Bisection Method

Computer Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

Bisection Method

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Basis of Bisection Method

Theorem An equation f(x)=0, where f(x) is a real continuous function, has at least one root between x_l and x_u if $f(x_l)\, f(x_u) < 0$.

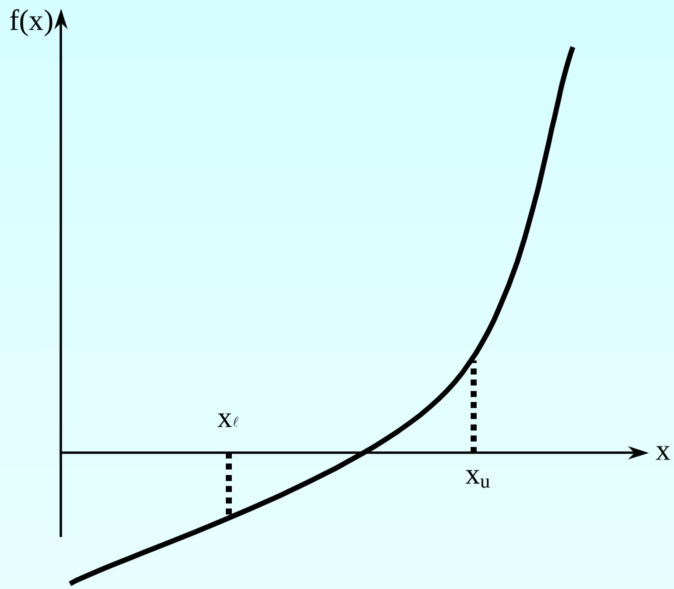


Figure 1 At least one root exists between the two points if the function is real, continuous, and changes sign.

Basis of Bisection Method

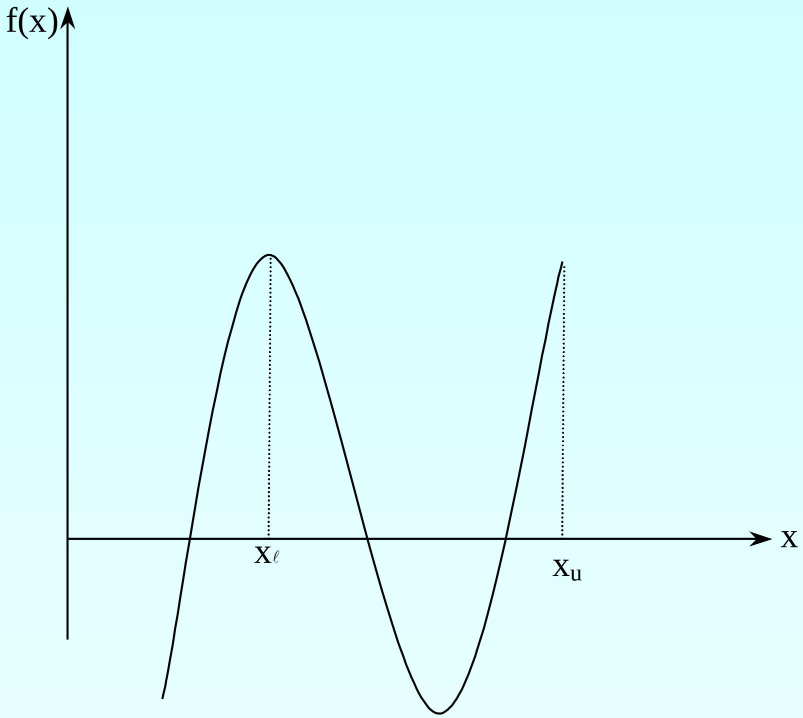


Figure 2 If function $f(x)$ does not change sign between two points, roots of the equation $f(x)=0$ may still exist between the two points.

