

Lagrangian Interpolation

Computer Engineering Majors

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<http://numericalmethods.eng.usf.edu>

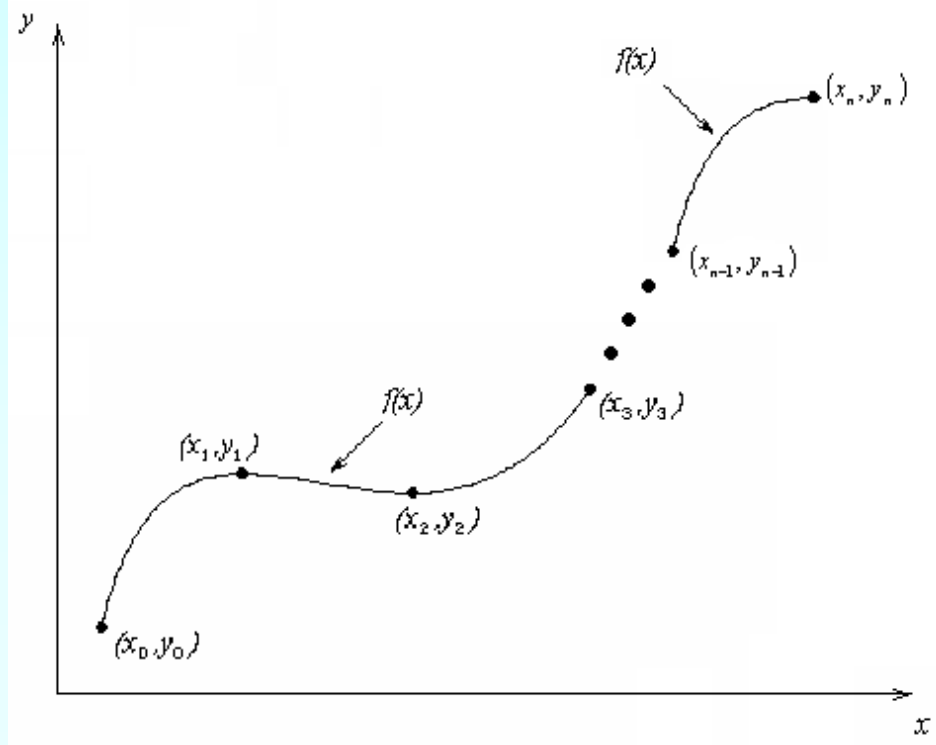
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Lagrange Method of Interpolation

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What is Interpolation ?

Given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$, find the value of 'y' at a value of 'x' that is not given.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Lagrangian Interpolation

Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^n L_i(x) f(x_i)$$

where ‘ n ’ in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $(n + 1)$ data points as $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $(n - 1)$ terms with terms of $j = i$ omitted.

Example

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a rectangular plate. The hole centers in the plate that describe the path the arm needs to take are given below.

If the laser is traversing from $x = 2$ to $x = 4.25$ in a linear path, find the value of y at $x = 4$ using the Lagrange method for linear interpolation.

x (m)	y (m)
2	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.2	3.5
10.6	5.0

