835) \leq 2 - m$$

$$m \leq 2 - \log(0.38835) = 2.4108$$

So

$$m = 2$$

The number of significant digits at least correct in the estimated root 14.585 is 2.

Advantages

- Always convergent
- The root bracket gets halved with each iteration - guaranteed.

Drawbacks

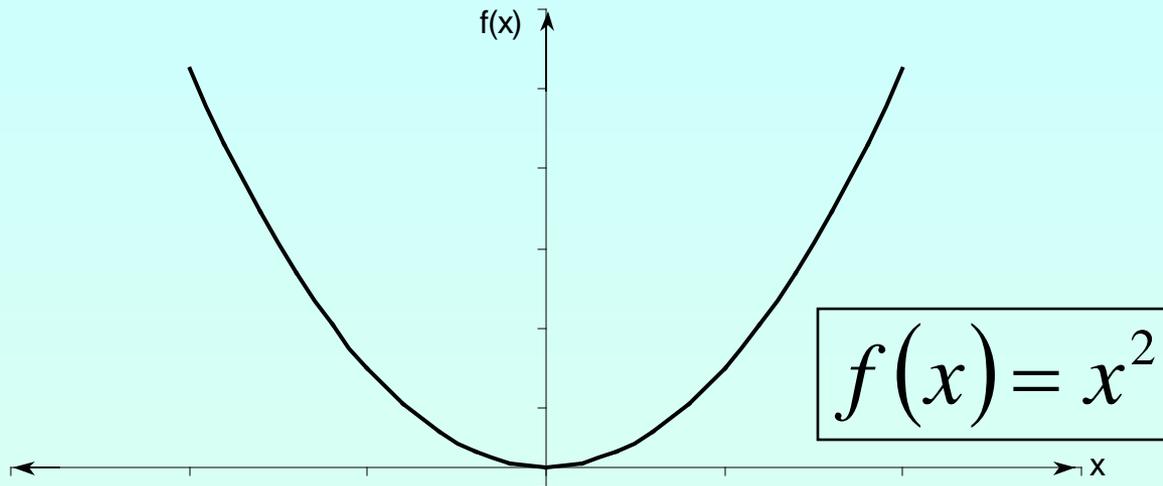
- Slow convergence

Drawbacks (continued)

- If one of the initial guesses is close to the root, the convergence is slower

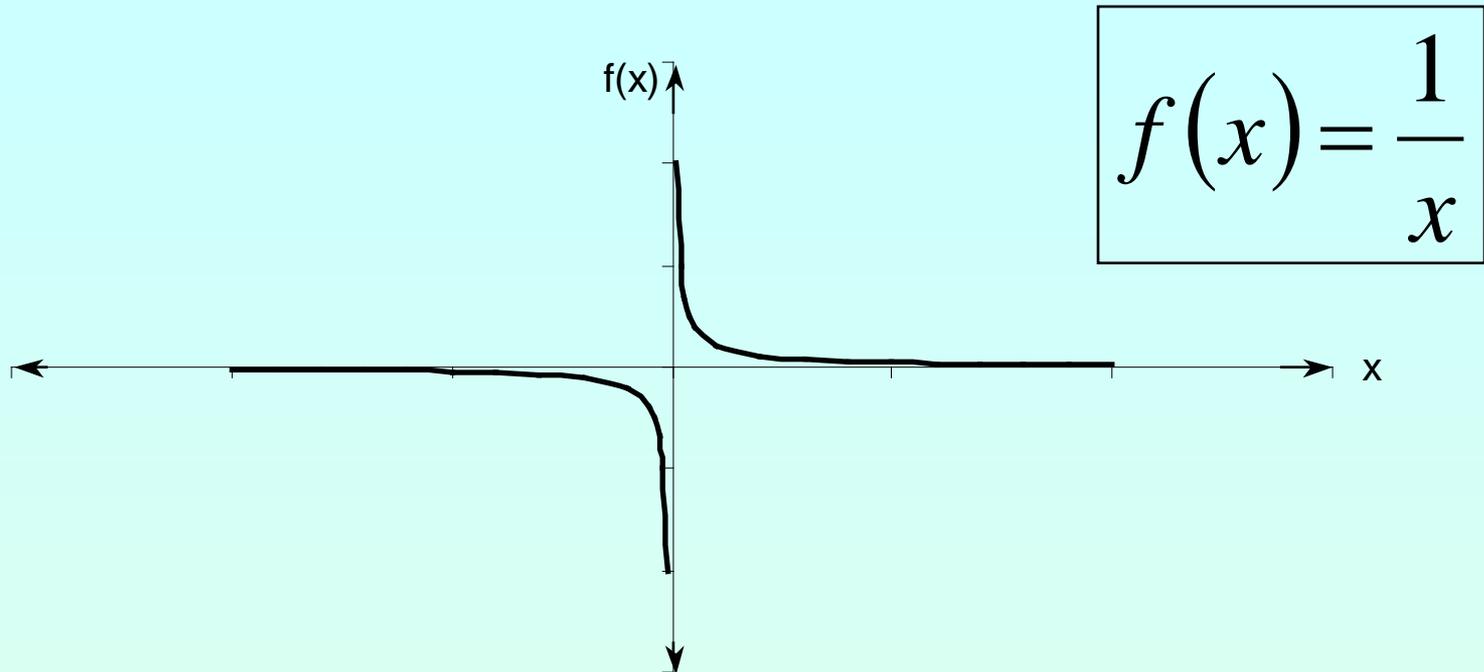
Drawbacks (continued)

- If a function $f(x)$ is such that it just touches the x -axis it will be unable to find the lower and upper guesses.



Drawbacks (continued)

- Function changes sign but root does not exist



Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/bisection_method.html

THE END

<http://numericalmethods.eng.usf.edu>