Lagrangian Interpolation

Civil Engineering Majors

Authors: Autar Kaw, Jai Paul

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Transforming Numerical Methods Education for STEM Undergraduates

Lagrange Method of Interpolation

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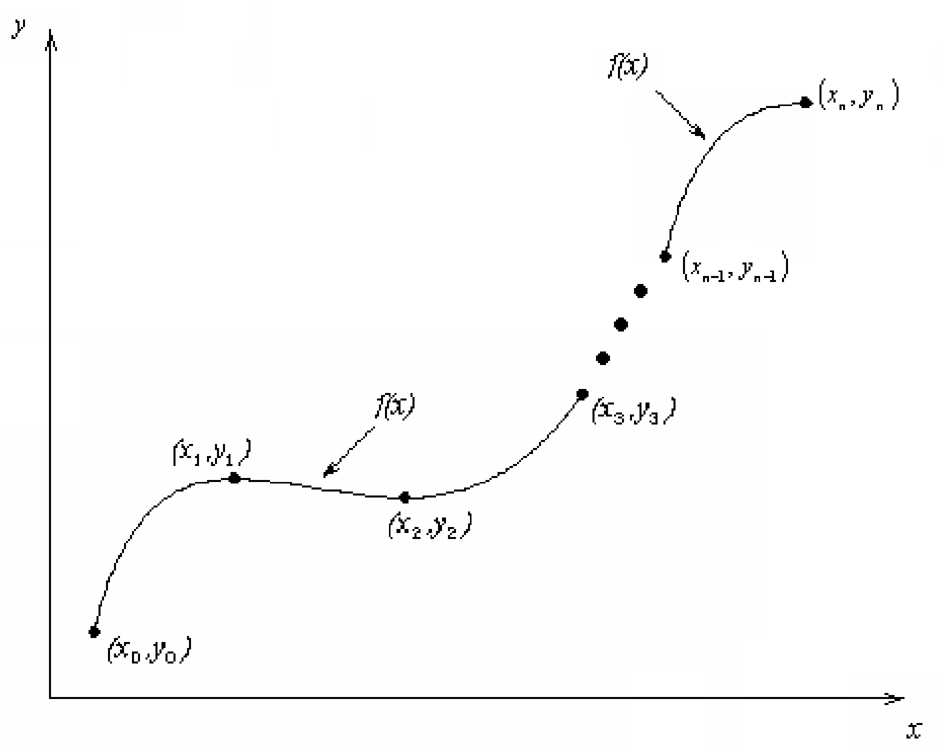
What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), …… (x_n,y_n), find the value of ‘y’ at a value of ‘x’ that is not given.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Lagrangian Interpolation

Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x)\, f(x_i)$$

where 'n' in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $(n+1)$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

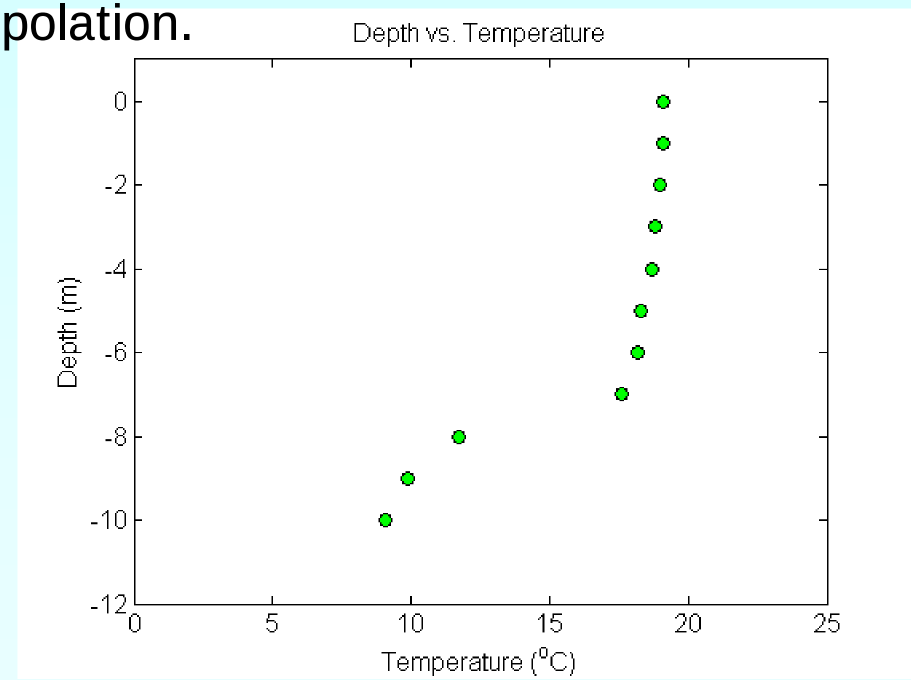
$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.

