# Gauss Quadrature Rule of Integration

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# What is Integration?

**Integration**

The process of measuring the area under a curve.

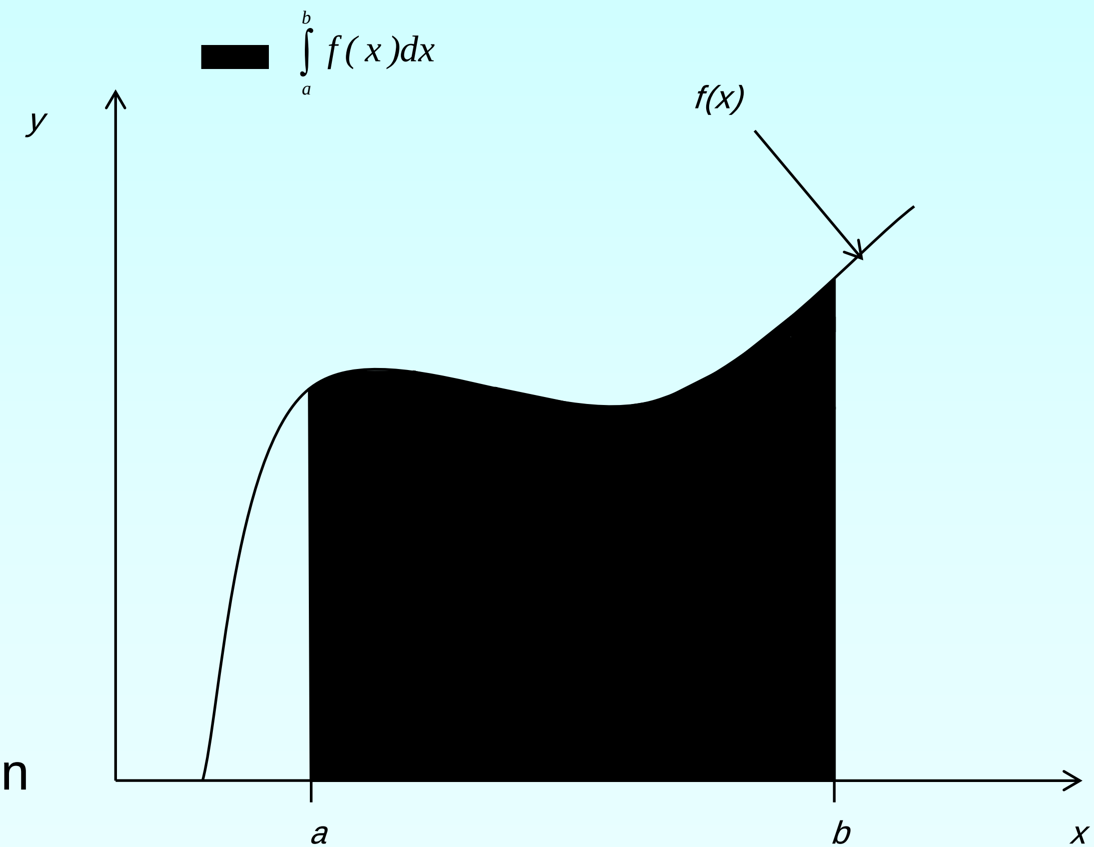
$$I = \int_a^b f(x)dx$$

Where:

*f(x)* is the integrand

a= lower limit of integration

b= upper limit of integration



# Two-Point Gaussian Quadrature Rule

# Basis of the Gaussian Quadrature Rule

Previously, the Trapezoidal Rule was developed by the method of undetermined coefficients. The result of that development is summarized below.

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b)$$

$$= \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b)$$

# Basis of the Gaussian Quadrature Rule

The two-point Gauss Quadrature Rule is an extension of the Trapezoidal Rule approximation where the arguments of the function are not predetermined as a and b but as unknowns $x_1$ and $x_2$. In the two-point Gauss Quadrature Rule, the integral is approximated as

$$I = \int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

# Basis of the Gaussian Quadrature Rule

The four unknowns $x_1$, $x_2$, $c_1$ and $c_2$ are found by assuming that the formula gives exact results for integrating a general third order polynomial, $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

Hence

$$\int_a^b f(x)dx = \int_a^b \left(a_0 + a_1x + a_2x^2 + a_3x^3\right)dx$$

$$= \left[a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + a_3\frac{x^4}{4}\right]_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2-a^2}{2}\right) + a_2\left(\frac{b^3-a^3}{3}\right) + a_3\left(\frac{b^4-a^4}{4}\right)$$

# Basis of the Gaussian Quadrature Rule

It follows that

$$\int_a^b f(x)dx = c_1\left(a_0 + a_1x_1 + a_2x_1^{\ 2} + a_3x_1^{\ 3}\right) + c_2\left(a_0 + a_1x_2 + a_2x_2^{\ 2} + a_3x_2^{\ 3}\right)$$