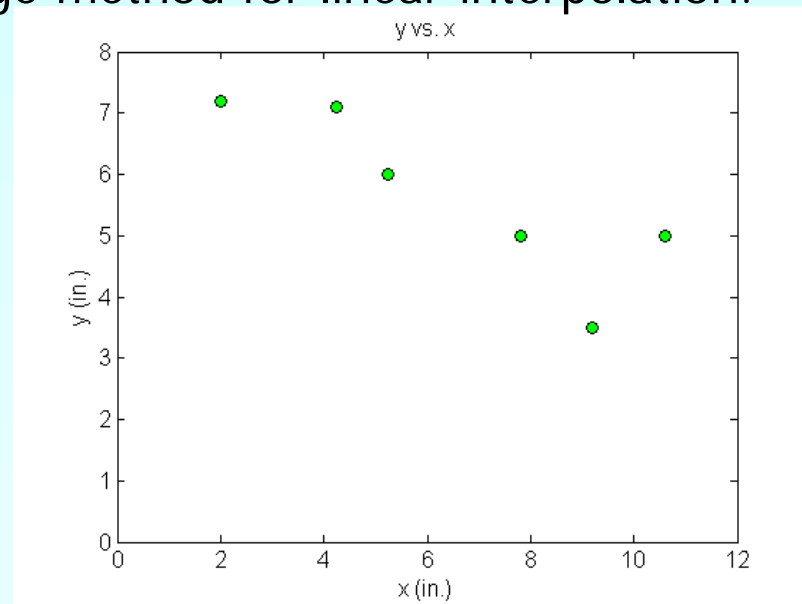


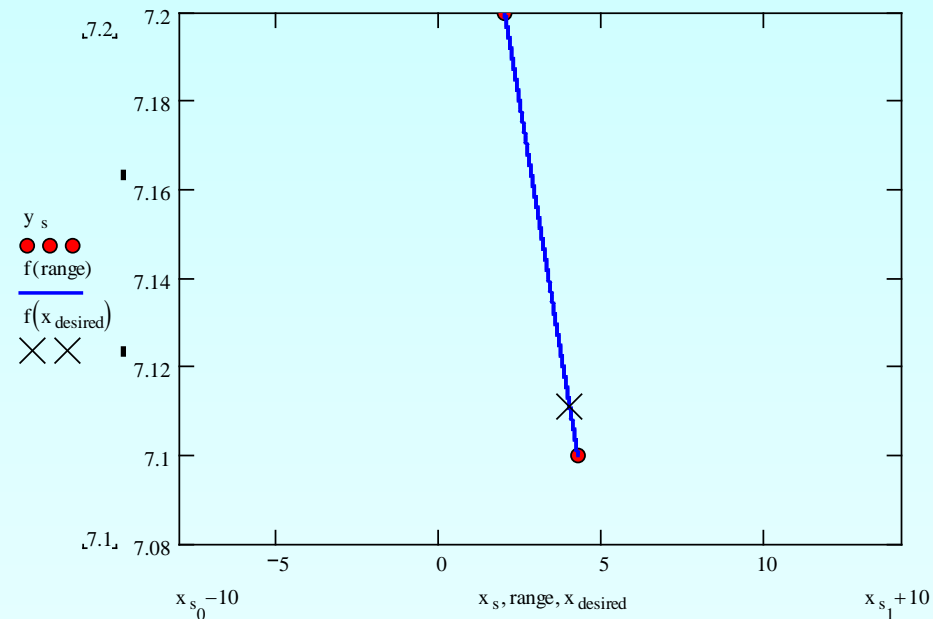
Figure 2 Location of holes on the rectangular plate.

Linear Interpolation

$$y(x) = \sum_{i=0}^1 L_i(x) y(x_i)$$
$$= L_0(x) y(x_0) + L_1(x) y(x_1)$$

$$x_0 = 2.00, y(x_0) = 7.2$$

$$x_1 = 4.25, y(x_1) = 7.1$$



Linear Interpolation (contd)

$$L_0(x) = \prod_{\substack{j=0 \\ j \neq 0}}^1 \frac{x - x_j}{x_0 - x_j} = \frac{x - x_1}{x_0 - x_1}$$

$$L_1(x) = \prod_{\substack{j=0 \\ j \neq 1}}^1 \frac{x - x_j}{x_1 - x_j} = \frac{x - x_0}{x_1 - x_0}$$

