Newton's Divided Difference Polynomial Method of Interpolation

Computer Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

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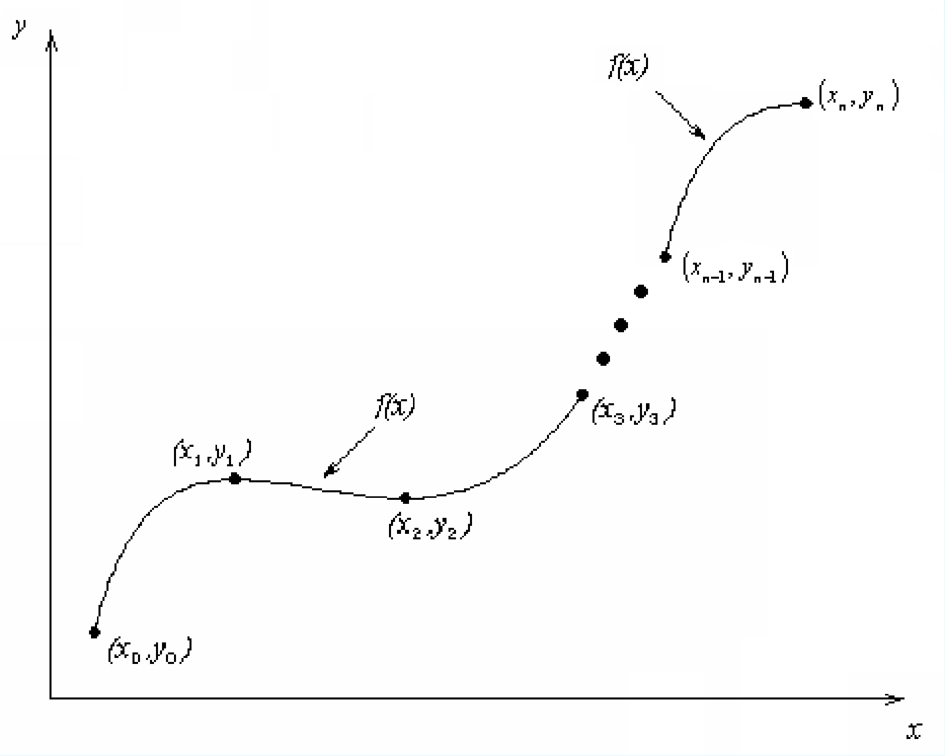
What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), ... (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



Interpolants

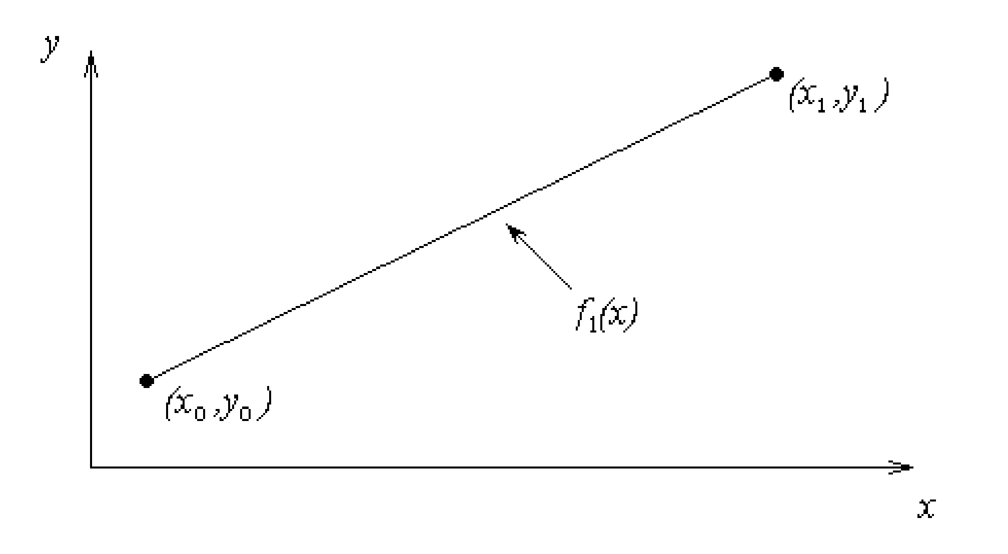
Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Newton's Divided Difference Method

Linear interpolation: Given $(x_0, y_0), (x_1, y_1),$ pass a linear interpolant through the data

$$f_1(x) = b_0 + b_1(x - x_0)$$



where

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Example

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a rectangular plate. The hole centers in the plate that describe the path the arm needs to take are given below.