Equating Equations the two previous two expressions yield

$$a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right)$$

$$= c_1\left(a_0 + a_1x_1 + a_2x_1^{\ 2} + a_3x_1^{\ 3}\right) + c_2\left(a_0 + a_1x_2 + a_2x_2^{\ 2} + a_3x_2^{\ 3}\right)$$

$$= a_0(c_1 + c_2) + a_1(c_1x_1 + c_2x_2) + a_2\left(c_1x_1^{\ 2} + c_2x_2^{\ 2}\right) + a_3\left(c_1x_1^{\ 3} + c_2x_2^{\ 3}\right)$$

# Basis of the Gaussian Quadrature Rule

Since the constants $a_0$, $a_1$, $a_2$, $a_3$ are arbitrary

$$b - a = c_1 + c_2$$

$$\frac{b^2 - a^2}{2} = c_1 x_1 + c_2 x_2$$

$$\frac{b^3 - a^3}{3} = c_1 x_1{}^2 + c_2 x_2{}^2$$

$$\frac{b^4 - a^4}{4} = c_1 x_1{}^3 + c_2 x_2{}^3$$

# Basis of Gauss Quadrature

The previous four simultaneous nonlinear Equations have only one acceptable solution,

$$x_1 = \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}$$

$$x_2 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}$$

$$c_1 = \frac{b-a}{2}$$

$$c_2 = \frac{b-a}{2}$$

# Basis of Gauss Quadrature

Hence Two-Point Gaussian Quadrature Rule

$$\int_a^b f(x)dx \approx c_1 f\left(x_1\right)+c_2 f\left(x_2\right)$$

$$=\frac{b-a}{2} f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}\right)+\frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}\right)$$

# Higher Point Gaussian Quadrature Formulas

# Higher Point Gaussian Quadrature Formulas

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$$

is called the three-point Gauss Quadrature Rule.

The coefficients $c_1$, $c_2$, and $c_3$, and the functional arguments $x_1$, $x_2$, and $x_3$ are calculated by assuming the formula gives exact expressions for integrating a fifth order polynomial

$$\int_a^b \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5\right)dx$$

General n-point rules would approximate the integral

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + \ldots\ldots\ldots + c_n f(x_n)$$

# Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

In handbooks, coefficients and arguments given for n-point Gauss Quadrature Rule are given for integrals

$$\int_{-1}^{1} g(x)dx \approx \sum_{i=1}^{n} c_i g(x_i)$$

as shown in Table 1.

**Table 1: Weighting factors c and function arguments x used in Gauss Quadrature Formulas.**

| Points | Weighting Factors | Function Arguments |
|---|---|---|
| 2 | $c_1$ = 1.000000000<br>$c_2$ = 1.000000000 | $x_1$ = -0.577350269<br>$x_2$ = 0.577350269 |
| 3 | $c_1$ = 0.555555556<br>$c_2$ = 0.888888889<br>$c_3$ = 0.555555556 | $x_1$ = -0.774596669<br>$x_2$ = 0.000000000<br>$x_3$ = 0.774596669 |
| 4 | $c_1$ = 0.347854845<br>$c_2$ = 0.652145155<br>$c_3$ = 0.652145155<br>$c_4$ = 0.347854845 | $x_1$ = -0.861136312<br>$x_2$ = -0.339981044<br>$x_3$ = 0.339981044<br>$x_4$ = 0.861136312 |

# Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

**Table 1 (cont.) : Weighting factors c and function arguments x used in Gauss Quadrature Formulas.**

| Points | Weighting Factors | Function Arguments |
|---|---|---|
| 5 | $c_1$ = 0.236926885<br>$c_2$ = 0.478628670<br>$c_3$ = 0.568888889<br>$c_4$ = 0.478628670<br>$c_5$ = 0.236926885 | $x_1$ = -0.906179846<br>$x_2$ = -0.538469310<br>$x_3$ = 0.000000000<br>$x_4$ = 0.538469310<br>$x_5$ = 0.906179846 |
| 6 | $c_1$ = 0.171324492<br>$c_2$ = 0.360761573<br>$c_3$ = 0.467913935<br>$c_4$ = 0.467913935<br>$c_5$ = 0.360761573<br>$c_6$ = 0.171324492 | $x_1$ = -0.932469514<br>$x_2$ = -0.661209386<br>$x_3$ = -0.2386191860<br>$x_4$ = 0.2386191860<br>$x_5$ = 0.661209386<br>$x_6$ = 0.932469514 |

# Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

So if the table is given for $\int_{-1}^{1} g(x)dx$ integrals, how does one solve $\int_{a}^{b} f(x)dx$? The answer lies in that any integral with limits of $[a, b]$ can be converted into an integral with limits $[-1, 1]$ Let

$$x = mt + c$$

If $x = a,$ then $t = -1$

If $x = b,$ then $t = 1$

Such that:

$$m = \frac{b-a}{2}$$

# Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

Then $$c = \frac{b+a}{2}$$ Hence

$$x = \frac{b-a}{2}t + \frac{b+a}{2}$$

$$dx = \frac{b-a}{2}dt$$

Substituting our values of x, and dx into the integral gives us

$$\int_a^b f(x)dx = \int_{-1}^{1} f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)\frac{b-a}{2}dt$$

# Example 1

For an integral $\int_a^b f(x)dx,$ derive the one-point Gaussian Quadrature Rule.