Example

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth plot for a lake. Determine the value of the temperature at $z = -7.5$ using the Lagragian method for linear interpolation.

Temperature	Depth
T (^oC)	z (m)
19.1	0
19.1	-1
19	-2
18.8	-3
18.7	-4
18.3	-5
18.2	-6
17.6	-7
11.7	-8
9.9	-9
9.1	-10



Temperature vs. depth of a lake

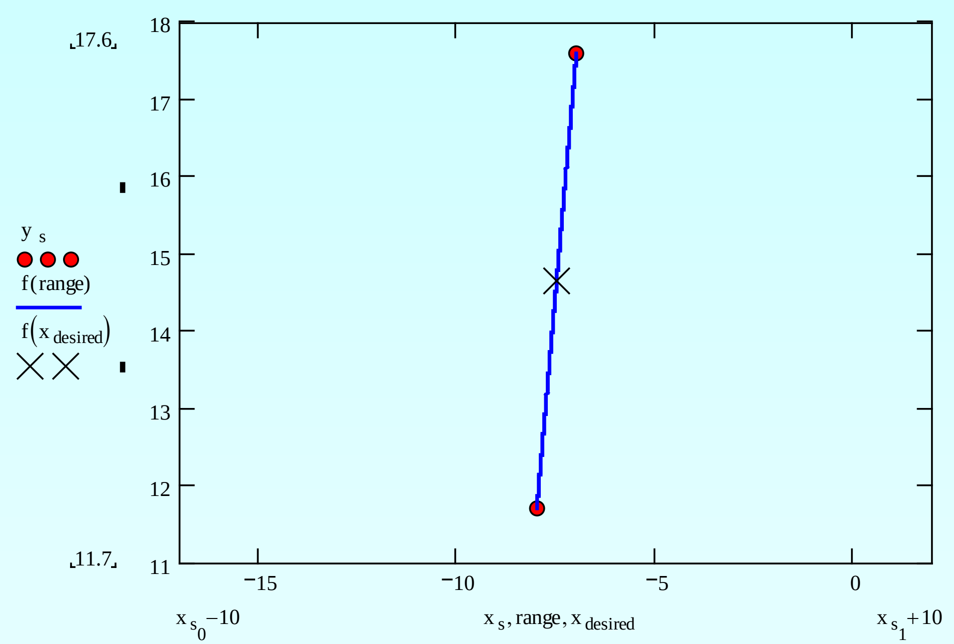
Linear Interpolation

$$T(z) = \sum_{i=0}^{1} L_i(z)T(z_i)$$

$$= L_0(z)T(z_0) + L_1(z)T(z_1)$$

$$z_0 = -8, T(z_0) = 11.7$$

$$z_1 = -7, T(z_1) = 17.6$$



Linear Interpolation (contd)

$$L_0(z) = \prod_{\substack{j=0 \\ j\neq 0}}^{1} \frac{z - z_j}{z_0 - z_j} = \frac{z - z_1}{z_0 - z_1}$$

$$L_1(z) = \prod_{\substack{j=0 \\ j\neq 1}}^{1} \frac{z - z_j}{z_1 - z_j} = \frac{z - z_0}{z_1 - z_0}$$

$$T(z) = \frac{z - z_1}{z_0 - z_1} T(z_0) + \frac{z - z_0}{z_1 - z_0} T(z_1) = \frac{z + 7}{-8 + 7}(11.7) + \frac{z + 8}{-7 + 8}(17.6), \quad -8 \leq z \leq -7$$

$$T(-7.5) = \frac{-7.5 + 7}{-8 + 7}(11.7) + \frac{-7.5 + 8}{-7 + 8}(17.6) = 0.5(11.7) + 0.5(17.6)$$

