Chapter 05.02
Direct Method of Interpolation

After reading this chapter, you should be able to:

1. *apply the direct method of interpolation,*
2. *solve problems using the direct method of interpolation, and*
3. *use the direct method interpolants to find derivatives and integrals of discrete functions.*

What is interpolation?

Many times, data is given only at discrete points such as (x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}), (x_n, y_n). So, how then does one find the value of y at any other value of x? Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points (Figure 1). Then one can find the value of y at any other value of x. This is called *interpolation*.

Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function $f(x)$ should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

(A) evaluate,
(B) differentiate, and
(C) integrate

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ points. One of the methods of interpolation is called the direct method. Other methods include Newton's divided difference polynomial method and the Lagrangian interpolation method. We will discuss the direct method in this chapter.

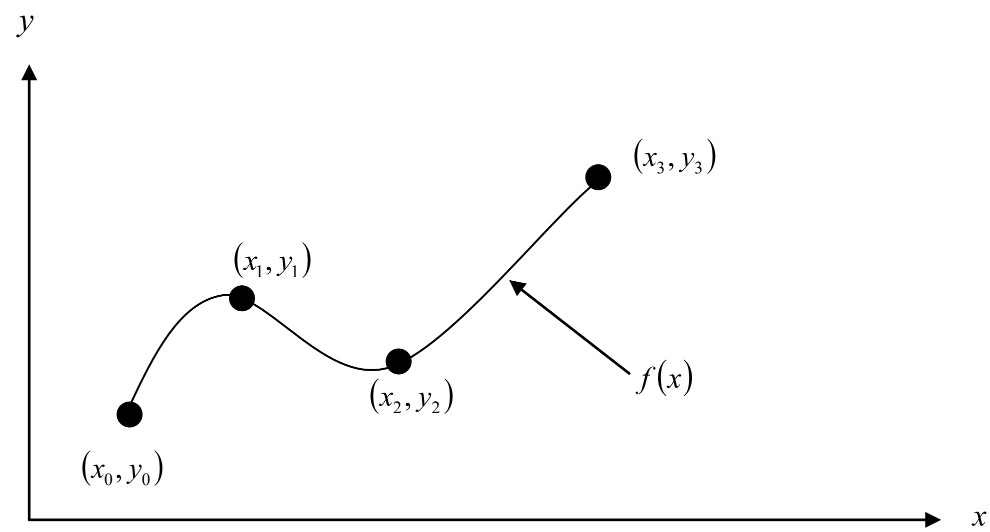

Figure 1 Interpolation of discrete data.

Direct Method

The direct method of interpolation is based on the following premise. Given $n+1$ data points, fit a polynomial of order n as given below

$$y = a_0 + a_1 x + ... + a_n x^n \tag{1}$$

through the data, where $a_0, a_1, ..., a_n$ are $n+1$ real constants. Since $n+1$ values of y are given at $n+1$ values of x, one can write $n+1$ equations. Then the $n+1$ constants, $a_0, a_1, ..., a_n$ can be found by solving the $n+1$ simultaneous linear equations. To find the value of y at a given value of x, simply substitute the value of x in Equation 1.

But, it is not necessary to use all the data points. How does one then choose the order of the polynomial and what data points to use? This concept and the direct method of interpolation are best illustrated using examples.

Example 1

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth data for a lake in Table 2.

Table 1 Temperature vs. depth for a lake.

Temperature, T $(°\text{C})$	Depth, z (m)
19.1	0
19.1	–1
19	–2
18.8	–3
18.7	–4
18.3	–5
18.2	–6
17.6	–7
11.7	–8
9.9	–9
9.1	–10

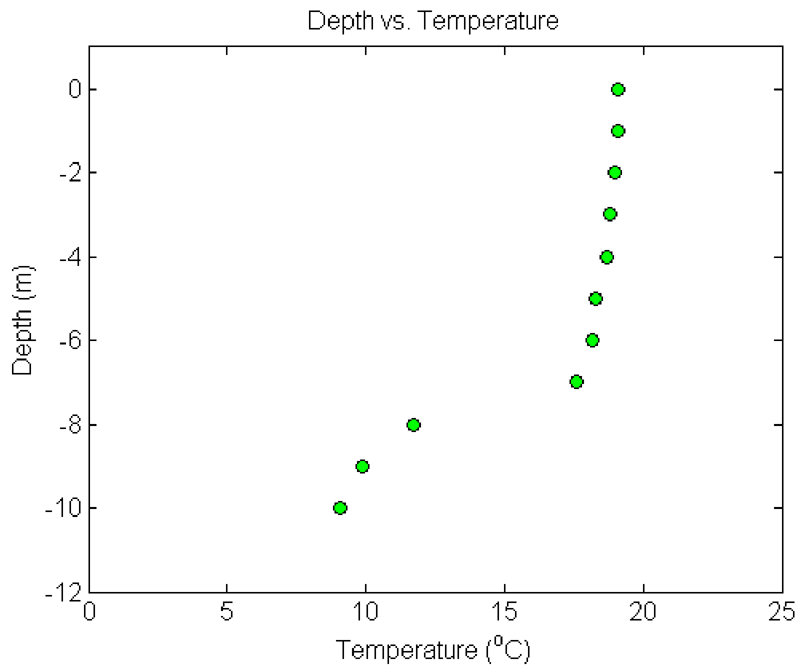

Figure 2 Temperature vs. depth of a lake.

