Newton’s Divided Difference Polynomial Method of Interpolation

Civil Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

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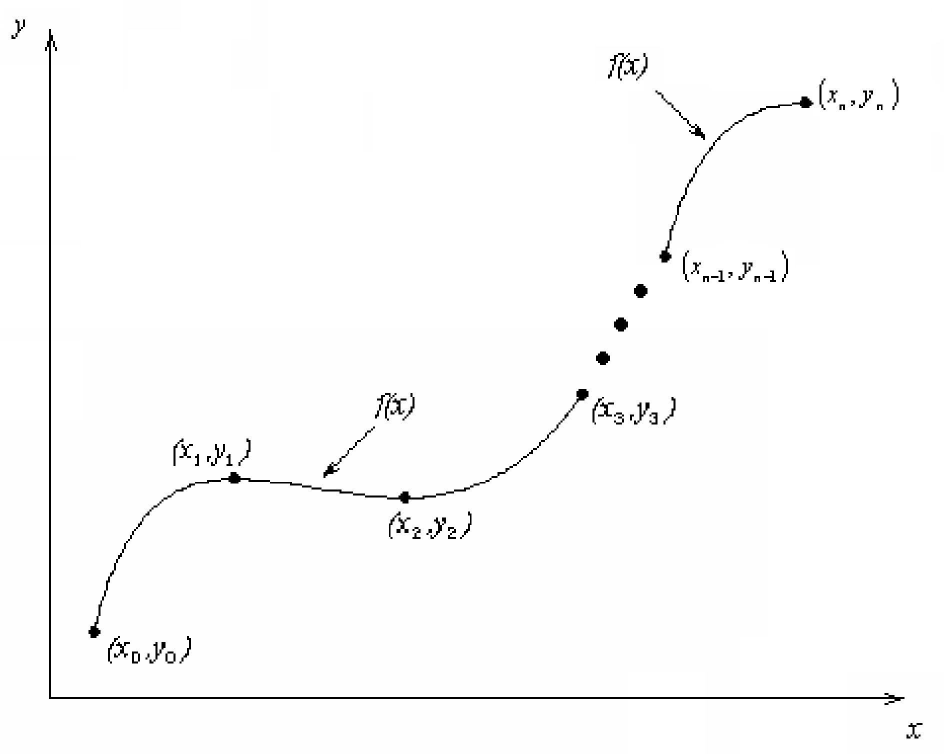
What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), …… (x_n,y_n), find the value of ‘y’ at a value of ‘x’ that is not given.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Newton's Divided Difference Method

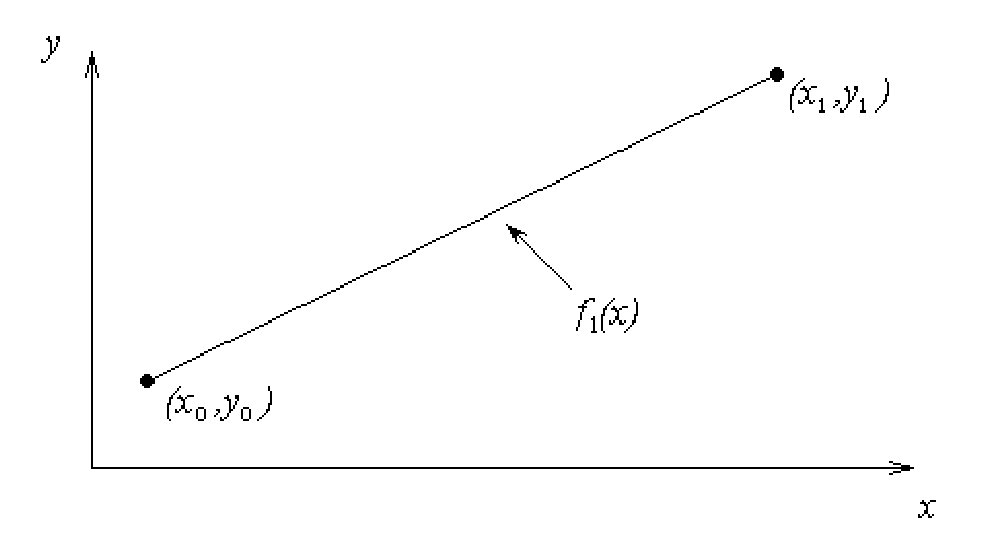
<u>Linear interpolation</u>: Given $(x_0, y_0), (x_1, y_1),$ pass a linear interpolant through the data