Basis of Bisection Method

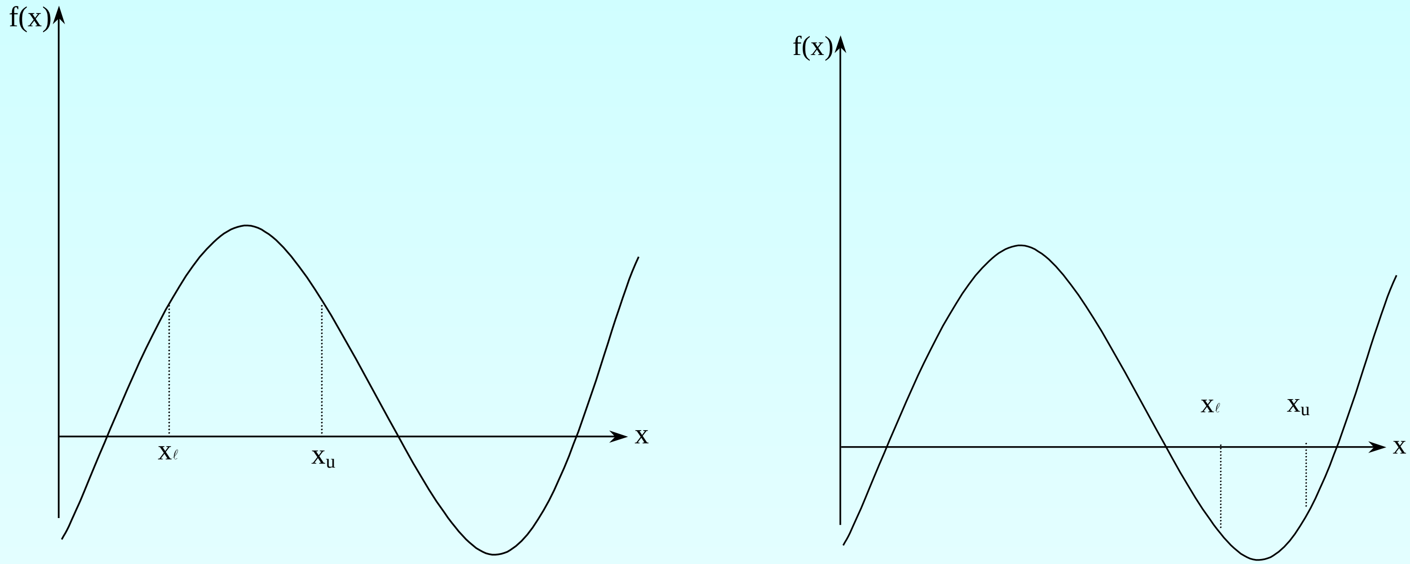


Figure 3 If the function $f(x)$ does not change sign between two points, there may not be any roots for the equation $f(x)=0$ between the two points.

Basis of Bisection Method

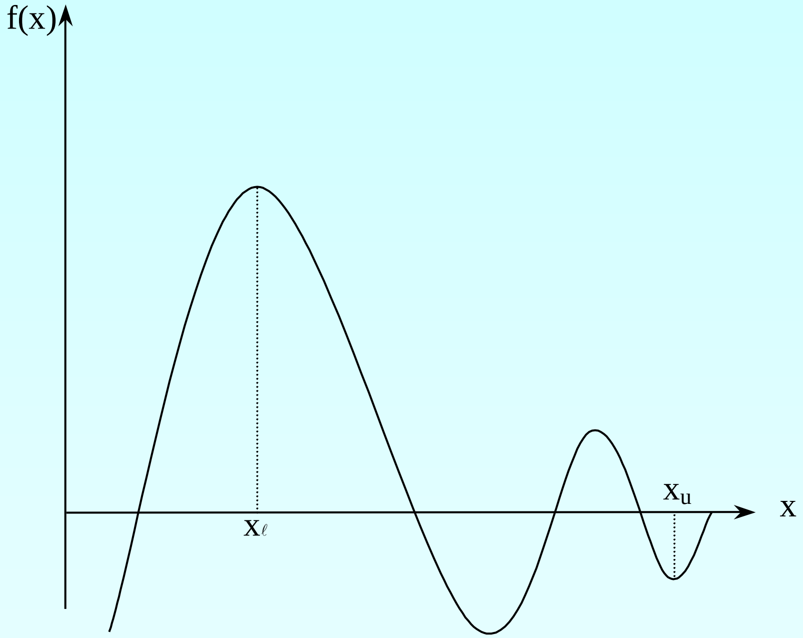


Figure 4 If the function $f(x)$ changes sign between two points, more than one root for the equation $f(x)=0$ may exist between the two points.

Algorithm for Bisection Method

Step 1

Choose x_ℓ and x_u as two guesses for the root such that $f(x_\ell)\, f(x_u) < 0$, or in other words, f(x) changes sign between x_ℓ and x_u. This was demonstrated in Figure 1.