If the laser is traversing from x = 2 to x = 4.25 in a linear path, find the value of y at x = 4 using the Newton's Divided Difference method for linear interpolation.

x (m)	y (m)
2	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.2	3.5
10.6	5.0

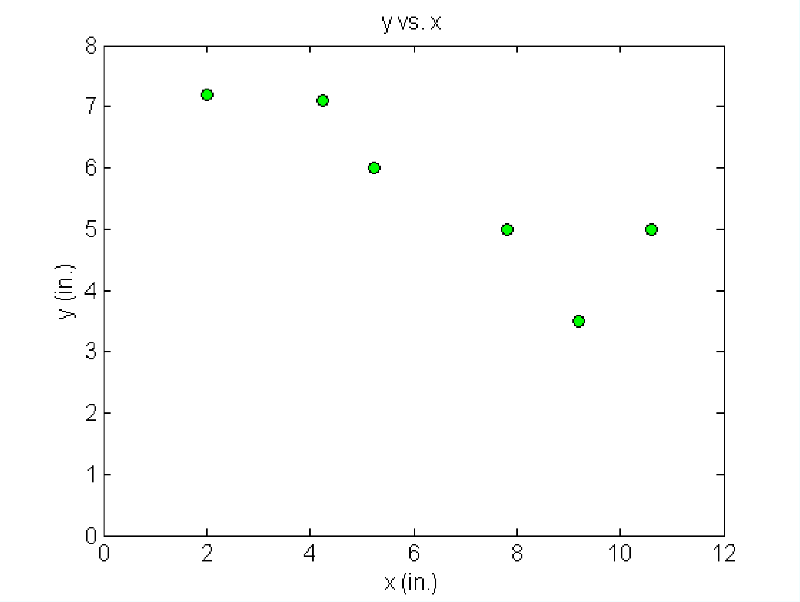


Figure 2 Location of holes on the rectangular plate.