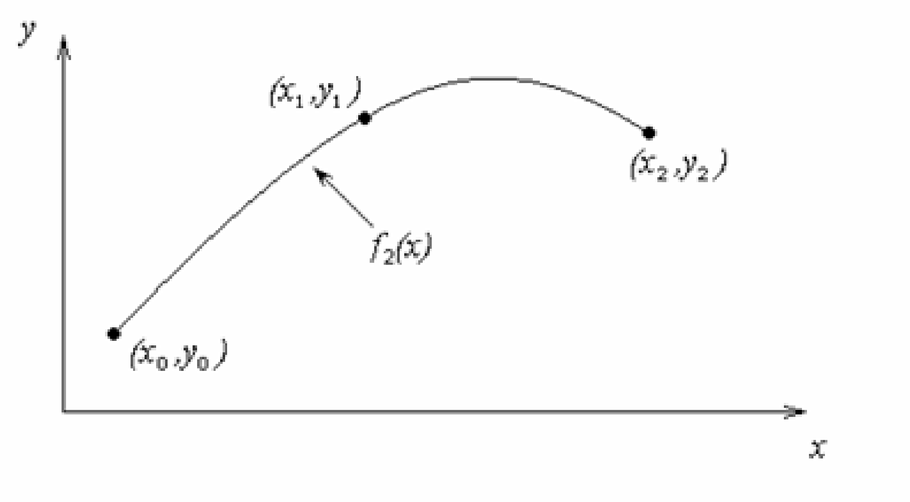
$$= 14.65\,^{\circ}\mathrm{C}$$

Quadratic Interpolation

For the second order polynomial interpolation (also called quadratic interpolation), we choose the velocity given by

$$v(t) = \sum_{i=0}^{2} L_i(t)v(t_i)$$

$$= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2)$$



Example

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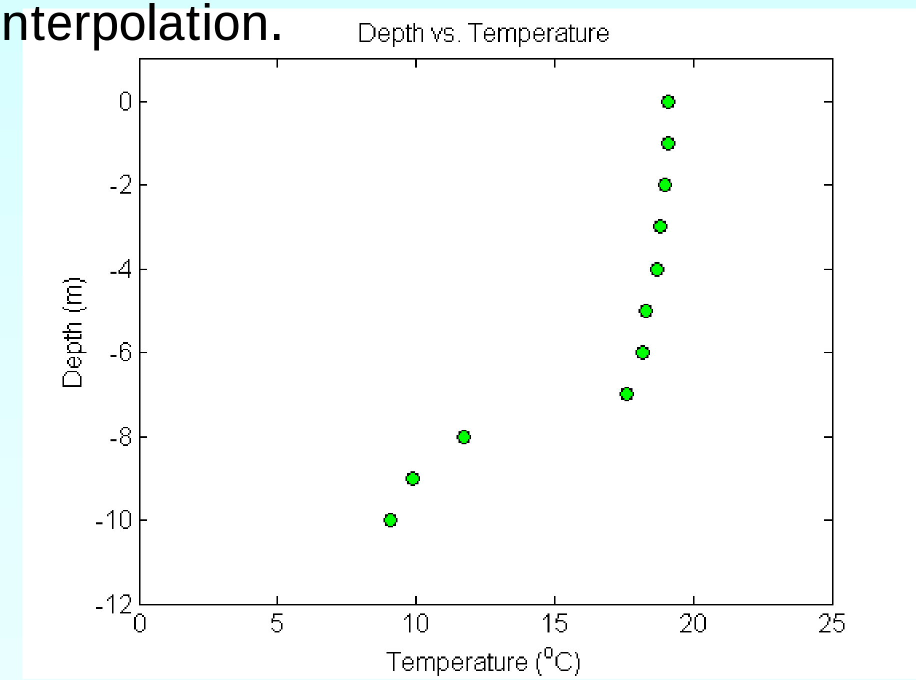
17.6	-7
11.7	-8
9.9	-9
9.1	-10