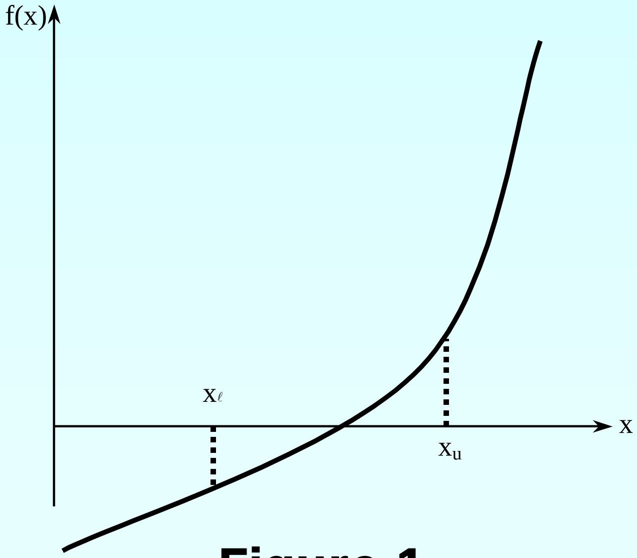


Figure 1

Step 2

Estimate the root, x_m of the equation f (x) = 0 as the mid point between x_ℓ and x_u as

$$x_m = \frac{x_\ell + x_u}{2}$$

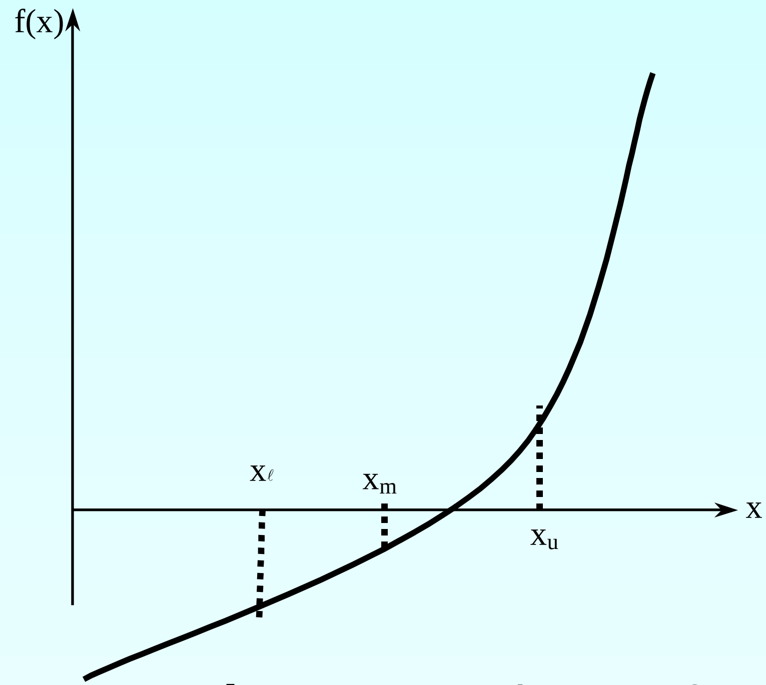


Figure 5 Estimate of x_m

Step 3

Now check the following

a) If $f(x_l)f(x_m) < 0$, then the root lies between x_ℓ and x_m; then $x_\ell = x_\ell$; $x_u = x_m$.

b) If $f(x_l)f(x_m) > 0$, then the root lies between x_m and x_u; then $x_\ell = x_m$; $x_u = x_u$.

c) If $f(x_l)f(x_m) = 0$; then the root is $x_{m.}$ Stop the algorithm if this is true.