Linear Interpolation

$$y(x) = b_0 + b_1(x - x_0)$$

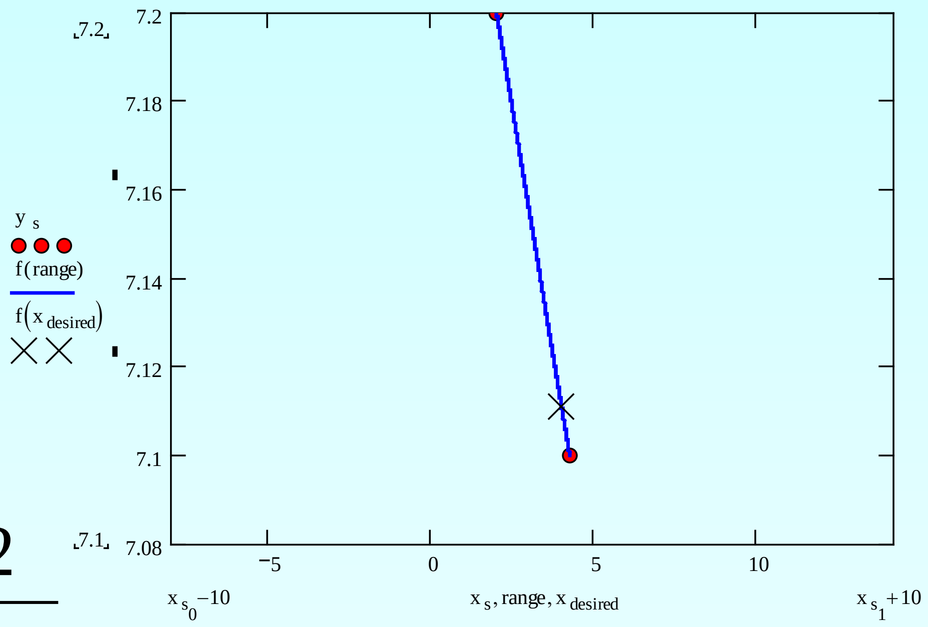
$$x_0 = 2.00,\ y(x_0) = 7.2$$

$$x_1 = 4.25,\ y(x_1) = 7.1$$

$$b_0 = y(x_0) = 7.2$$

$$b_1 = \frac{y(x_1) - y(x_0)}{x_1 - x_0} = \frac{7.1 - 7.2}{4.25 - 2.00}$$

$$= -0.044444$$



Linear Interpolation (contd)

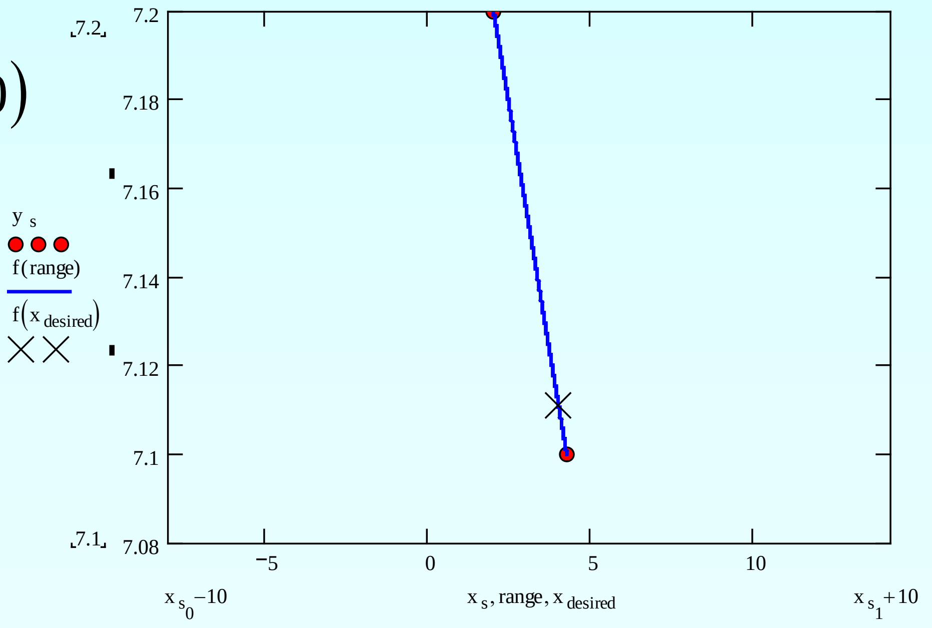
$$y(x) = b_0 + b_1(x - x_0)$$

$$= 7.2 - 0.044444(x - 2.00), \qquad 2.00 \le x \le 4.25$$

At $x = 4$

$$x(4.00) = 7.2 - 0.044444(4.00 - 2.00)$$

$$= 7.1111\ in.$$



Quadratic Interpolation

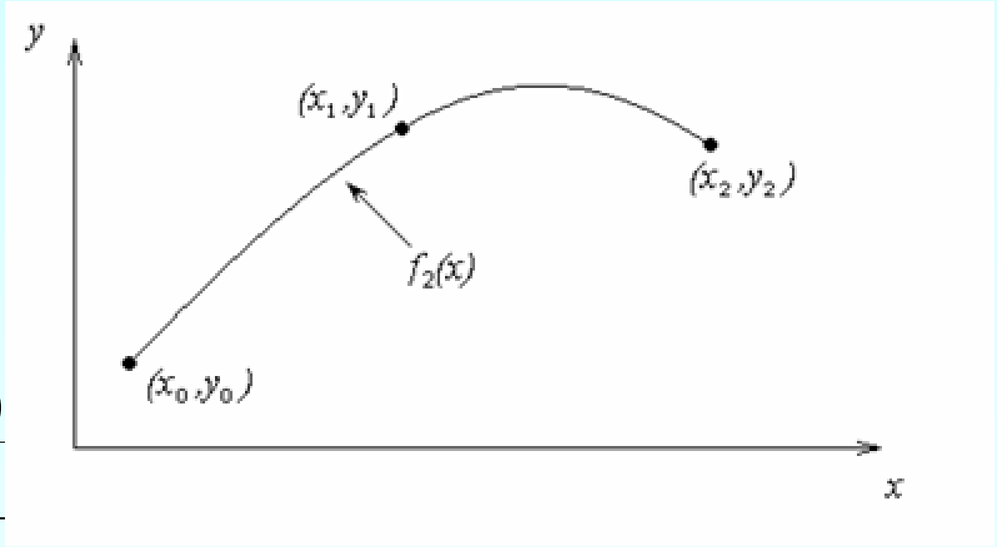
Given (x_0, y_0), (x_1, y_1), and (x_2, y_2), fit a quadratic interpolant through the data.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