$$f_1(x) = b_0 + b_1(x - x_0)$$

where

$$b_0 = f(x_0)$$

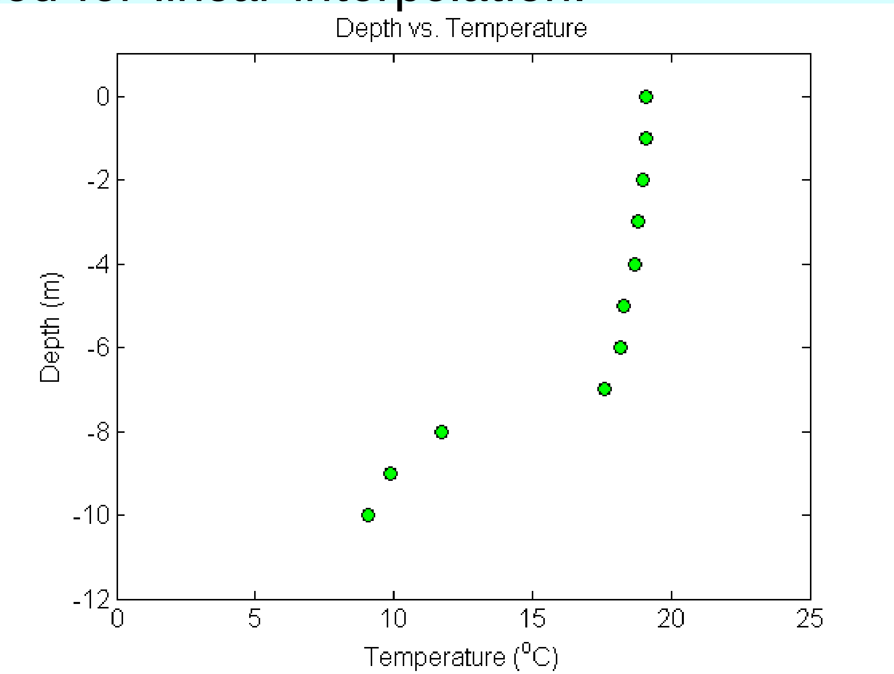
$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



Example

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth plot for a lake. Determine the value of the temperature at $z = -7.5$ using Newton's Divided Difference method for linear interpolation.

Temperature	Depth
T (^oC)	z (m)
19.1	0
19.1	-1
19	-2
18.8	-3
18.7	-4
18.3	-5
18.2	-6
17.6	-7
11.7	-8
9.9	-9
9.1	-10