Temperature vs. depth of a lake

Quadratic Interpolation (contd)

$$z_o = -9,\ T(z_o) = 9.9$$

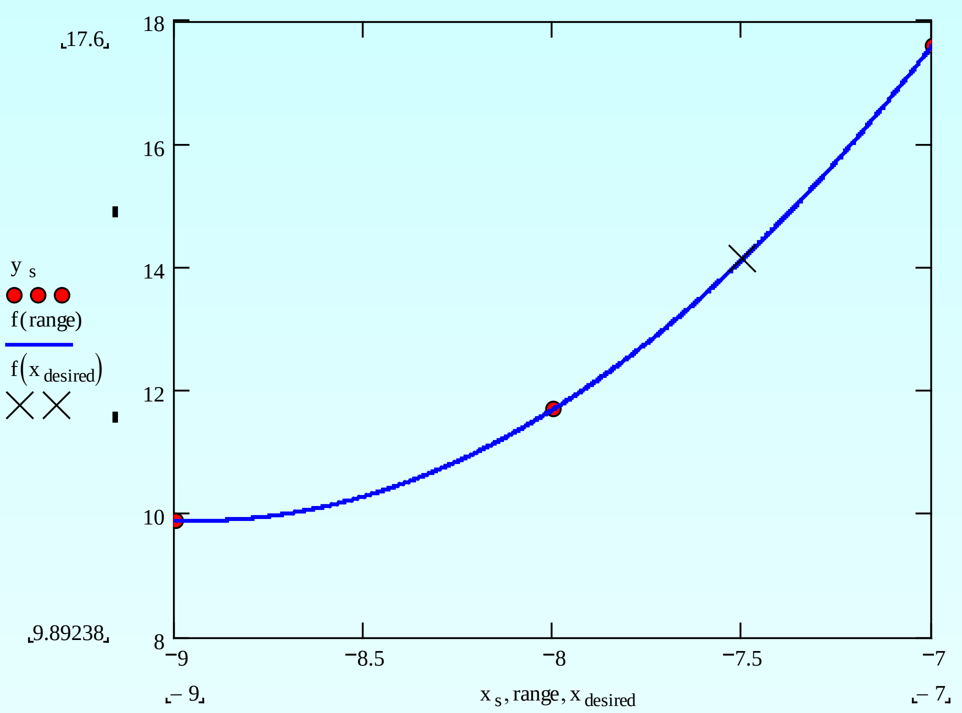
$$z_1 = -8,\ T(z_1) = 11.7$$

$$z_2 = -7,\ T(z_2) = 17.6$$

$$L_0(z) = \prod_{\substack{j=0 \\ j\neq 0}}^{2} \frac{z - z_j}{z_0 - z_j} = \left(\frac{z - z_1}{z_0 - z_1}\right)\left(\frac{z - z_2}{z_0 - z_2}\right)$$

$$L_1(z) = \prod_{\substack{j=0 \\ j\neq 1}}^{2} \frac{z - z_j}{z_1 - z_j} = \left(\frac{z - z_0}{z_1 - z_0}\right)\left(\frac{z - z_2}{z_1 - z_2}\right)$$

$$L_2(z) = \prod_{\substack{j=0 \\ j\neq 2}}^{2} \frac{z - z_j}{z_2 - z_j} = \left(\frac{z - z_0}{z_2 - z_0}\right)\left(\frac{z - z_1}{z_2 - z_1}\right)$$



