Direct Method of Interpolation

Civil Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

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What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), ... (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.
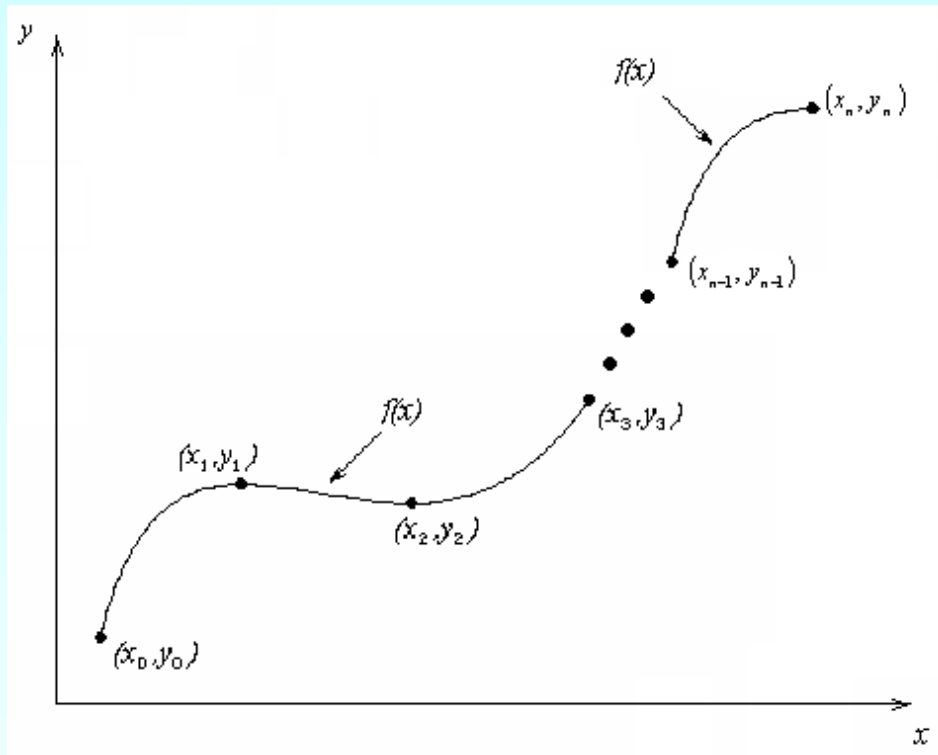

Figure 1 Interpolation of discrete.

Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate

Direct Method

Given 'n+1' data points (x_0,y_0), (x_1,y_1),............ (x_n,y_n), pass a polynomial of order 'n' through the data as given below:

$$y = a_0 + a_1 x + ... + a_n x^n.$$

where a_0, a_1,................. a_n are real constants.

- Set up 'n+1' equations to find 'n+1' constants.
- To find the value 'y' at a given value of 'x', simply substitute the value of 'x' in the above polynomial.

Example

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth plot for a lake. Determine the value of the temperature at z = −7.5 using the direct method for linear interpolation.

Temperature	Depth
T (^{o}C)	z (m)
19.1	0
19.1	-1
19	-2
18.8	-3
18.7	-4
18.3	-5
18.2	-6
17.6	-7
11.7	-8
9.9	-9
9.1	-10
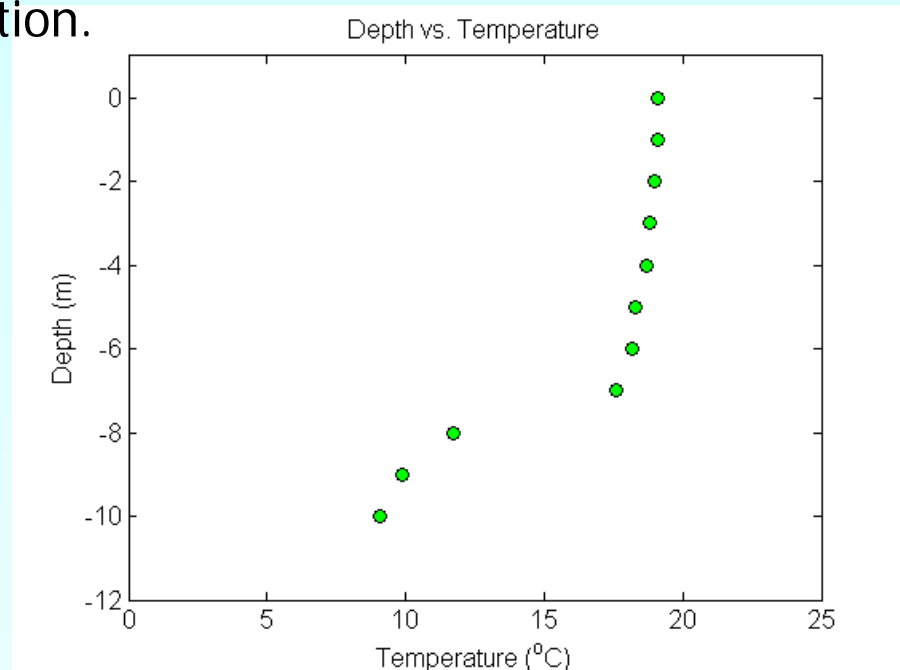
Temperature vs. depth of a lake