Step 4

Find the new estimate of the root

$$x_m = \frac{x_\ell + x_u}{2}$$

Find the absolute relative approximate error

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

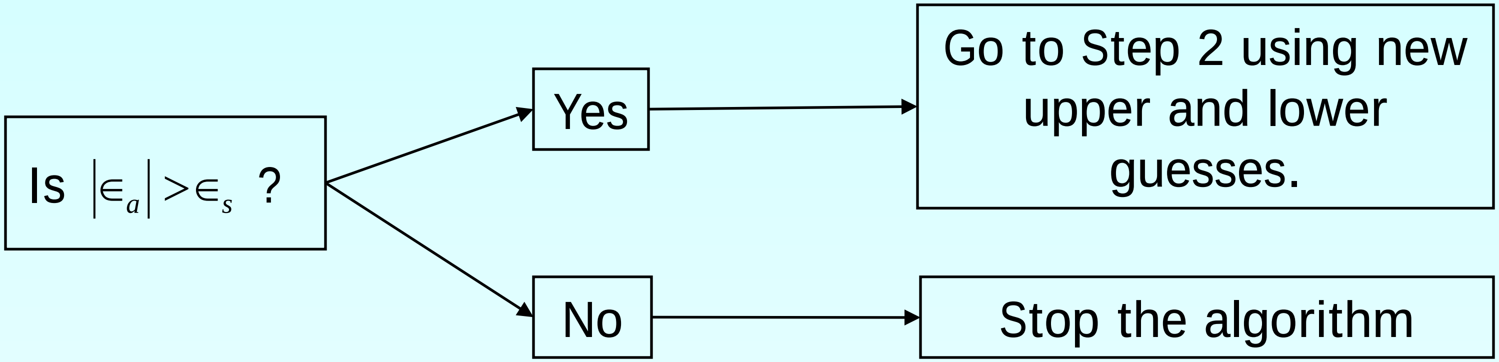
where

$$x_m^{old} = \text{previous estimate of root}$$

$$x_m^{new} = \text{current estimate of root}$$

Step 5

Compare the absolute relative approximate error $|\in_a|$ with the pre-specified error tolerance $\in_s$.



Note one should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user about it.

Example 1

To find the inverse of a value, a, one can use the equation

$$f(x) = a - \frac{1}{x} = 0$$

where x is the inverse of a.

Use the bisection method of finding roots of equations to find the inverse of a = 2.5. Conduct three iterations to estimate the root of the above equation.

Find the absolute relative approximate error at the end of each iteration and the number of significant digits at least correct at the end of each iteration.

Example 1 Cont.

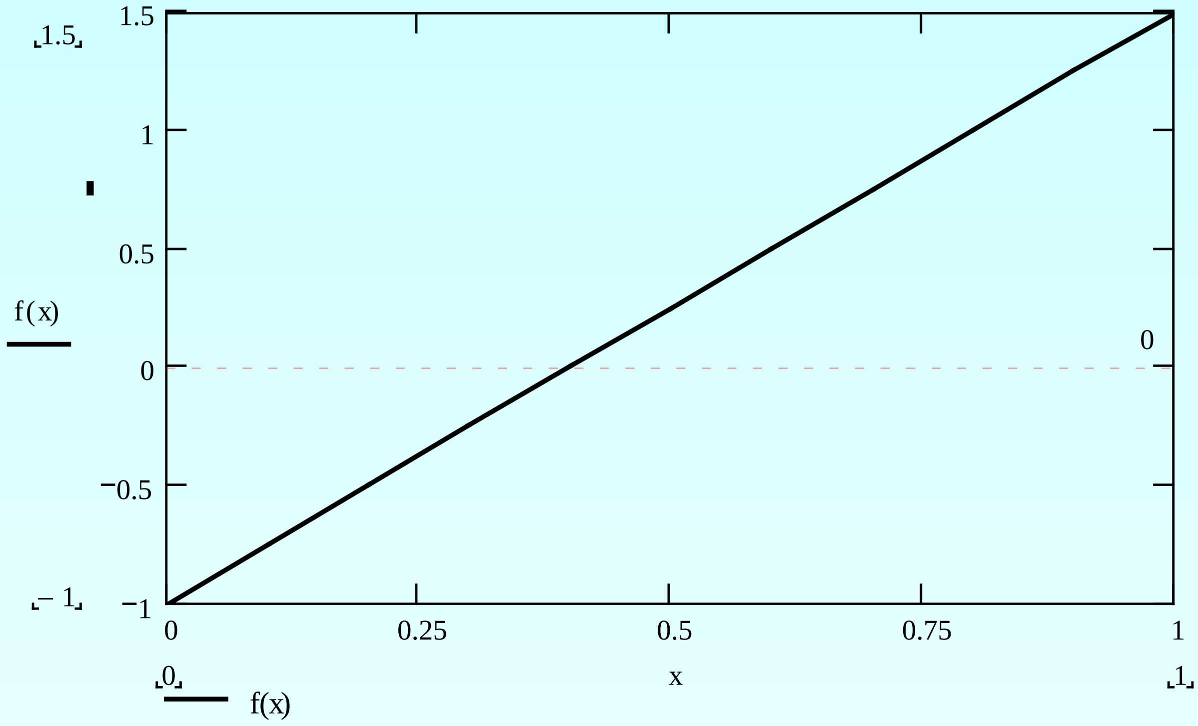


Figure 8 Graph of the function f(x).

$$f(x) = a - \frac{1}{x} = 0$$

Example 1 Cont.