Quadratic Interpolation (contd)

$$T(z) = \left(\frac{z - z_1}{z_0 - z_1}\right)\left(\frac{z - z_2}{z_0 - z_2}\right)T(z_0) + \left(\frac{z - z_0}{z_1 - z_0}\right)\left(\frac{z - z_2}{z_1 - z_2}\right)T(z_1) + \left(\frac{z - z_0}{z_2 - z_0}\right)\left(\frac{z - z_1}{z_2 - z_1}\right)T(z_2)$$

$$T(-7.5) = \frac{(-7.5+8)(-7.5+7)}{(-9+8)(-9+7)}(9.9) + \frac{(-7.5+9)(-7.5+7)}{(-8+9)(-8+7)}(11.7) + \frac{(-7.5+9)(-7.5+8)}{(-7+9)(-7+8)}(17.6)$$

$$= (-0.125)(9.9) + (0.75)(11.7) + (0.375)(17.6)$$

$$= 14.138°\mathrm{C}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{14.138 - 14.65}{14.138}\right| \times 100$$

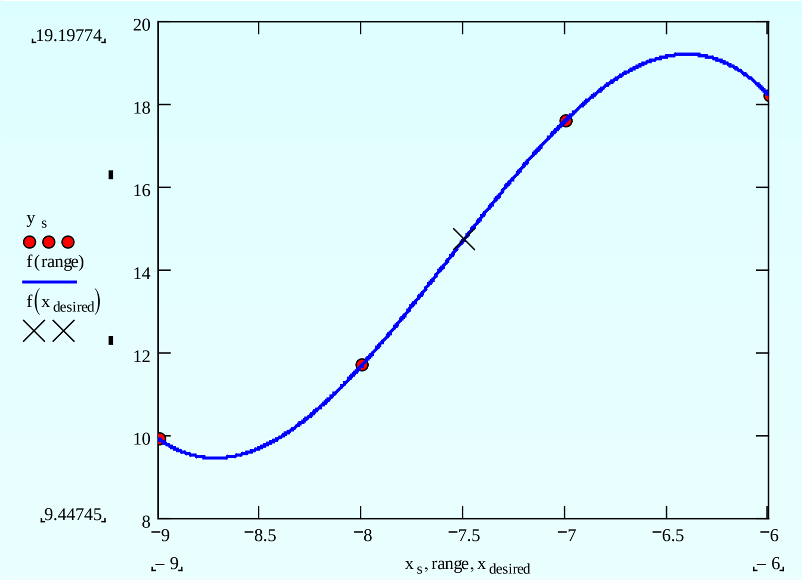
$$= 3.6251\%$$

Cubic Interpolation

For the third order polynomial (also called cubic interpolation), we choose the temperature given by

$$T(z) = \sum_{i=0}^{3} L_i(z) T(z_i)$$

$$= L_0(z)T(z_0) + L_1(z)T(z_1) + L_2(z)T(z_2) + L_3(z)T(z_3)$$



Example

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth plot for a lake. Determine the value of the temperature at $z = -7.5$ using the Lagragian method for cubic interpolation.