Example

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a rectangular plate. The hole centers in the plate that describe the path the arm needs to take are given below.

If the laser is traversing from x = 2 to x = 4.25 in a linear path, find the value of y at x = 4 using the Newton's Divided Difference method for quadratic interpolation.

x (m)	y (m)
2	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.2	3.5
10.6	5.0

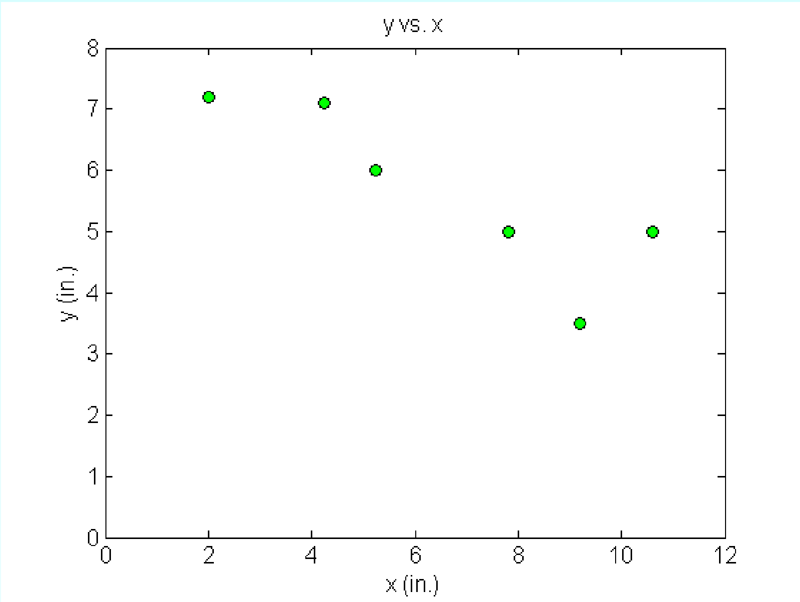


Figure 2 Location of holes on the rectangular plate.

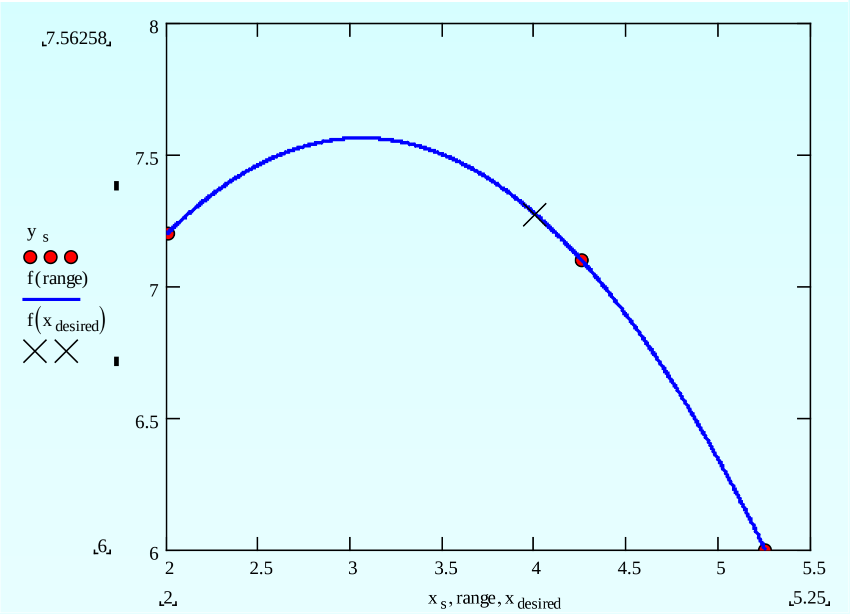
Quadratic Interpolation (contd)