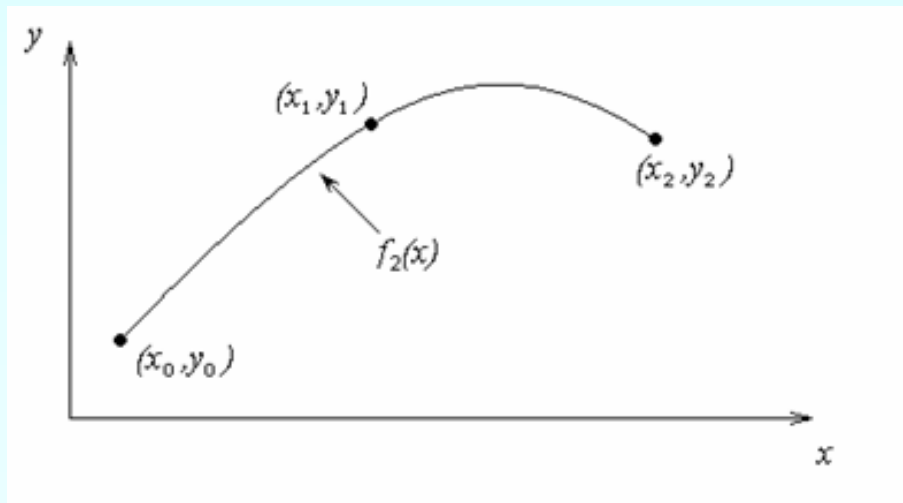
$$\begin{aligned} y(x) &= \frac{x - x_1}{x_0 - x_1} y(x_0) + \frac{x - x_0}{x_1 - x_0} y(x_1) \\ &= \frac{x - 4.25}{2.00 - 4.25} (7.2) + \frac{x - 2.00}{4.25 - 2.00} (7.1), \quad 2.00 \leq x \leq 4.25 \end{aligned}$$

$$\begin{aligned} y(4.00) &= \frac{4.00 - 4.25}{2.00 - 4.25} (7.2) + \frac{4.00 - 2.00}{4.25 - 2.00} (7.1) \\ &= 0.11111(7.2) + 0.88889(7.1) \\ &= 7.1111 \text{ in.} \end{aligned}$$

Quadratic Interpolation

For the second order polynomial interpolation (also called quadratic interpolation), we choose the velocity given by

$$\begin{aligned}v(t) &= \sum_{i=0}^2 L_i(t)v(t_i) \\ &= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2)\end{aligned}$$



Example

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a rectangular plate. The hole centers in the plate that describe the path the arm needs to take are given below.

If the laser is traversing from $x = 2$ to $x = 4.25$ in a linear path, find the value of y at $x = 4$ using the Lagrange method for quadratic interpolation.

x (m)	y (m)
2	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.2	3.5
10.6	5.0

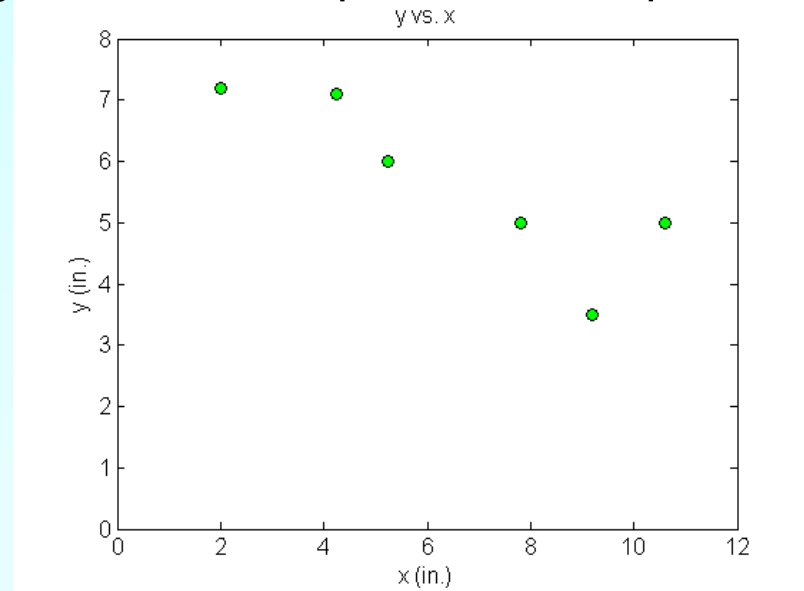


Figure 2 Location of holes on the rectangular plate.