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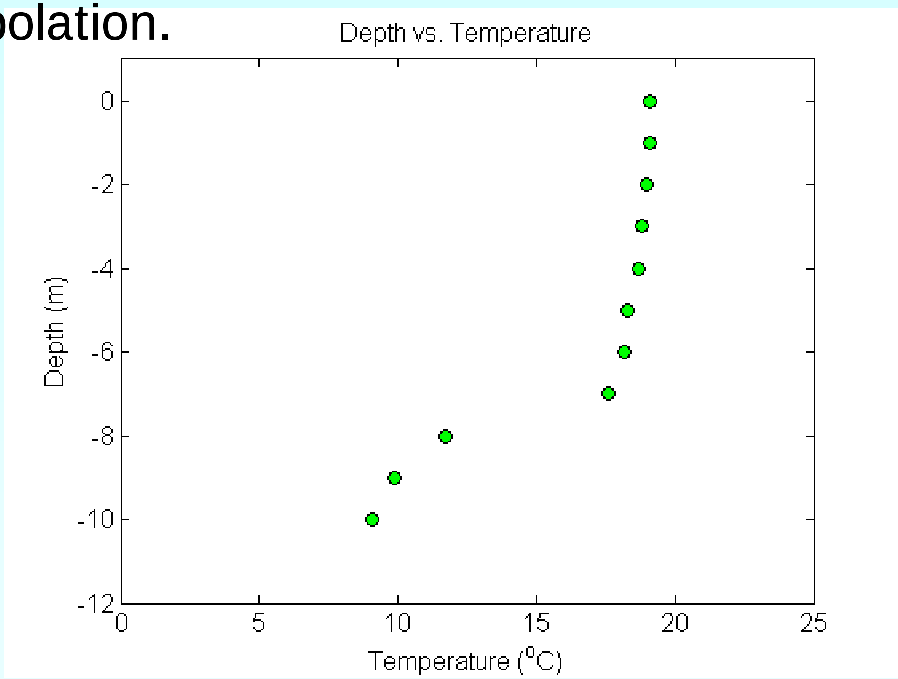
18.2	-6
17.6	-7
11.7	-8
9.9	-9
9.1	-10



Temperature vs. depth of a lake

Cubic Interpolation (contd)

$z_o = -9,\ T(z_o) = 9.9$ $\quad$ $z_1 = -8,\ T(z_1) = 11.7$

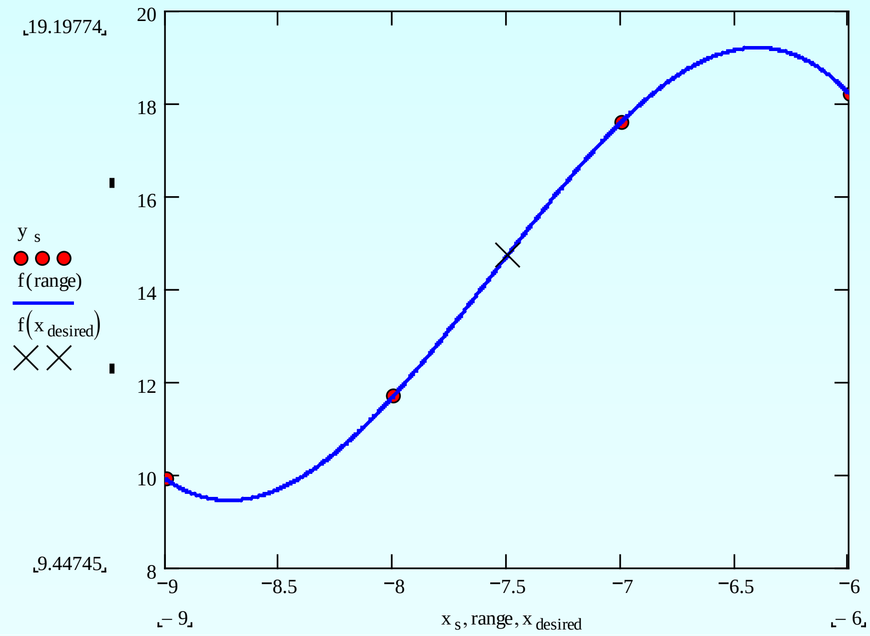
$z_2 = -7,\ T(z_2) = 17.6$ $\quad$ $z_3 = -6,\ T(z_3) = 18.2$

$$L_0(z) = \prod_{\substack{j=0\\j\neq 0}}^{3} \frac{z - z_j}{z_0 - z_j} = \left(\frac{z - z_1}{z_0 - z_1}\right)\left(\frac{z - z_2}{z_0 - z_2}\right)\left(\frac{z - z_3}{z_0 - z_3}\right)$$

$$L_1(z) = \prod_{\substack{j=0\\j\neq 1}}^{3} \frac{z - z_j}{z_1 - z_j} = \left(\frac{z - z_0}{z_1 - z_0}\right)\left(\frac{z - z_2}{z_1 - z_2}\right)\left(\frac{z - z_3}{z_1 - z_3}\right)$$