Solution

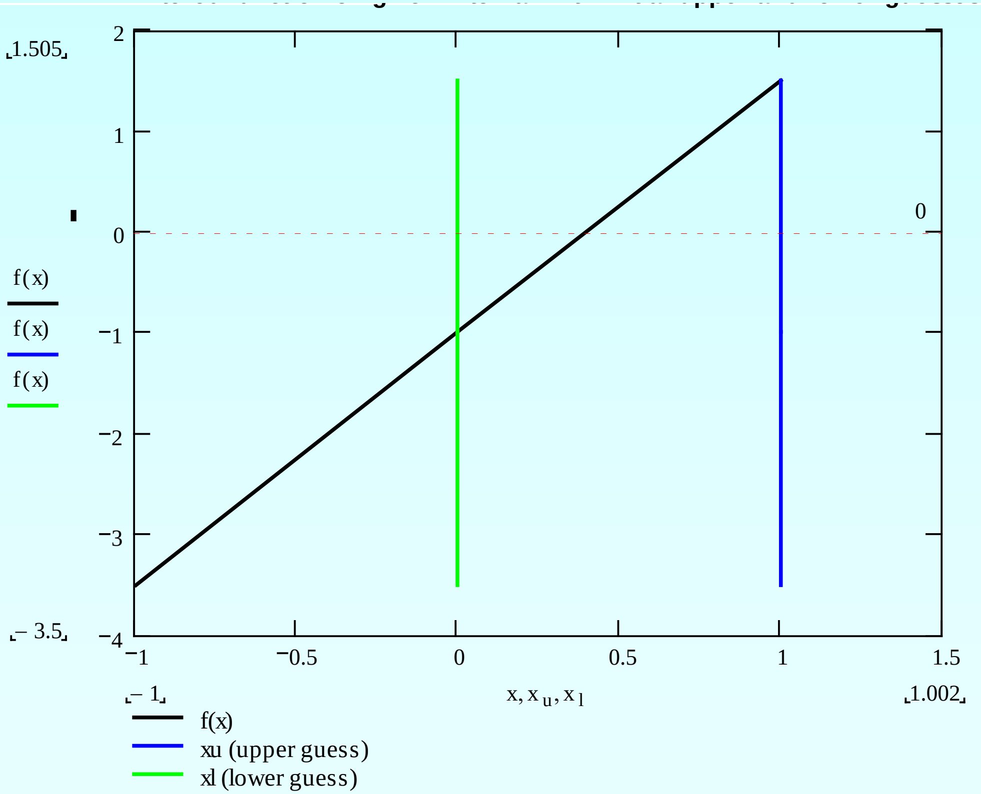


Figure 9 Checking that the bracket is valid.

$$f(x) = a - \frac{1}{x} = 0$$
$$= ax - 1$$
$$= 2.5x - 1$$

Let us assume $x_l = 0,\ x_u = 1$

Check if the function changes sign between x_l and x_u.

$$f(x_l) = f(0) = 2.5(0) - 1 = -1$$
$$f(x_u) = f(1) = 2.5(1) - 1 = 1.5$$
$$f(x_l)f(x_u) = f(0)f(1) = (-1)(1.5) < 0$$

There is at least one root between the brackets.

Example 1 Cont.