Using the given data, we see the largest change in temperature is between $z = -8$ m and $z = -7$ m. Determine the value of the temperature at $z = -7.5$ m using the direct method of interpolation and a first order polynomial.

Solution

For first order polynomial interpolation (also called linear interpolation), we choose the temperature given by

$$T(z) = a_0 + a_1 z$$

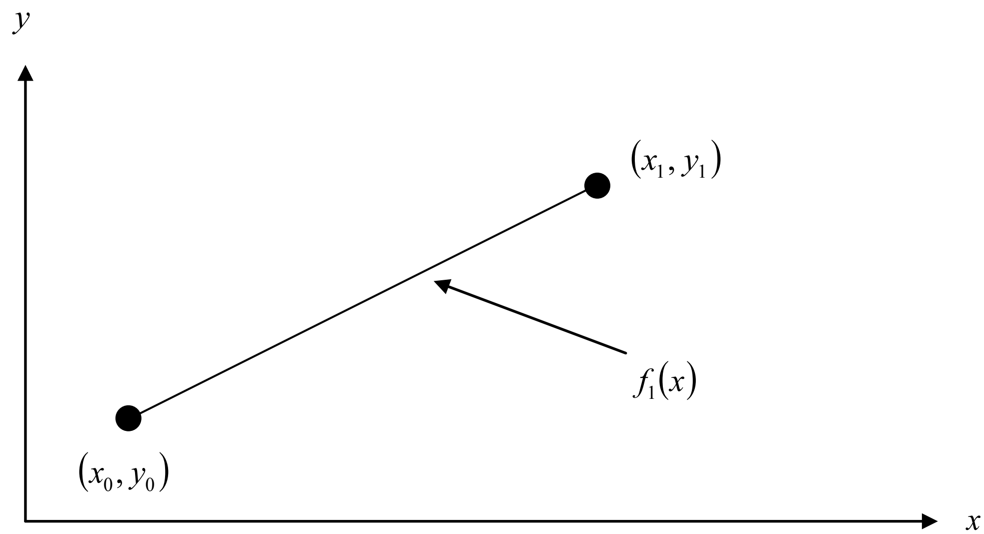

Figure 3 Linear interpolation.

Since we want to find the temperature at $z = -7.5$ m, and we are using a first order polynomial, we need to choose the two data points that are closest to $z = -7.5$ m that also bracket $z = -7.5$ m to evaluate it. The two points are $z_0 = -8$ and $z_1 = -7$.

Then

$$z_0 = -8, T(z_0) = 11.7$$
$$z_1 = -7, T(z_1) = 17.6$$

gives

$$T(-8) = a_0 + a_1(-8) = 11.7$$
$$T(-7) = a_0 + a_1(-7) = 17.6$$

Writing the equations in matrix form, we have

$$\begin{bmatrix} 1 & -8 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 11.7 \\ 17.6 \end{bmatrix}$$

Solving the above two equations gives

$$a_0 = 58.9 \text{ and } a_1 = 5.9$$

Hence

$$T(z) = a_0 + a_1 z$$
$$T(z) = 58.9 + 5.9z, \ -8 \le z \le -7$$
$$T(-7.5) = 58.9 + 5.9(-7.5)$$
$$= 14.65\,^\circ\text{C}$$

Example 2

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth data for a lake in Table 4.

Table 2 Temperature vs. depth for a lake.

Temperature, T $(°\text{C})$	Depth, z (m)
19.1	0
19.1	–1
19	–2
18.8	–3
18.7	–4
18.3	–5
18.2	–6
17.6	–7
11.7	–8
9.9	–9
9.1	–10

Using the given data, we see the largest change in temperature is between $z = -8\ \text{m}$ and $z = -7\ \text{m}$. Determine the value of the temperature at $z = -7.5\ \text{m}$ using the direct method of interpolation and a second order polynomial. Find the absolute relative approximate error for the second order polynomial approximation.