Temperature vs. depth of a lake

Linear Interpolation

$$T(z) = b_0 + b_1(z - z_0)$$

$$z_0 = -8,\ T(z_0) = 11.7$$

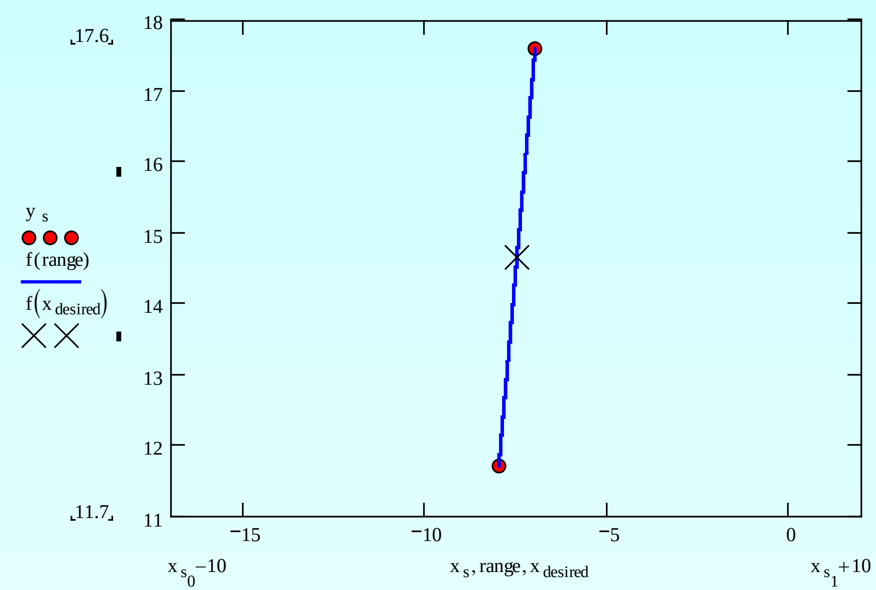
$$z_1 = -7,\ T(z_1) = 17.6$$

$$b_0 = T(z_0)$$

$$= 11.7$$

$$b_1 = \frac{T(z_1) - T(z_0)}{z_1 - z_0} = \frac{17.6 - 11.7}{-7 + 8}$$

$$= 5.9$$



Linear Interpolation (contd)

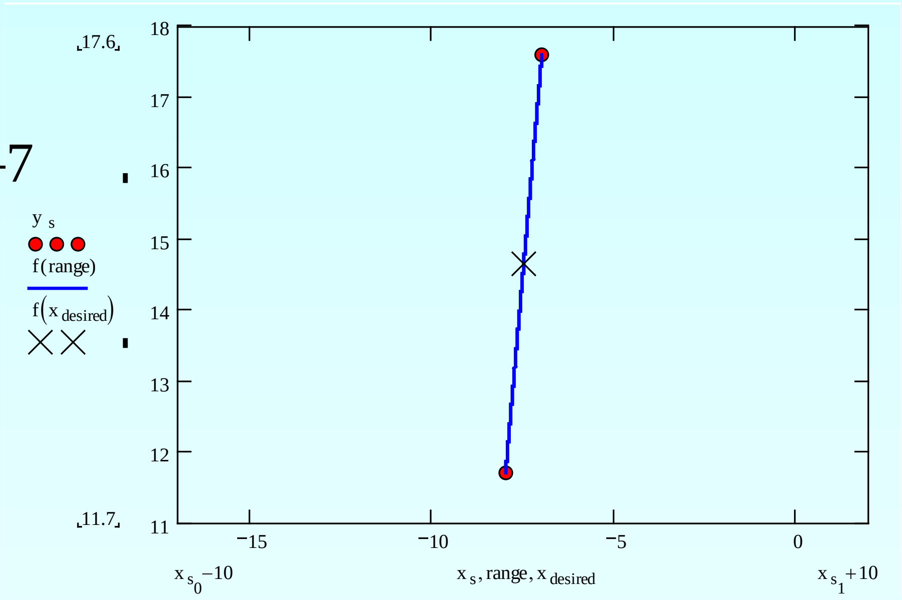
$$T(z) = b_0 + b_1(z - z_0)$$

$$= 11.7 + 5.9(z + 8), \qquad -8 \le z \le -7$$

At $z = -7.5$

$$T(-7.5) = 11.7 + 5.9(-7.5 + 8)$$

$$= 14.65°C$$



Quadratic Interpolation

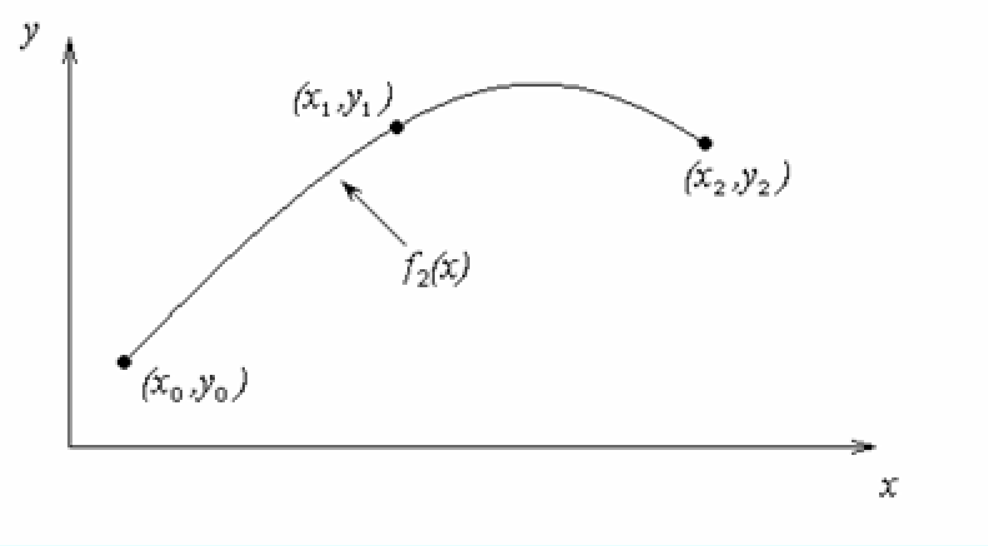
Given (x_0, y_0), (x_1, y_1), and (x_2, y_2), fit a quadratic interpolant through the data.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