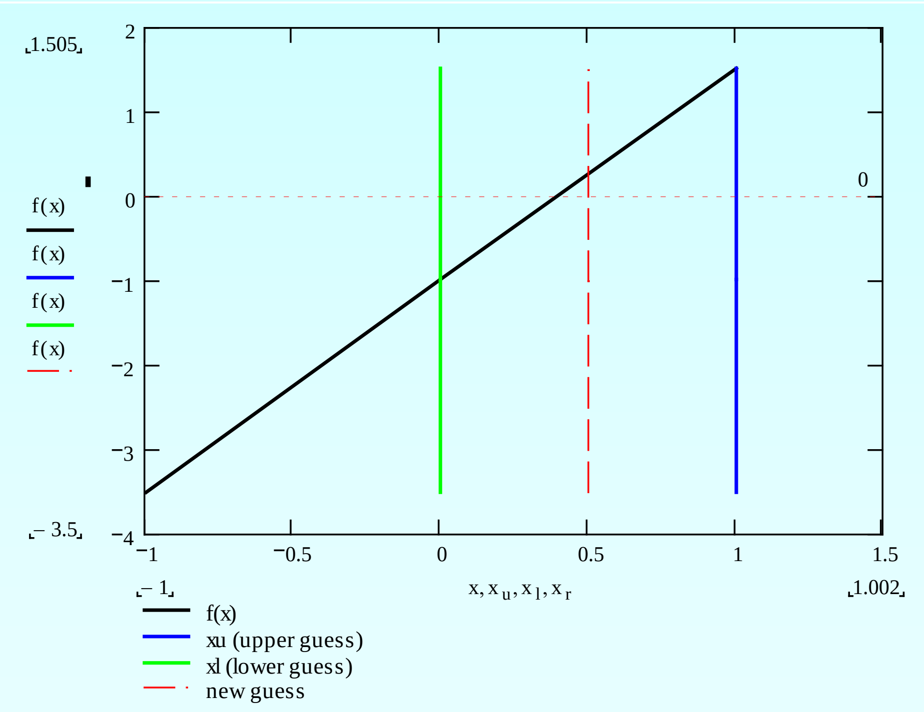


Figure 10 Graph of the estimated root after Iteration 1.

<u>Iteration 1</u>

The estimate of the root is

$$x_m = \frac{x_l + x_u}{2} = \frac{0+1}{2} = 0.5$$

$$f(x_m) = f(0.5) = 2.5(0.5) - 1 = 0.25$$

$$f(x_l)f(x_m) = f(0)f(0.5) = (-1)(0.25) < 0$$

The root is bracketed between x_l and x_m. The lower and upper limits of the new bracket are

$$x_l = 0,\ x_u = 0.5$$

The absolute relative approximate error $|\in_a|$ cannot be calculated as we do not have a previous approximation.

Example 1 Cont.

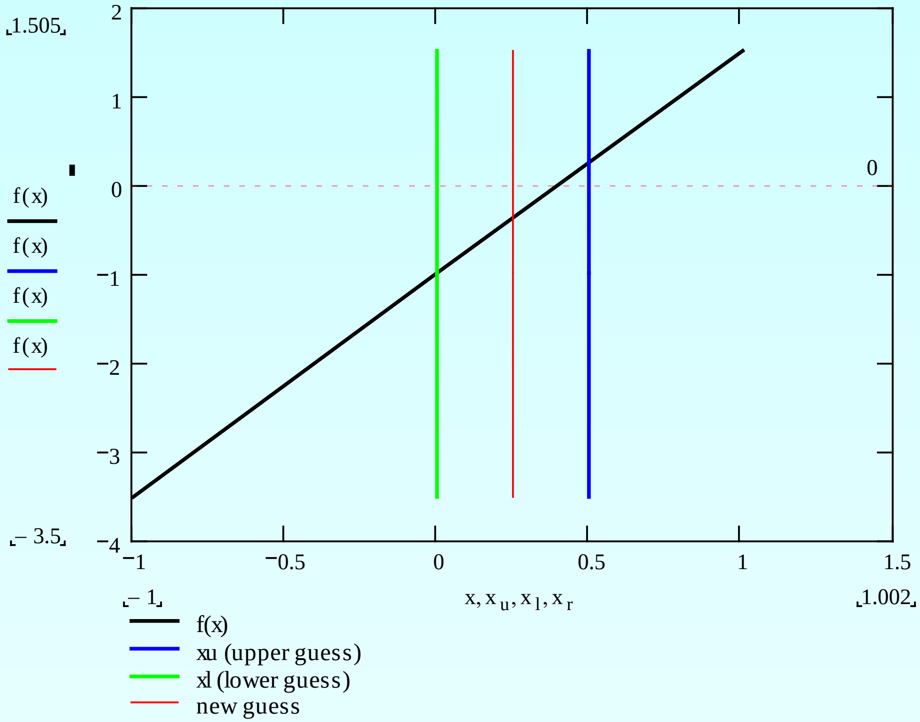


Figure 11 Graph of the estimated root after Iteration 2.