Quadratic Interpolation

$$x_0 = 2.00, \quad y(x_0) = 7.2$$

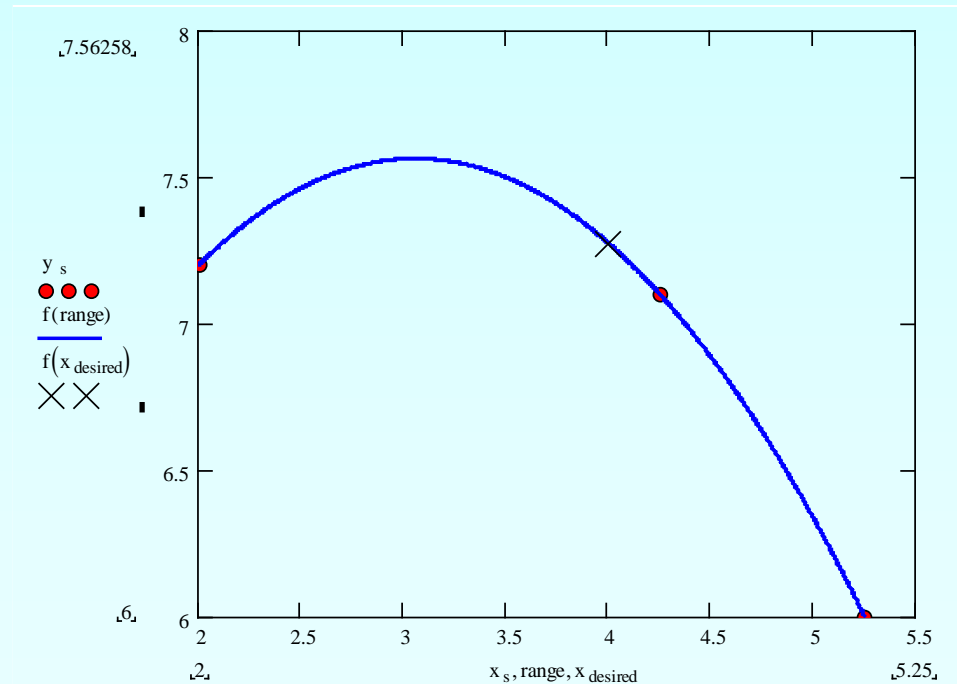
$$x_1 = 4.25, \quad y(x_1) = 7.1$$

$$x_2 = 5.25, \quad y(x_2) = 6.0$$

$$L_0(x) = \prod_{\substack{j=0 \\ j \neq 0}}^2 \frac{x - x_j}{x_0 - x_j} = \left(\frac{x - x_1}{x_0 - x_1} \right) \left(\frac{x - x_2}{x_0 - x_2} \right)$$

$$L_1(x) = \prod_{\substack{j=0 \\ j \neq 1}}^2 \frac{x - x_j}{x_1 - x_j} = \left(\frac{x - x_0}{x_1 - x_0} \right) \left(\frac{x - x_2}{x_1 - x_2} \right)$$

$$L_2(x) = \prod_{\substack{j=0 \\ j \neq 2}}^2 \frac{x - x_j}{x_2 - x_j} = \left(\frac{x - x_0}{x_2 - x_0} \right) \left(\frac{x - x_1}{x_2 - x_1} \right)$$