Linear Interpolation

$$T(z) = a_0 + a_1 z$$

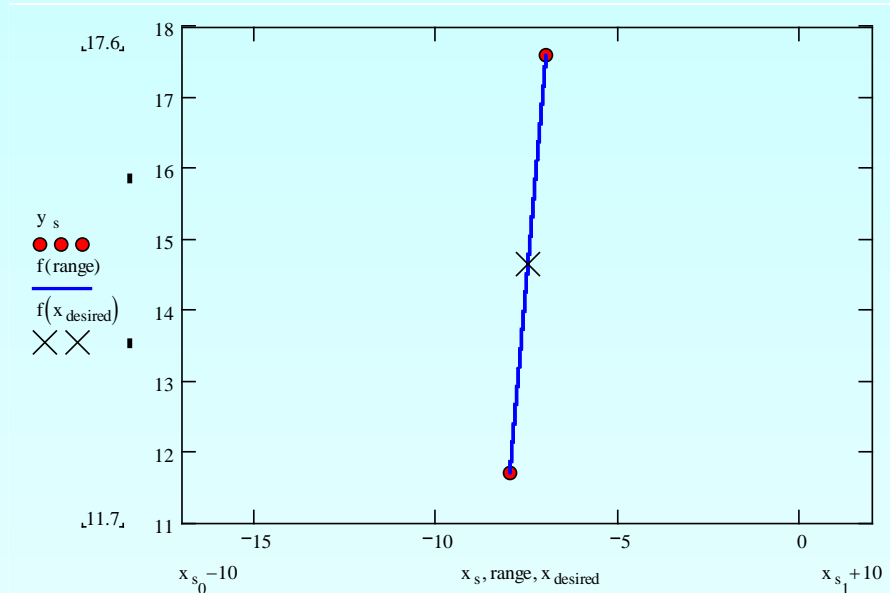$$T(-8) = a_0 + a_1(-8) = 11.7$$

$$T(-7) = a_0 + a_1(-7) = 17.6$$

Solving the above two equations gives,

$$a_0 = 58.9 \qquad a_1 = 5.9$$

Hence

$$T(z) = 58.9 + 5.9z, \quad -8 \le z \le -7$$

$$T(-7.5) = 58.9 + 5.9(-7.5) = 14.65°C$$

Example

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth plot for a lake. Determine the value of the temperature at z = −7.5 using the direct method for quadratic interpolation.

Temperature	Depth
T (^{o}C)	z (m)
19.1	0
19.1	-1
19	-2
18.8	-3
18.7	-4
18.3	-5
18.2	-6
17.6	-7
11.7	-8
9.9	-9
9.1	-10
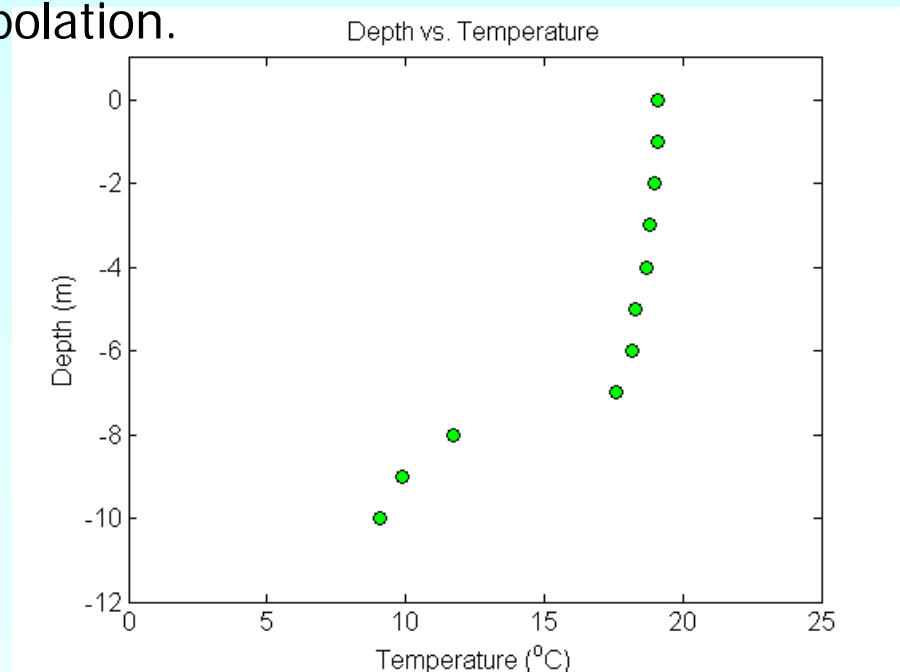
Temperature vs. depth of a lake