<u>Iteration 2</u>

The estimate of the root is

$$x_m = \frac{x_l + x_u}{2} = \frac{0 + 0.5}{2} = 0.25$$

$$f(x_m) = f(0.25) = 2.5(0.25) - 1 = -0.375$$

$$f(x_m) f(x_u) = f(0.25) f(0.5)$$
$$= (-0.375)(0.25) < 0$$

The root is bracketed between x_m and x_u.

The lower and upper limits of the new bracket are

$$x_l = 0.25, \; x_u = 0.5$$

Example 1 Cont.

The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 2 is

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$
$$= \left|\frac{0.25 - 0.5}{0.25}\right| \times 100$$
$$= 100\%$$

None of the significant digits are at least correct in the estimated root of

$$x_m = 0.25$$

as the absolute relative approximate error is greater than 5%.

Example 1 Cont.

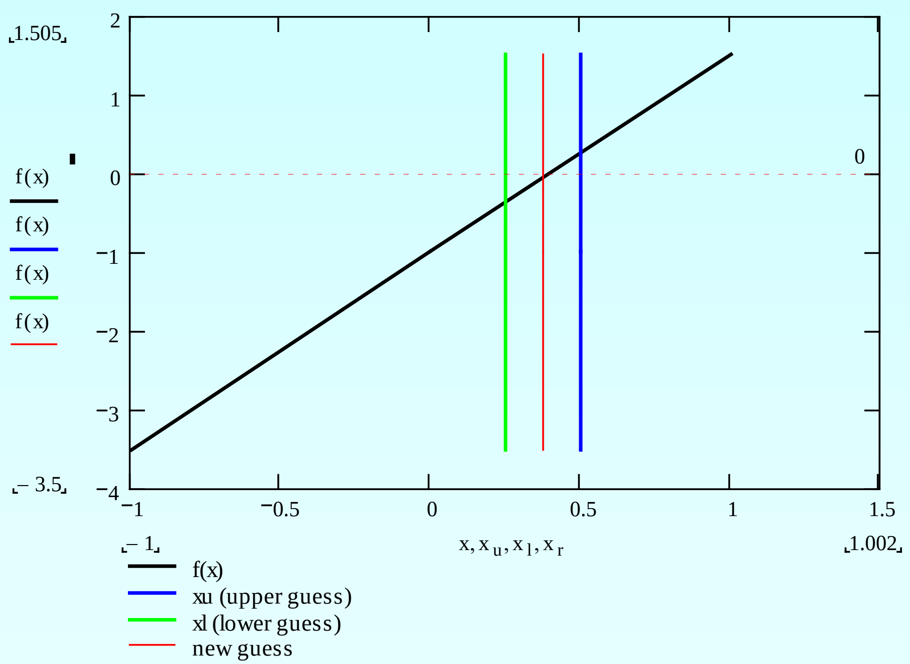


Figure 12 Graph of the estimated root after Iteration 3.

<u>Iteration 3</u>

The estimate of the root is

$$x_m = \frac{x_l + x_u}{2} = \frac{0.25 + 0.5}{2} = 0.375$$

$$f(x_m) = f(0.375) = 2.5(0.375) - 1 = -0.0625$$

$$f(x_m) f(x_u) = f(0.375) f(0.5)$$
$$= (-0.0625)(0.25) < 0$$

The root is bracketed between x_m and x_u .

The lower and upper limits of the new bracket are

$$x_l = 0.25, \ \ x_u = 0.5$$

Example 1 Cont.

The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 3 is

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$
$$= \left|\frac{0.375 - 0.25}{0.375}\right| \times 100$$
$$= 33.333\%$$

Still none of the significant digits are at least correct in the estimated root of the equation as the absolute relative approximate error is greater than 5%. Seven more iterations were conducted and these iterations are shown in the table below.

Example 1 Cont.

Table 1 Root of $f(x)=0$ as function of number of iterations for bisection method.