Quadratic Interpolation (contd)

$$y(x) = \left(\frac{x - x_1}{x_0 - x_1} \right) \left(\frac{x - x_2}{x_0 - x_2} \right) y(x_0) + \left(\frac{x - x_0}{x_1 - x_0} \right) \left(\frac{x - x_2}{x_1 - x_2} \right) y(x_1) + \left(\frac{x - x_0}{x_2 - x_0} \right) \left(\frac{x - x_1}{x_2 - x_1} \right) y(x_2)$$

$$\begin{aligned} y(4.00) &= \frac{(4.00 - 4.25)(4.00 - 5.25)}{(2.00 - 4.25)(2.00 - 5.25)} (7.2) + \frac{(4.00 - 2.00)(4.00 - 5.25)}{(4.25 - 2.00)(4.25 - 5.25)} (7.1) + \frac{(4.00 - 2.00)(4.00 - 4.25)}{(5.25 - 2.00)(5.25 - 4.25)} (6.0) \\ &= (0.042735)(7.2) + (1.1111)(7.1) + (-0.15385)(6.0) \\ &= 7.2735 \text{ in.} \end{aligned}$$

The absolute relative approximate error $|\epsilon_a|$ obtained between the results from the first and second order polynomial is

$$\begin{aligned} |\epsilon_a| &= \left| \frac{7.2735 - 7.1111}{7.2735} \right| \times 100 \\ &= 2.2327\% \end{aligned}$$

Comparison Table

Order of Polynomial	1	2
Location (in.)	7.1111	7.2735
Absolute Relative Approximate Error	-----	2.2327%

Example

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a rectangular plate. The hole centers in the plate that describe the path the arm needs to take are given below.

If the laser is traversing from $x = 2$ to $x = 4.25$ in a linear path, find the value of y at $x = 4$ using a fifth order Lagrange polynomial.

x (m)	y (m)
2	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.2	3.5
10.6	5.0

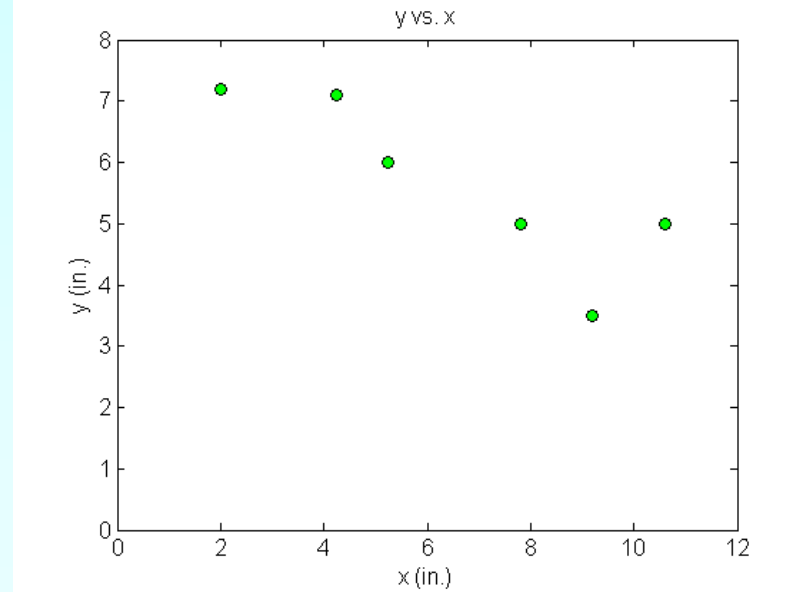


Figure 2 Location of holes on the rectangular plate.