**Solution**

The one-point Gaussian Quadrature Rule is

$$\int_a^b f(x)dx \approx c_1 f\left(x_1\right)$$

# Solution

The two unknowns $x_1$, and $c_1$ are found by assuming that the formula gives exact results for integrating a general first order polynomial,

$$f(x) = a_0 + a_1 x.$$

$$\int_a^b f(x)dx = \int_a^b (a_0 + a_1 x)dx$$

$$= \left[ a_0 x + a_1 \frac{x^2}{2} \right]_a^b$$

$$= a_0(b - a) + a_1\left(\frac{b^2 - a^2}{2}\right)$$

# Solution

It follows that

$$\int_a^b f(x)dx = c_1\left(a_0 + a_1 x_1\right)$$

Equating Equations, the two previous two expressions yield

$$a_0(b-a)+a_1\left(\frac{b^2-a^2}{2}\right) = c_1\left(a_0 + a_1 x_1\right) = a_0(c_1)+a_1(c_1 x_1)$$

# Basis of the Gaussian Quadrature Rule

Since the constants $a_0$, and $a_1$ are arbitrary

$$b - a = c_1$$

$$\frac{b^2 - a^2}{2} = c_1 x_1$$

giving

$$c_1 = b - a$$

$$x_1 = \frac{b + a}{2}$$

# Solution

Hence One-Point Gaussian Quadrature Rule

$$\int_a^b f(x)dx \approx c_1 f\left(x_1\right) = (b-a)\ f\left(\frac{b+a}{2}\right)$$

# Example 2

The concentration of benzene at a critical location is given by

$$c = 1.75\left[erfc(0.6560) + e^{32.73}erfc(5.758)\right]$$

where

$$erfc(x) = \int_{\infty}^{x} e^{-z^2}\,dz$$

So in the above formula

$$erfc(0.6560) = \int_{\infty}^{0.6560} e^{-z^2}\,dz$$

Since $e^{-z^2}$ decays rapidly as $z \to \infty$, we will approximate

$$erfc(0.6560) = \int_{5}^{0.6560} e^{-z^2}\,dz$$

a) Use two-point Gauss Quadrature Rule to approximate the value of *erfc*(0.6560).

b) Find the true error, $E_t$ for part (a).

c) Also, find the absolute relative true error, $\left|\in_a\right|$ for part (a).

# Solution

a) First, change the limits of integration from [5,0.6560] to [1,−1] by previous relations as follows

$$\int_{5}^{0.6560} f(z)dz = \frac{0.6560-5}{2}\int_{-1}^{1} f\left(\frac{0.6560-5}{2}z+\frac{0.6560+5}{2}\right)dz$$

$$= -2.1720\int_{-1}^{1} f\left(-2.1720z+2.8280\right)dz$$

# Solution (cont)

Next, get weighting factors and function argument values from Table 1 for the two point rule,

$$c_1 = \ 1.0000$$

$$z_1 = -0.57735$$

$$c_2 = \ 1.0000$$

$$z_2 = \ 0.57735$$

# Solution (cont.)

Now we can use the Gauss Quadrature formula

$$-2.1720\int_{-1}^{1} f\left(-2.1720z + 2.8280\right)dz$$

$$\approx -2.1720\left[c_1 f\left(-2.1720z_1 + 2.8280\right) + c_2 f\left(-2.1720z_2 + 2.8280\right)\right]$$

$$\approx -2.1720\left[f\left(-2.1720\left(-0.57735\right) + 2.8280\right) + f\left(-2.1720\left(0.57735\right) + 2.8280\right)\right]$$

$$\approx -2.1720\left[f(4.0820) + f(1.5740)\right]$$

$$\approx -2.1720\left[\left(5.8003\times 10^{-8}\right) + \left(0.083955\right)\right]$$

$$\approx -0.18235$$

# Solution (cont)

since

$$f(4.0820) = e^{-4.0820^2} = 5.8003\times10^{-8}$$

$$f(1.5740) = e^{-1.5740^2} = 0.083955$$

# Solution (cont)

b) The true error, $E_t$ , is

$$E_t = True\ Value - Approximate\ Value$$

$$= -0.31333 - (-0.18235)$$

$$= -0.13098$$

c) The absolute relative true error, $\left|\in_t\right|$ , is (Exact value = −0.31333)

$$\left|\in_t\right| = \left|\frac{-0.31333 - (-0.18235)}{-0.31333}\right| \times 100\%$$

$$= 41.801\%$$

# Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/gauss_quadrature.html

# THE END

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