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$x_0 = 2.00, \; y(x_0) = 7.2$$

$$x_1 = 4.25, \; y(x_1) = 7.1$$

$$x_2 = 5.25, \; y(x_2) = 6.0$$



Quadratic Interpolation (contd)

$$b_0 = y(x_0)$$

$$= 7.2$$

$$b_1 = \frac{y(x_1) - y(x_0)}{x_1 - x_0} = \frac{7.1 - 7.2}{4.25 - 2.00}$$

$$= -0.044444$$

$$b_2 = \frac{\dfrac{y(x_2) - y(x_1)}{x_2 - x_1} - \dfrac{y(x_1) - y(x_0)}{x_1 - x_0}}{x_2 - x_0} = \frac{\dfrac{6.0 - 7.1}{5.25 - 4.25} - \dfrac{7.1 - 7.2}{4.25 - 2.00}}{5.25 - 2.00}$$

$$= \frac{-1.1 + 0.044444}{3.25}$$

$$= -0.32479$$

Quadratic Interpolation (contd)

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$= 7.2 - 0.044444(x - 2.00) - 0.32479(x - 2.00)(x - 4.25), \qquad 2.00 \le x \le 5.25$$

At $x = 4$,

$$y(4.00) = 7.2 - 0.044444(4.00 - 2.00) - 0.32479(4.00 - 2.00)(4.00 - 4.25)$$

$$= 7.2735 \text{ } in.$$

The absolute relative approximate error $|\in_a|$ obtained between the

results from the first and second order polynomial is

$$|\in_a| = \left| \frac{7.2735 - 7.1111}{7.2735} \right| \times 100$$

$$= 2.2327\%$$

General Form

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f[x_0] = f(x_0)$$

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Rewriting

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

General Form

Given $(n+1)$ data points, $(x_0, y_0),(x_1, y_1),......,(x_{n-1}, y_{n-1}),(x_n, y_n)$ as

$$f_n(x) = b_0 + b_1(x - x_0) + ... + b_n(x - x_0)(x - x_1)...(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

$$\vdots$$

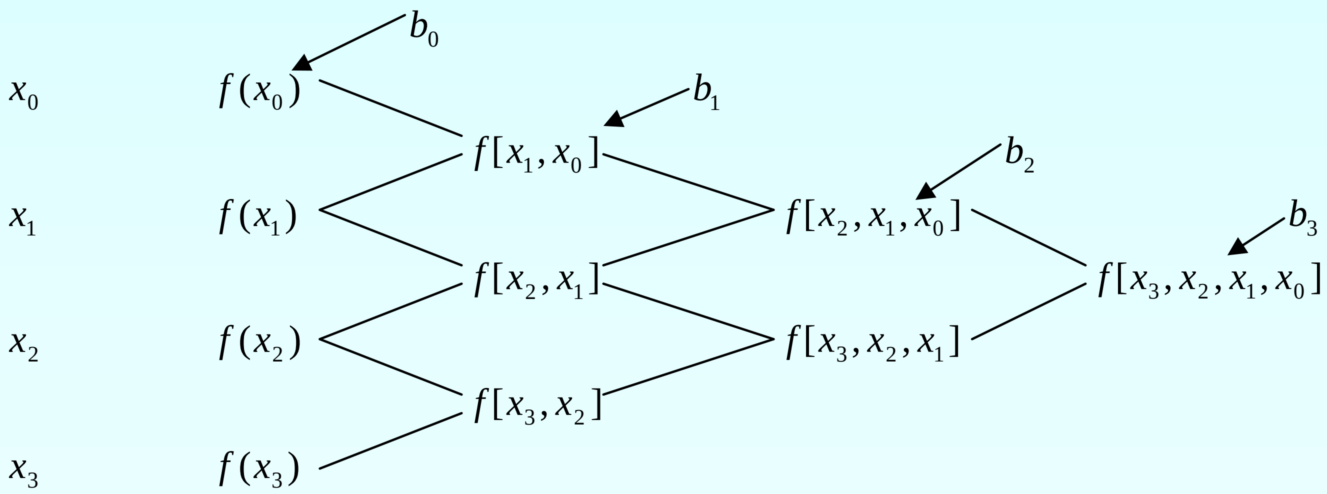
$$b_{n-1} = f[x_{n-1}, x_{n-2}, ..., x_0]$$