Fifth Order Interpolation

$$y(x) = \sum_{i=0}^5 L_i(x) y(x_i)$$

$$= L_0(x)y(x_0) + L_1(x)y(x_1) + L_2(x)y(x_2) + L_3(x)y(x_3) + L_4(x)y(x_4) + L_5(x)y(x_5)$$

$$x_0 = 2.00, \quad y(x_0) = 7.2$$

$$x_1 = 4.25, \quad y(x_1) = 7.1$$

$$x_2 = 5.25, \quad y(x_2) = 6.0$$

$$x_3 = 7.81, \quad y(x_3) = 5.0$$

$$x_4 = 9.20, \quad y(x_4) = 3.5$$

$$x_5 = 10.60, \quad y(x_5) = 5.0$$

Fifth Order Interpolation (contd)

$$L_0(x) = \prod_{\substack{j=0 \\ j \neq 0}}^5 \frac{x - x_j}{x_0 - x_j} = \left(\frac{x - x_1}{x_0 - x_1} \right) \left(\frac{x - x_2}{x_0 - x_2} \right) \left(\frac{x - x_3}{x_0 - x_3} \right) \left(\frac{x - x_4}{x_0 - x_4} \right) \left(\frac{x - x_5}{x_0 - x_5} \right)$$

$$L_1(x) = \prod_{\substack{j=0 \\ j \neq 1}}^5 \frac{x - x_j}{x_1 - x_j} = \left(\frac{x - x_0}{x_1 - x_0} \right) \left(\frac{x - x_2}{x_1 - x_2} \right) \left(\frac{x - x_3}{x_1 - x_3} \right) \left(\frac{x - x_4}{x_1 - x_4} \right) \left(\frac{x - x_5}{x_1 - x_5} \right)$$

$$L_2(x) = \prod_{\substack{j=0 \\ j \neq 2}}^5 \frac{x - x_j}{x_2 - x_j} = \left(\frac{x - x_0}{x_2 - x_0} \right) \left(\frac{x - x_1}{x_2 - x_1} \right) \left(\frac{x - x_3}{x_2 - x_3} \right) \left(\frac{x - x_4}{x_2 - x_4} \right) \left(\frac{x - x_5}{x_2 - x_5} \right)$$

$$L_3(x) = \prod_{\substack{j=0 \\ j \neq 3}}^5 \frac{x - x_j}{x_3 - x_j} = \left(\frac{x - x_0}{x_3 - x_0} \right) \left(\frac{x - x_1}{x_3 - x_1} \right) \left(\frac{x - x_2}{x_3 - x_2} \right) \left(\frac{x - x_4}{x_3 - x_4} \right) \left(\frac{x - x_5}{x_3 - x_5} \right)$$

$$L_4(x) = \prod_{\substack{j=0 \\ j \neq 4}}^5 \frac{x - x_j}{x_4 - x_j} = \left(\frac{x - x_0}{x_4 - x_0} \right) \left(\frac{x - x_1}{x_4 - x_1} \right) \left(\frac{x - x_2}{x_4 - x_2} \right) \left(\frac{x - x_3}{x_4 - x_3} \right) \left(\frac{x - x_5}{x_4 - x_5} \right)$$

$$L_5(x) = \prod_{\substack{j=0 \\ j \neq 5}}^5 \frac{x - x_j}{x_5 - x_j} = \left(\frac{x - x_0}{x_5 - x_0} \right) \left(\frac{x - x_1}{x_5 - x_1} \right) \left(\frac{x - x_2}{x_5 - x_2} \right) \left(\frac{x - x_3}{x_5 - x_3} \right) \left(\frac{x - x_4}{x_5 - x_4} \right)$$

Fifth Order Polynomial (contd)