Solution

For second order polynomial interpolation (also called quadratic interpolation), we choose the velocity given by

$$v(t) = a_0 + a_1 t + a_2 t^2$$

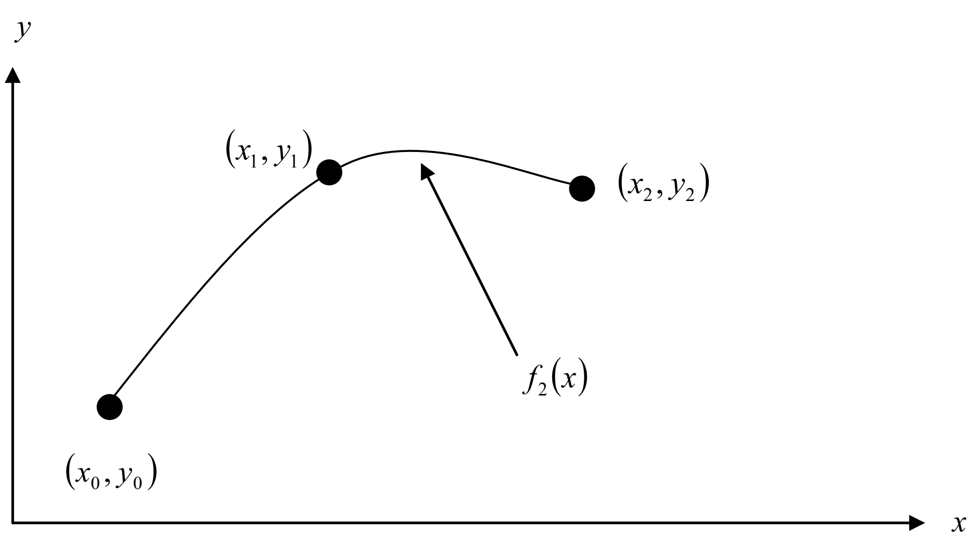

Figure 4 Quadratic interpolation.

Since we want to find the temperature at $z=-7.5$, and we are using a second order polynomial, we need to choose the three data points that are closest to $z=-7.5$ that also bracket $z=-7.5$ to evaluate it. The three points are $z_0=-9$, $z_1=-8$ and $z_2=-7$. (Choosing the three points as $z_0=-8$, $z_1=-7$ and $z_2=-6$ is equally valid.)

Then

$$z_0=-9,\ T(z_0)=9.9$$
$$z_1=-8,\ T(z_1)=11.7$$
$$z_2=-7,\ T(z_2)=17.6$$

gives

$$T(-9)=a_0+a_1(-9)+a_2(-9)^2=9.9$$
$$T(-8)=a_0+a_1(-8)+a_2(-8)^2=11.7$$
$$T(-7)=a_0+a_1(-7)+a_2(-7)^2=17.6$$

Writing the three equations in matrix form

$$\begin{bmatrix}1 & -9 & 81\\ 1 & -8 & 64\\ 1 & -7 & 49\end{bmatrix}\begin{bmatrix}a_0\\ a_1\\ a_2\end{bmatrix}=\begin{bmatrix}9.9\\ 11.7\\ 17.6\end{bmatrix}$$

and the solution of the above three equations gives

$$a_0=173.7$$
$$a_1=36.65$$
$$a_2=2.05$$

Hence

$$T(z)=173.7+36.65z+2.05z^2,\ -9\le z\le -7$$

At $z=-7.5$,

$$\begin{aligned}T(-7.5)&=173.7+36.65(-7.5)+2.05(-7.5)^2\\ &=14.138\,^\circ\mathrm{C}\end{aligned}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$\begin{aligned}|\in_a|&=\left|\frac{14.138-14.65}{14.138}\right|\times 100\\ &=3.6251\%\end{aligned}$$

Example 3

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth data for a lake in Table 5.

Table 3 Temperature vs. depth for a lake.

Temperature, T $(°\text{C})$	Depth, z (m)
19.1	0
19.1	–1
19	–2
18.8	–3
18.7	–4
18.3	–5
18.2	–6
17.6	–7
11.7	–8
9.9	–9
9.1	–10

Using the given data, we see the largest change in temperature is between $z = -8 \text{ m}$ and $z = -7 \text{ m}$.

a) Determine the value of the temperature at $z = -7.5 \text{ m}$ using the direct method of interpolation and a third order polynomial. Find the absolute relative approximate error for the third order polynomial approximation.

b) The position where the thermocline exists is given where $\frac{d^2T}{dz^2} = 0$. Using the expression from part (a), what is the value of the depth at which the thermocline exists?