$$L_2(z) = \prod_{\substack{j=0\\j\neq 2}}^{3} \frac{z - z_j}{z_2 - z_j} = \left(\frac{z - z_0}{z_2 - z_0}\right)\left(\frac{z - z_1}{z_2 - z_1}\right)\left(\frac{z - z_3}{z_2 - z_3}\right)$$

$$L_3(z) = \prod_{\substack{j=0\\j\neq 3}}^{3} \frac{z - z_j}{z_3 - z_j} = \left(\frac{z - z_0}{z_3 - z_0}\right)\left(\frac{z - z_1}{z_3 - z_1}\right)\left(\frac{z - z_2}{z_3 - z_2}\right)$$



Cubic Interpolation (contd)

$$T(z)=\left(\frac{z-z_1}{z_0-z_1}\right)\left(\frac{z-z_2}{z_0-z_2}\right)\left(\frac{z-z_3}{z_0-z_3}\right)T(z_0)+\left(\frac{z-z_0}{z_1-z_0}\right)\left(\frac{z-z_2}{z_1-z_2}\right)\left(\frac{z-z_3}{z_1-z_3}\right)T(z_1)$$

$$+\left(\frac{z-z_0}{z_2-z_0}\right)\left(\frac{z-z_1}{z_0-z_1}\right)\left(\frac{z-z_3}{z_0-z_3}\right)T(z_2)+\left(\frac{z-z_0}{z_3-z_0}\right)\left(\frac{z-z_1}{z_3-z_1}\right)\left(\frac{z-z_2}{z_3-z_2}\right)T(z_3)$$

$$z_0 \le z \le z_3$$

$$T(-7.5)=\frac{(-7.5+8)(-7.5+7)(-7.5+6)}{(-9+8)(-9+7)(-9+6)}(9.9)+\frac{(-7.5+9)(-7.5+7)(-7.5+6)}{(-8+9)(-8+7)(-8+6)}(11.7)$$

$$+\frac{(-7.5+9)(-7.5+8)(-7.5+6)}{(-7+9)(-7+8)(-7+6)}(17.6)+\frac{(-7.5+9)(-7.5+8)(-7.5+7)}{(-6+9)(-6+8)(-6+7)}(18.2)$$

$$=(-0.0625)(9.9)+(0.5625)(11.7)+(0.5625)(17.6)+(-0.0625)(18.2)$$

$$=14.725°C$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the second and third order polynomial is

$$|\in_a|=\left|\frac{14.725-14.138}{14.725}\right|\times 100$$

$$=3.9898\%$$

Comparison Table

Order of Polynomial	1	2	3
Temperature °C	14.65	14.138	14.725
Absolute Relative Approximate Error	----------	3.6251 %	3.9898 %

Thermocline

What is the value of depth at which the thermocline exists?

The position where the thermocline exists is given where $\dfrac{d^2T}{dz^2} = 0$

$$T(z) = \frac{(z+8)(z+7)(z+6)}{(-9+8)(-9+7)(-9+6)}(9.9) + \frac{(z+9)(z+7)(z+6)}{(-8+9)(-8+7)(-8+6)}(11.7)$$

$$+ \frac{(z+9)(z+8)(z+6)}{(-7+9)(-7+8)(-7+6)}(17.6) + \frac{(z+9)(z+8)(z+7)}{(-6+9)(-6+8)(-6+7)}(18.2)$$

$$= -615.9 - 262.58z - 35.55z^2 - 1.5667z^3, \qquad -9 \le z \le -6$$

$$\frac{dT}{dz} = 262.58 - 71.1z - 4.7z^2, \qquad -9 \le z \le -6$$

$$\frac{d^2T}{dz^2} = -71.1 - 9.4z, \qquad -9 \le z \le -6$$

Simply setting this expression equal to zero, we get

$$0 = -71.1 - 9.4z, \qquad -9 \le z \le -6$$

$$z = -7.5638 \text{ m}$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/lagrange_method.html

THE END

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