$$b_n = f[x_n, x_{n-1}, ..., x_0]$$

General form

The third order polynomial, given (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3), is

$$
\begin{aligned}
f_3(x) = {} & f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) \\
& + f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)
\end{aligned}
$$



Example

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a rectangular plate. The hole centers in the plate that describe the path the arm needs to take are given below.

If the laser is traversing from x = 2 to x = 4.25 in a linear path, find the value of y at x = 4 using the Newton's Divided Difference method for a fifth order polynomial.

x (m)	y (m)
2	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.2	3.5
10.6	5.0

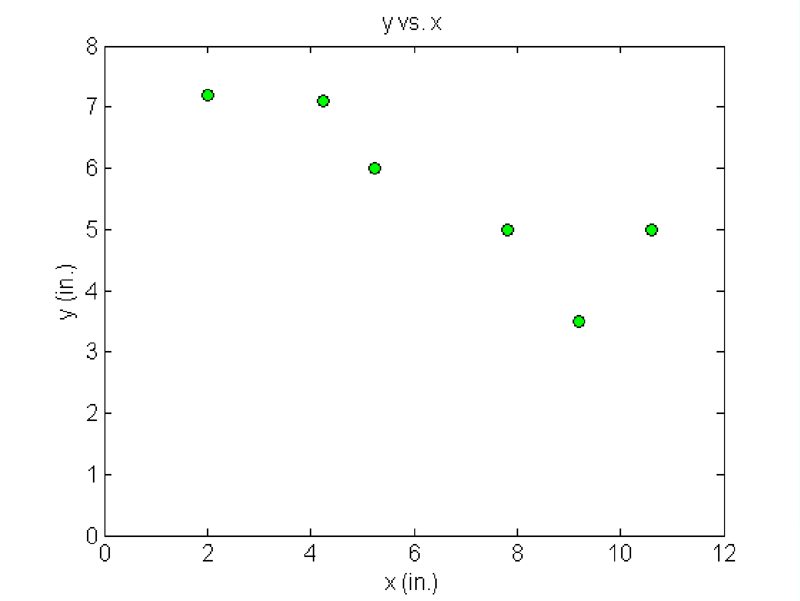


Figure 2 Location of holes on the rectangular plate.

Example

The value of y profile is chosen as

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$
$$+ b_4(x - x_0)(x - x_1)(x - x_2)(x - x_3) + b_5(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