$$\begin{aligned}y(x) = & \left(\frac{x-x_1}{x_0-x_1} \right) \left(\frac{x-x_2}{x_0-x_2} \right) \left(\frac{x-x_3}{x_0-x_3} \right) \left(\frac{x-x_4}{x_0-x_4} \right) \left(\frac{x-x_5}{x_0-x_5} \right) y(x_0) \\ & + \left(\frac{x-x_0}{x_1-x_0} \right) \left(\frac{x-x_2}{x_1-x_2} \right) \left(\frac{x-x_3}{x_1-x_3} \right) \left(\frac{x-x_4}{x_1-x_4} \right) \left(\frac{x-x_5}{x_1-x_5} \right) y(x_1) \\ & + \left(\frac{x-x_0}{x_2-x_0} \right) \left(\frac{x-x_1}{x_2-x_1} \right) \left(\frac{x-x_3}{x_2-x_3} \right) \left(\frac{x-x_4}{x_2-x_4} \right) \left(\frac{x-x_5}{x_2-x_5} \right) y(x_2) \\ & + \left(\frac{x-x_0}{x_3-x_0} \right) \left(\frac{x-x_1}{x_3-x_1} \right) \left(\frac{x-x_2}{x_3-x_2} \right) \left(\frac{x-x_4}{x_3-x_4} \right) \left(\frac{x-x_5}{x_3-x_5} \right) y(x_3) \\ & + \left(\frac{x-x_0}{x_4-x_0} \right) \left(\frac{x-x_1}{x_4-x_1} \right) \left(\frac{x-x_2}{x_4-x_2} \right) \left(\frac{x-x_3}{x_4-x_3} \right) \left(\frac{x-x_5}{x_4-x_5} \right) y(x_4) \\ & + \left(\frac{x-x_0}{x_5-x_0} \right) \left(\frac{x-x_1}{x_5-x_1} \right) \left(\frac{x-x_2}{x_5-x_2} \right) \left(\frac{x-x_3}{x_5-x_3} \right) \left(\frac{x-x_4}{x_5-x_4} \right) y(x_5)\end{aligned}$$

Fifth Order Polynomial (contd)

$$\begin{aligned}
 y(x) = & \frac{(x-4.25)(x-5.25)(x-7.81)(x-9.20)(x-10.60)}{(2.00-4.25)(2.00-5.25)(2.00-7.81)(2.00-9.20)(2.00-10.60)} \quad (7.2) \\
 + & \frac{(x-2.00)(x-5.25)(x-7.81)(x-9.20)(x-10.60)}{(4.25-2.00)(4.25-5.25)(4.25-7.81)(4.25-9.20)(4.25-10.60)} \quad (7.1) \\
 + & \frac{(x-2.00)(x-4.25)(x-7.81)(x-9.20)(x-10.60)}{(5.25-2.00)(5.25-4.25)(5.25-7.81)(5.25-9.20)(5.25-10.60)} \quad (6.0) \\
 + & \frac{(x-2.00)(x-4.25)(x-5.25)(x-9.20)(x-10.60)}{(7.81-2.00)(7.81-4.25)(7.81-5.25)(7.81-9.20)(7.81-10.60)} \quad (5.0) \\
 + & \frac{(x-2.00)(x-4.25)(x-5.25)(x-7.81)(x-10.60)}{(9.20-2.00)(9.20-4.25)(9.20-5.25)(9.20-7.81)(9.20-10.60)} \quad (3.5) \\
 + & \frac{(x-2.00)(x-4.25)(x-5.25)(x-7.81)(x-9.20)}{(10.60-2.00)(10.60-4.25)(10.60-5.25)(10.60-7.81)(10.60-9.20)} \quad (5.0)
 \end{aligned}$$

Fifth Order Polynomial (contd)