Example

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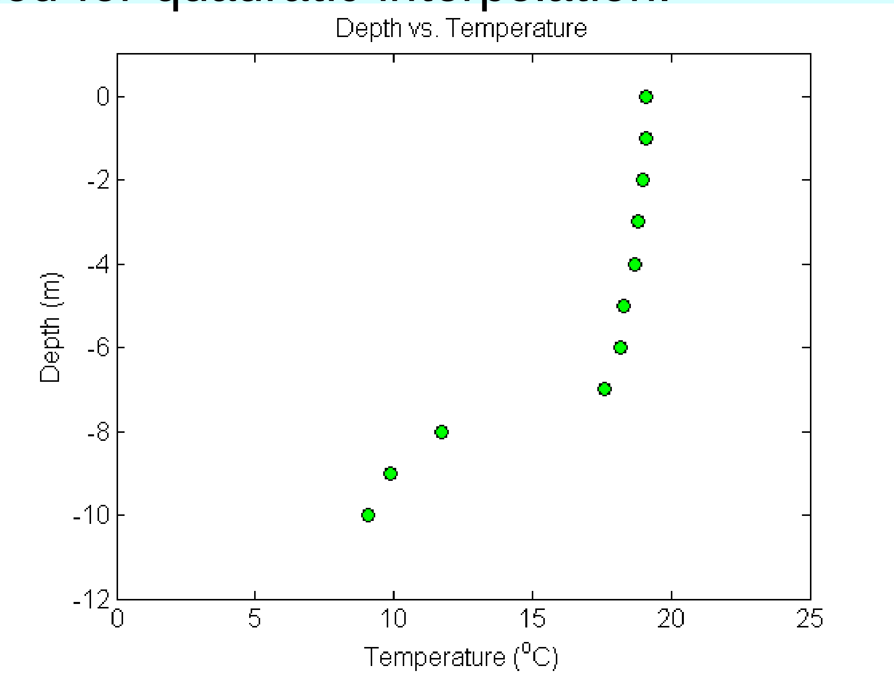
Temperature	Depth
T (^oC)	z (m)
19.1	0
19.1	-1
19	-2
18.8	-3
18.7	-4
18.3	-5
18.2	-6
17.6	-7
11.7	-8
9.9	-9
9.1	-10



Temperature vs. depth of a lake

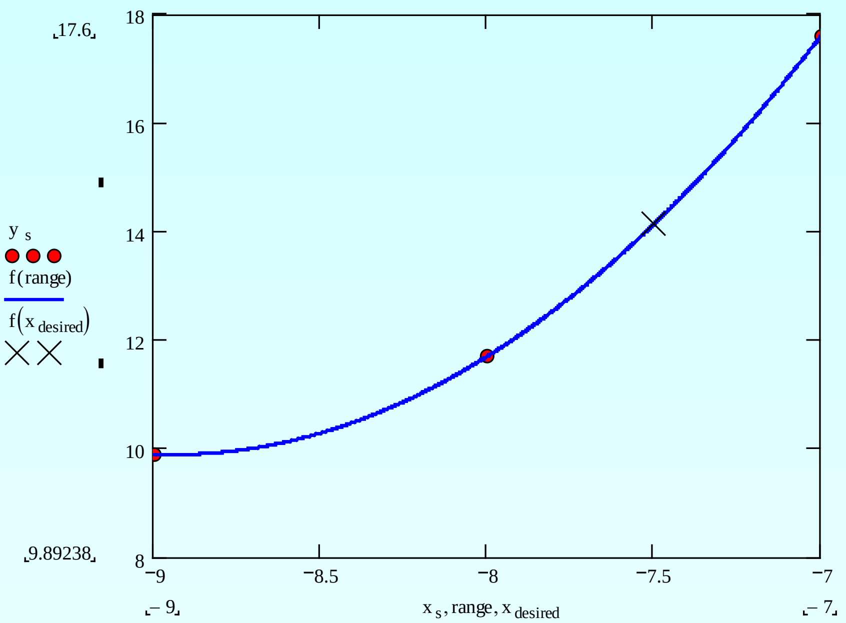
Quadratic Interpolation (contd)