Iteration	x_l	x_u	x_m	$\lvert\in_a\rvert\%$	$f(x_m)$
1	0	1	0.5	----------	0.25
2	0	0.5	0.25	100	−0.375
3	0.25	0.5	0.375	33.33	−0.0625
4	0.375	0.5	0.4375	14.2857	0.09375
5	0.375	0.4375	0.40625	7.6923	0.01563
6	0.375	0.40625	0.39063	4.00	−0.02344
7	0.39063	0.40625	0.39844	1.9608	$-3.90625 \ 10^{-3}$
8	0.39844	0.40625	0.40234	0.97087	$5.8594 \ 10^{-3}$
9	0.39844	0.40234	0.40039	0.48780	$9.7656 \ 10^{-4}$
10	0.39844	0.40039	0.39941	0.24450	$-1.4648 \ 10^{-3}$

Advantages

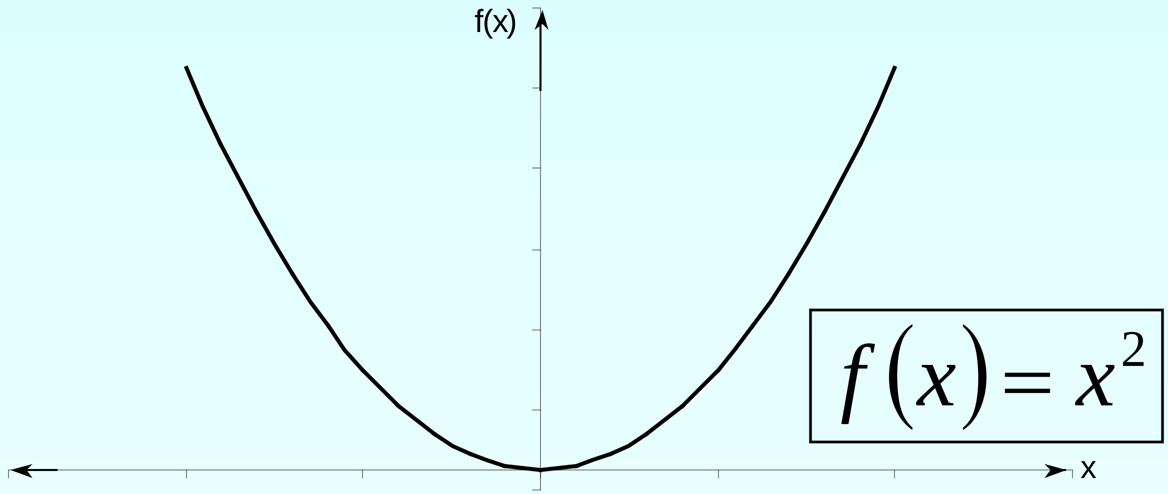
- Always convergent
- The root bracket gets halved with each iteration - guaranteed.

Drawbacks

- Slow convergence
- If one of the initial guesses is close to the root, the convergence is slower

Drawbacks (continued)

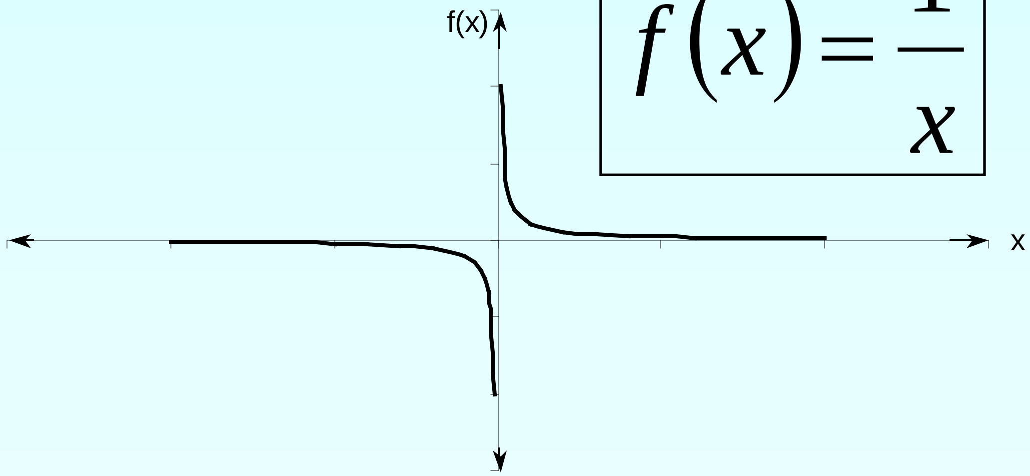
- If a function f(x) is such that it just touches the x-axis it will be unable to find the lower and upper guesses.



Drawbacks (continued)

- Function changes sign but root does not exist

$$f(x) = \frac{1}{x}$$



Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/bisection_method.html

THE END

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