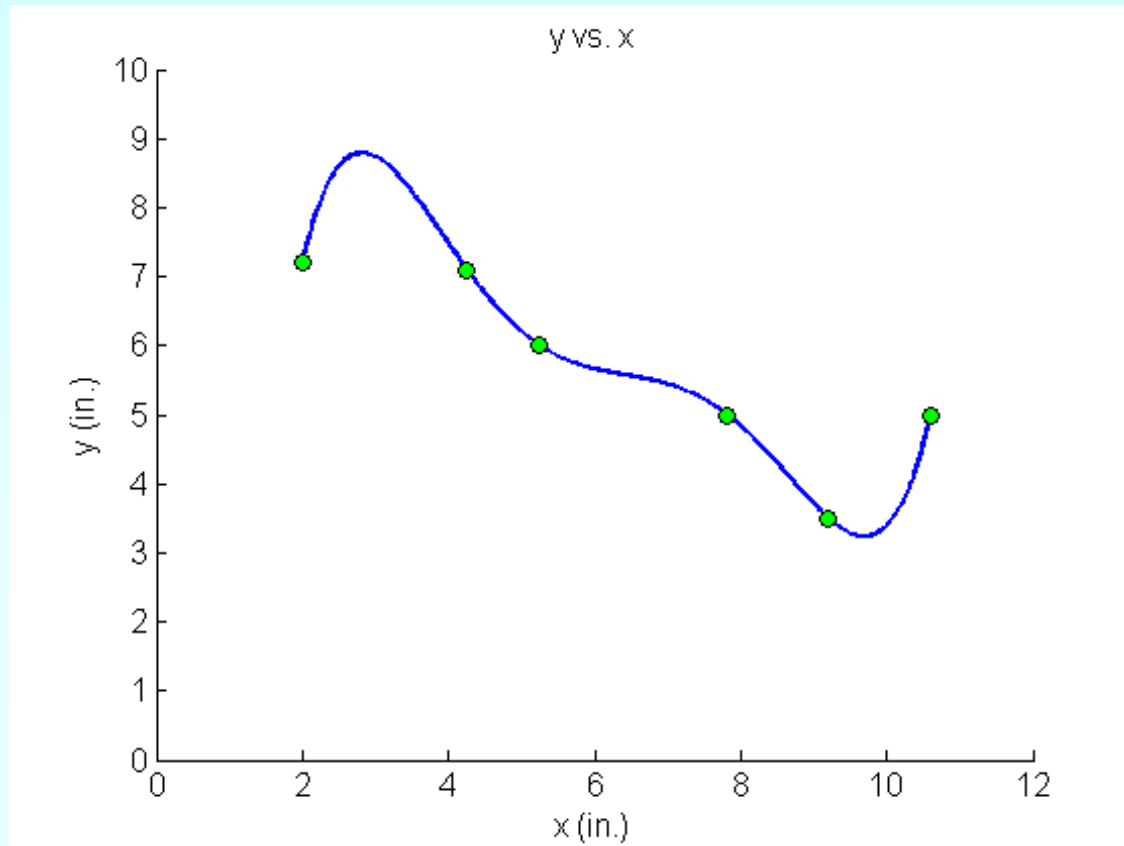
$$\begin{aligned}y(x) = & \frac{x^5 - 37.11x^4 + 536.77x^3 - 3773.2x^2 + 12862x - 16994}{-365.38} \\ & + \frac{x^5 - 34.86x^4 + 462.83x^3 - 2879.7x^2 + 8169.5x - 7997.1}{35.461} \\ & + \frac{x^5 - 33.86x^4 + 433.22x^3 - 2572.3x^2 + 6903.5x - 6473.9}{-29.304} \\ & + \frac{x^5 - 31.3x^4 + 366.53x^3 - 1984.1x^2 + 4912.4x - 4351.8}{41.069} \\ & + \frac{x^5 - 29.91x^4 + 335.81x^3 - 1757.2x^2 + 4241.6x - 3694.3}{-78.273} \\ & + \frac{x^5 - 28.51x^4 + 308.78x^3 - 1573.7x^2 + 3727.5x - 3206.4}{228.24}\end{aligned}$$

$$y(x) = -30.898 + 41.344x - 15.855x^2 + 2.7862x^3 - 0.23091x^4 + 0.0072923x^5, \\ 2 \leq x \leq 10.6$$

Fifth Order Polynomial (contd)

$$y(x) = -30.900 + 41.351x - 15.858x^2 + 2.7865x^3 - 0.23093x^4 + 0.0072929x^5,$$

$$2 \leq x \leq 10.6$$



Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/lagrange_method.html

THE END

<http://numericalmethods.eng.usf.edu>