Solution

a) For third order polynomial interpolation (also called cubic interpolation), we choose the temperature given by

$$T(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$$

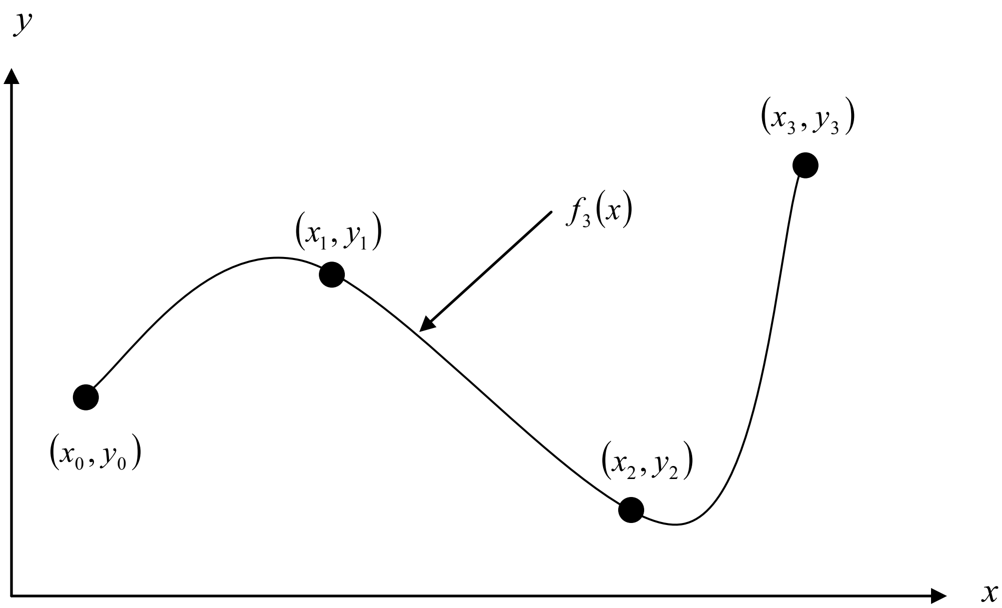

Figure 5 Cubic interpolation.

Since we want to find the temperature at $z = -7.5$, and we are using a third order polynomial, we need to choose the four data points closest to $z = -7.5$ that also bracket $z = -7.5$ to evaluate it. The four points are $z_0 = -9$, $z_1 = -8$, $z_2 = -7$ and $z_3 = -6$.

Then

$$z_0 = -9,\ T(z_0) = 9.9$$
$$z_1 = -8,\ T(z_1) = 11.7$$
$$z_2 = -7,\ T(z_2) = 17.6$$
$$z_3 = -6,\ T(z_3) = 18.2$$

gives

$$T(-9) = a_0 + a_1(-9) + a_2(-9)^2 + a_3(-9)^3 = 9.9$$
$$T(-8) = a_0 + a_1(-8) + a_2(-8)^2 + a_3(-8)^3 = 11.7$$
$$T(-7) = a_0 + a_1(-7) + a_2(-7)^2 + a_3(-7)^3 = 17.6$$
$$T(-6) = a_0 + a_1(-6) + a_2(-6)^2 + a_3(-6)^3 = 18.2$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & -9 & 81 & -729 \\ 1 & -8 & 64 & -512 \\ 1 & -7 & 49 & -343 \\ 1 & -6 & 36 & -216 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 9.9 \\ 11.7 \\ 17.6 \\ 18.2 \end{bmatrix}$$

Solving the above four equations gives

$$a_0 = -615.9$$
$$a_1 = -262.58$$

$$a_2 = -35.55$$
$$a_3 = -1.5667$$

Hence

$$T(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$$
$$= -615.9 - 262.58z - 35.55z^2 - 1.5667z^3, \quad -9 \le z \le -6$$
$$T(-7.5) = -615.9 - 262.58(-7.5) - 35.55(-7.5)^2 - 1.5667(-7.5)^3$$
$$= 14.725\,°\text{C}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the second and third order polynomial is

$$|\in_a| = \left|\frac{14.725 - 14.138}{14.725}\right| \times 100$$
$$= 3.9898\%$$

b) To find the position of the thermocline, we must find the points of inflection of the third order polynomial, given by $\frac{d^2T}{dz^2} = 0$

$$T(z) = -615.9 - 262.58z - 35.55z^2 - 1.5667z^3, \quad -9 \le z \le -6$$
$$\frac{dT}{dz} = -262.58 - 71.10z - 4.7z^2, \quad -9 \le z \le -6$$
$$\frac{d^2T}{dz^2} = -71.1 - 9.4z, \quad -9 \le z \le -6$$

Simply setting this expression equal to zero, we get

$$0 = -71.10 - 9.4z$$
$$z = -7.5638 \text{ m}$$

This answer can be verified due to the fact that it falls within the specified range of the third order polynomial and it also falls within the region of the greatest temperature change in the collected data from the lake.

INTERPOLATION	
Topic	Direct Method of Interpolation
Summary	Examples of direct method of interpolation.
Major	Civil Engineering
Authors	Autar Kaw
Date	November 8, 2012
Web Site	http://numericalmethods.eng.usf.edu