Using the six points,

$x_0 = 2.00, \quad y(x_0) = 7.2$ $\qquad$ $x_1 = 4.25, \quad y(x_1) = 7.1$

$x_2 = 5.25, \quad y(x_2) = 6.0$ $\qquad$ $x_3 = 7.81, \quad y(x_3) = 5.0$

$x_4 = 9.20, \quad y(x_4) = 3.5$ $\qquad$ $x_5 = 10.60, \quad y(x_5) = 5.0$

The values of the constants are found to be

$b_0 = 7.2$ $\qquad$ $b_1 = -0.044444$ $\qquad$ $b_2 = -0.32479$

$b_3 = 0.090198$ $\qquad$ $b_4 = -0.023009$ $\qquad$ $b_5 = 0.0072923$

Example

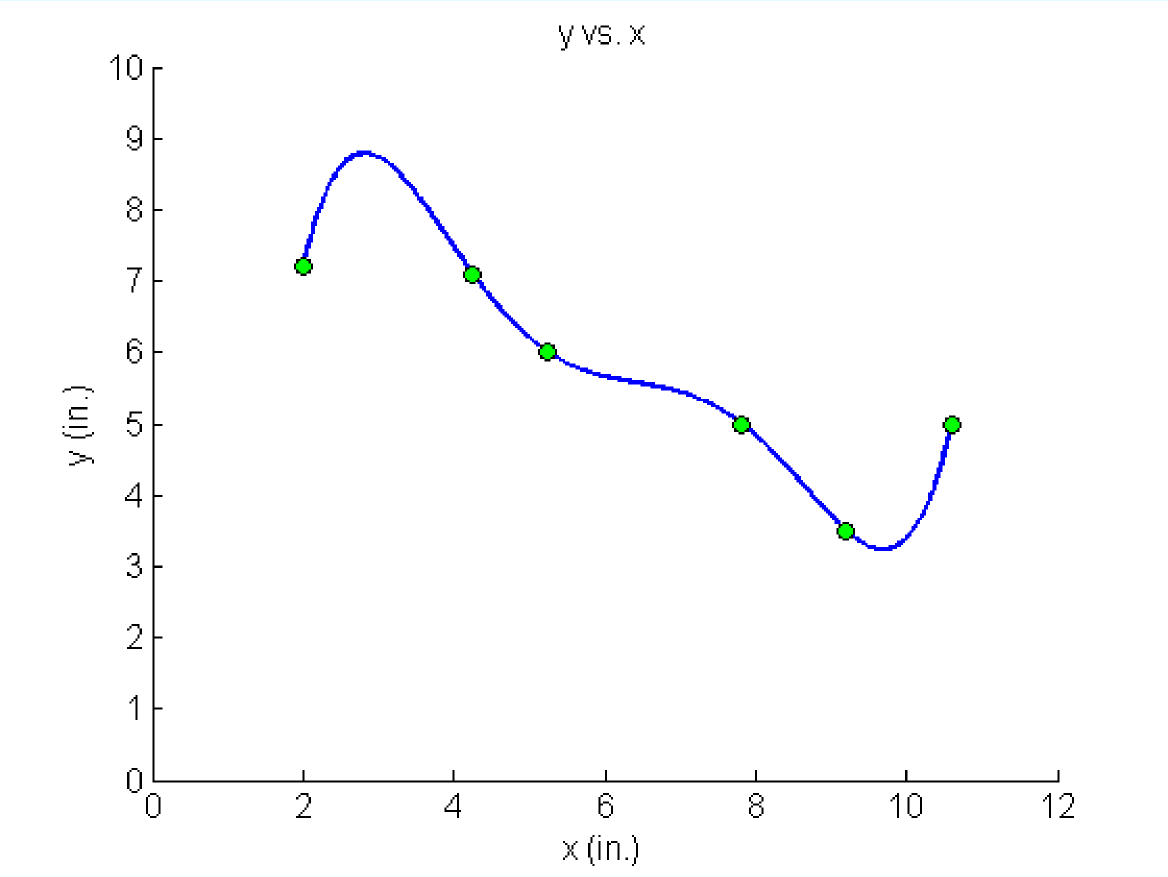
Hence

$$y(x)=b_0+b_1(x-x_0)+b_2(x-x_0)(x-x_1)+b_3(x-x_0)(x-x_1)(x-x_2)$$
$$+b_4(x-x_0)(x-x_1)(x-x_2)(x-x_3)+b_5(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)$$

$$=7.2-0.04444(x-2)-0.32479(x-2)(x-4.25)$$
$$+0.090198(x-2)(x-4.25)(x-5.25)$$
$$-0.023009(x-2)(x-4.25)(x-5.25)(x-7.81)$$
$$+0.0072923(x-2)(x-4.25)(x-5.25)(x-7.81)(x-9.2)$$

$$y(x)=-30.898+41.344x-15.855x^2+2.7862x^3-0.23091x^4+0.0072923x^5, \quad 2\le x\le 10.6$$

Example



$$y(x) = -30.898 + 41.344x - 15.855x^2 + 2.7862x^3 - 0.23091x^4 + 0.0072922x^5,$$

$$2 \le x \le 10.6$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/newton_divided_difference_method.html

THE END

http://numericalmethods.eng.usf.edu