Quadratic Interpolation

$$T(z) = a_0 + a_1 z + a_2 z^2$$

$$T(-9) = a_0 + a_1(-9) + a_2(-9)^2 = 9.9$$

$$T(-8) = a_0 + a_1(-8) + a_2(-8)^2 = 11.7$$

$$T(-7) = a_0 + a_1(-7) + a_2(-7)^2 = 17.6$$

Solving the above three equations gives
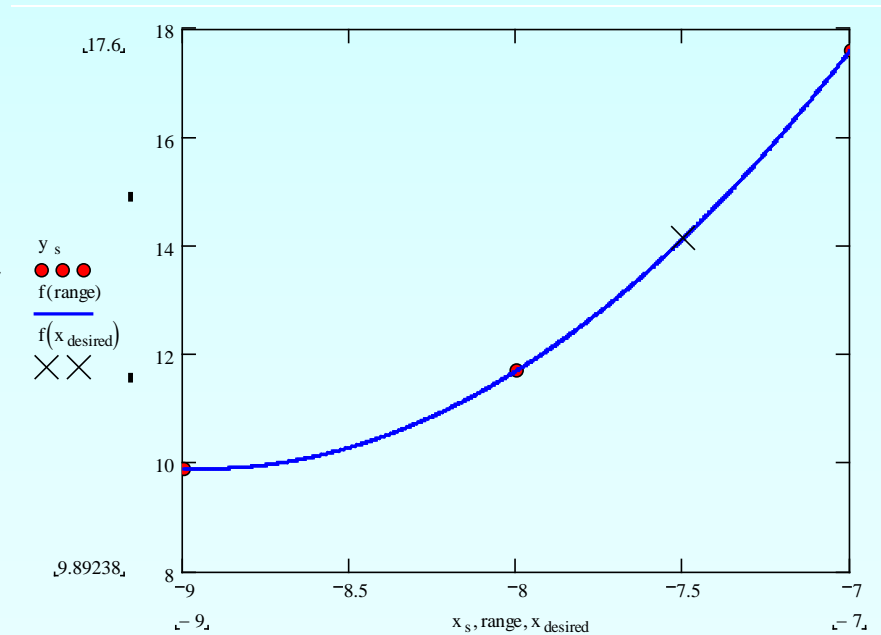
$$a_0 = 173.7 \qquad a_1 = 36.65 \qquad a_2 = 2.05$$

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Quadratic Interpolation (contd)

$$T(z) = 173.7 + 36.65z + 2.05z^2, \quad -9 \le z \le -7$$

$$T(-7.5) = 173.7 + 36.65(-7.5) + 2.05(-7.5)^2$$
$$= 14.138°C$$

The absolute relative approximate error obtained between the results from the first and second order polynomial is

$$\left| \in_a \right| = \left| \frac{14.138 - 14.65}{14.138} \right| \times 100$$
$$= 3.6251\%$$

Example

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth plot for a lake. Determine the value of the temperature at z = −7.5 using the direct method for cubic interpolation.