$$T(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)(z - z_1)$$

$$z_0 = -9, T(z_0) = 9.9$$

$$z_1 = -8, \ T(z_1) = 11.7$$

$$z_2 = -7, T(z_2) = 17.6$$



Quadratic Interpolation (contd)

$$b_0 = T(z_0) = 9.9$$

$$b_1 = \frac{T(z_1) - T(z_0)}{z_1 - z_0} = \frac{11.7 - 9.9}{-8 + 7} = 1.8$$

$$b_2 = \frac{\dfrac{T(z_2) - T(z_1)}{z_2 - z_1} - \dfrac{T(z_1) - T(z_0)}{z_1 - z_0}}{z_2 - z_0} = \frac{\dfrac{17.6 - 11.7}{-7 + 8} - \dfrac{11.7 - 9.9}{-8 + 9}}{-7 + 9}$$

$$= \frac{5.9 - 1.8}{2}$$

$$= 2.05$$

Quadratic Interpolation (contd)

$$T(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)(z - z_1)$$

$$= 9.9 + 1.8(z + 9) + 2.05(z + 9)(z + 8), \quad -9 \le z \le -7$$

At $z = -7.5,$

$$T(-7.5) = 9.9 + 1.8(-7.5 + 9) + 2.05(-7.5 + 9)(-7.5 + 8)$$

$$= 14.138°C$$

The absolute relative approximate error $\left|\in_a\right|$ obtained between the results

from the first and second order polynomial is

$$\left|\in_a\right| = \left|\frac{14.138 - 14.65}{14.138}\right| \times 100$$

$$= 3.6251\%$$

General Form

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f[x_0] = f(x_0)$$

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Rewriting

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

General Form

Given $(n+1)$ data points, $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ as

$$f_n(x) = b_0 + b_1(x - x_0) + \ldots. + b_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

$$\vdots$$

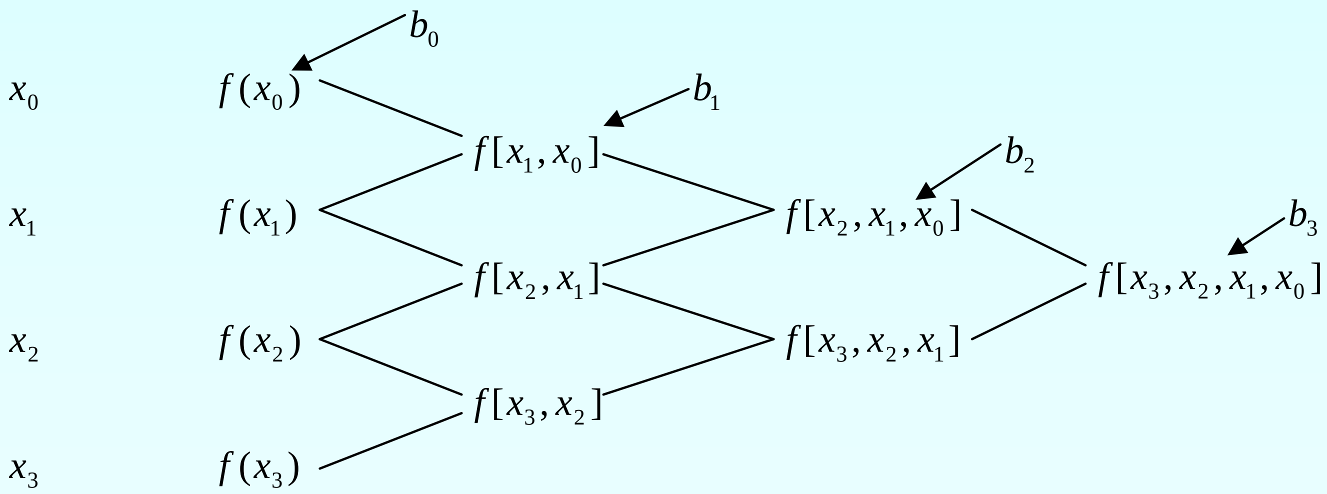
$$b_{n-1} = f[x_{n-1}, x_{n-2}, \ldots., x_0]$$

$$b_n = f[x_n, x_{n-1}, \ldots., x_0]$$

General form

The third order polynomial, given $(x_0, y_0), (x_1, y_1), (x_2, y_2),$ and $(x_3, y_3),$ is

$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$
$$+ f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$



Example

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth plot for a lake. Determine the value of the temperature at $z = -7.5$ using Newton's Divided Difference method for cubic interpolation.

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T (^oC)	z (m)
19.1	0
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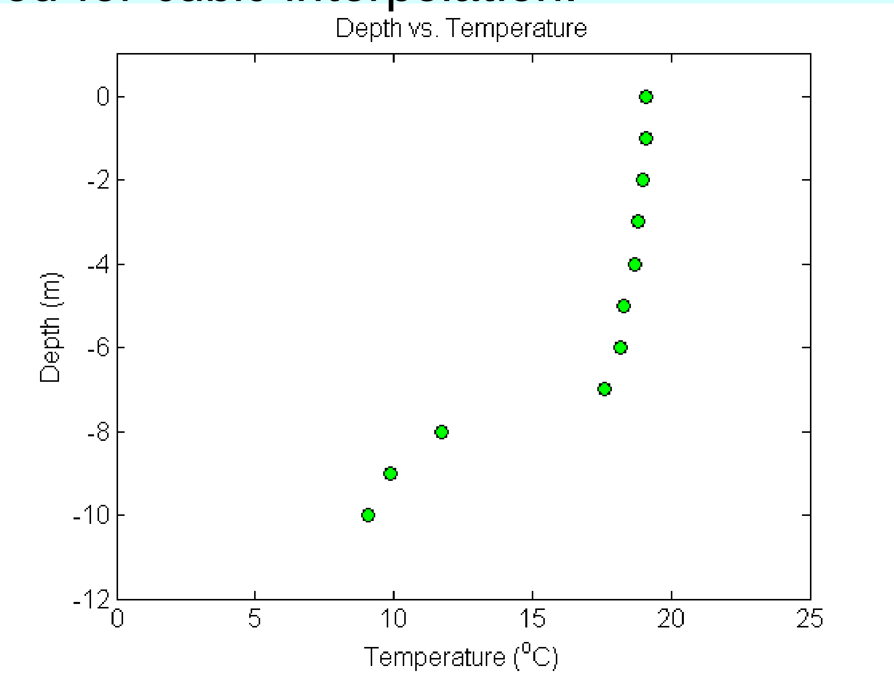
18.7	-4
18.3	-5
18.2	-6
17.6	-7
11.7	-8
9.9	-9
9.1	-10



Temperature vs. depth of a lake

Example

The temperature profile is chosen as

$$T(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)(z - z_1) + b_3(z - z_0)(z - z_1)(z - z_2)$$