Temperature T (^{o}C)	Depth z (m)
19.1	0
19.1	-1
19	-2
18.8	-3
18.7	-4
18.3	-5
18.2	-6
17.6	-7
11.7	-8
9.9	-9
9.1	-10
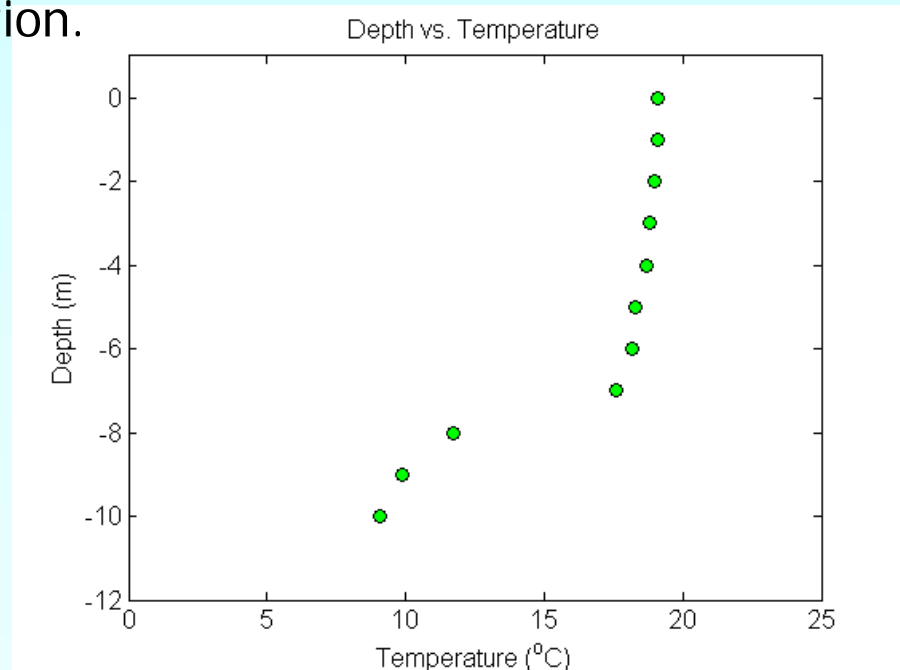
Temperature vs. depth of a lake

Cubic Interpolation

$$T(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$$

$$T(-9) = 9.9 = a_0 + a_1(-9) + a_2(-9)^2 + a_3(-9)^3$$

$$T(-8) = 11.7 = a_0 + a_1(-8) + a_2(-8)^2 + a_3(-8)^3$$

$$T(-7) = 17.6 = a_0 + a_1(-7) + a_2(-7)^2 + a_3(-7)^3$$

$$T(-6) = 18.2 = a_0 + a_1(-6) + a_2(-6)^2 + a_3(-6)^3$$

$$a_0 = -615.9 \quad a_1 = -262.58 \quad a_2 = -35.55 \quad a_3 = -1.5667$$
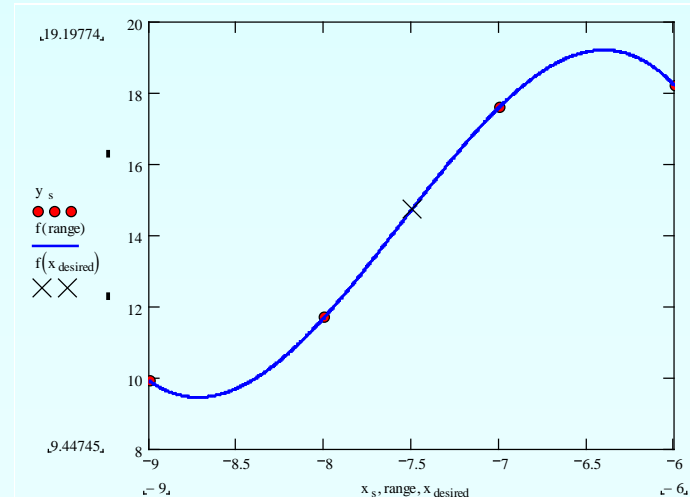
Cubic Interpolation (contd)

$$T(z) = -615.9 - 262.58z - 35.55z^2 - 1.5667z^3, \quad -9 \leq z \leq -6$$

$$T(-7.5) = -615.9 - 262.58(-7.5) - 35.55(-7.5)^2 - 1.5667(-7.5)^3$$

$$= 14.725°C$$

The absolute relative approximate error $\left| \in_a \right|$ obtained between the results from the second and third order polynomial is

$$\left| \in_a \right| = \left| \frac{14.725 - 14.138}{14.725} \right| \times 100$$

$$= 3.9898\%$$

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Comparison Table

Order of Polynomial	1	2	3
Temperature (^{o}C)	14.65	14.138	14.725
Absolute Relative Approximate Error	----------	3.6251 %	3.9898 %

Thermocline

What is the value of depth at which the thermocline exists?

To find the position of the thermocline we must find the points of inflection of the third order polynomial, given by: $\dfrac{d^2T}{dz^2} = 0$

$$T(z) = -615.9 - 262.58z - 35.55z^2 - 1.5667z^3, \quad -9 \le z \le -6$$

$$\frac{dT}{dz} = -262.58 - 71.10z - 4.7z^2, \quad -9 \le z \le -6$$

$$\frac{d^2T}{dz^2} = -71.10 - 9.4z, \quad -9 \le z \le -6$$

Simply setting this expression equal to zero, we get

$$0 = -71.10 - 9.4z, \quad -9 \le z \le -6$$

$$z = -7.5638 \ m$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/direct_method.html

THE END