We need to choose four data points that are closest to $z = -7.5$

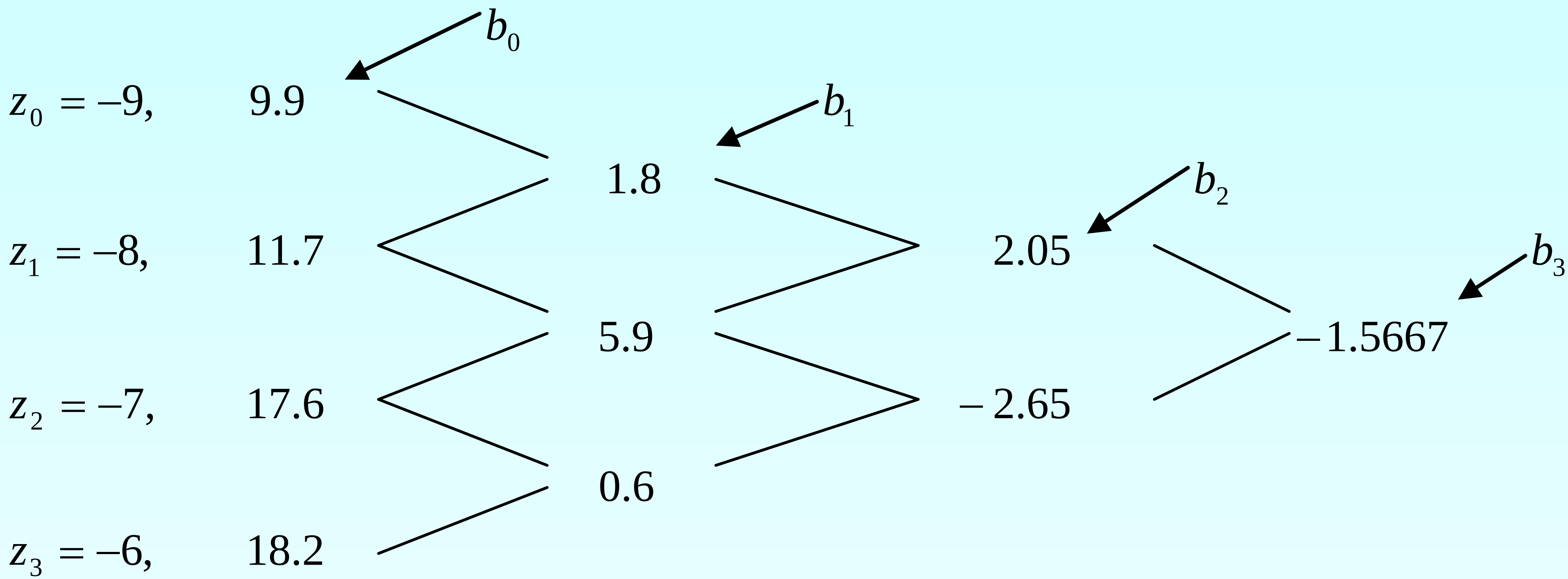
$$z_0 = -9, \quad T(z_0) = 9.9$$

$$z_1 = -8, \quad T(z_1) = 11.7$$

$$z_2 = -7, \quad T(z_2) = 17.6$$

$$z_3 = -6, \quad T(z_3) = 18.2$$

Example



The values of the constants are obtained as

$$b_0 = 9.9 \qquad b_1 = 1.8 \qquad b_2 = 2.05 \qquad b_3 = -1.5667$$

Example

$$T(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)(z - z_1) + b_3(z - z_0)(z - z_1)(z - z_2)$$

$$= 9.9 + 1.8(z+9) + 2.05(z+9)(z+8) - 1.5667(z+9)(z+8)(z+7)\,,\ -9 \le z \le -6$$

At $z = -7.5$,

$$T(-7.5) = 9.9 + 1.8(-7.5+9) + 2.05(-7.5+9)(-7.5+8)$$

$$-1.5667(-7.5+9)(-7.5+8)(-7.5+7)$$

$$= 14.725°C$$

The absolute relative approximate error $\left|\in_a\right|$ obtained between the results from the second and third order polynomial is

$$\left|\in_a\right| = \left|\frac{14.725 - 14.138}{14.725}\right| \times 100$$

$$= 3.9898\%$$

Comparison Table

Order of Polynomial	1	2	3
Temperature (^{o}C)	14.65	14.138	14.725
Absolute Relative Approximate Error	----------	3.6251%	3.9898%

Thermocline

What is the value of depth at which the thermocline exists?

The position where the thermocline exists is given where $\frac{d^2T}{dz^2} = 0$

$$T(z) = 9.9 + 1.8(z+9) + 2.05(z+9)(z+8) - 1.5667(z+9)(z+8)(z+7)$$

$$= -615.9 - 262.58z - 35.55z^2 - 1.5667z^3, \quad -9 \le z \le -6$$

$$\frac{dT}{dz} = -262.58 - 71.1z - 4.7z^2, \quad -9 \le z \le -6$$

$$\frac{d^2T}{dz^2} = -71.1 - 9.4z, \quad -9 \le z \le -6$$

Simply setting this expression equal to zero, we get

$$0 = -71.10 - 9.4z, \quad -9 \le z \le -6$$

$$z = -7.5638 \text{ m}$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/newton_divided_difference_method.html

THE END

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