Spline Interpolation Method

Computer Engineering Majors

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Spline Method of Interpolation

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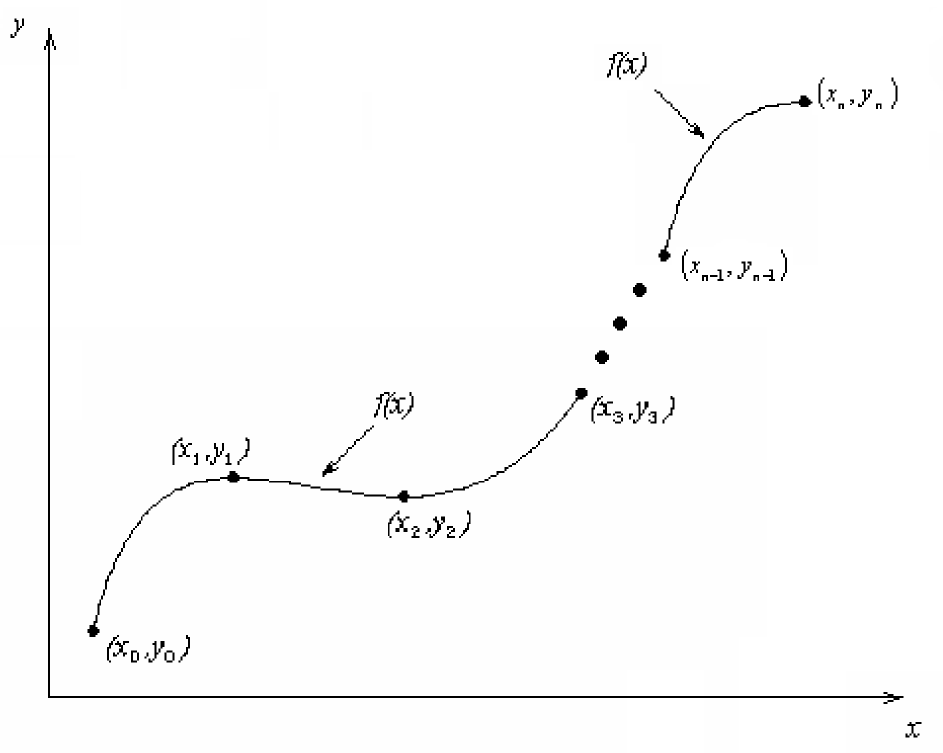
What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), ... (x_n,y_n), find the value of ‘y’ at a value of ‘x’ that is not given.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Why Splines ?

$$f(x) = \frac{1}{1+25x^2}$$

Table : Six equidistantly spaced points in [-1, 1]

x	$y = \frac{1}{1+25x^2}$
-1.0	0.038461
-0.6	0.1
-0.2	0.5
0.2	0.5
0.6	0.1
1.0	0.038461

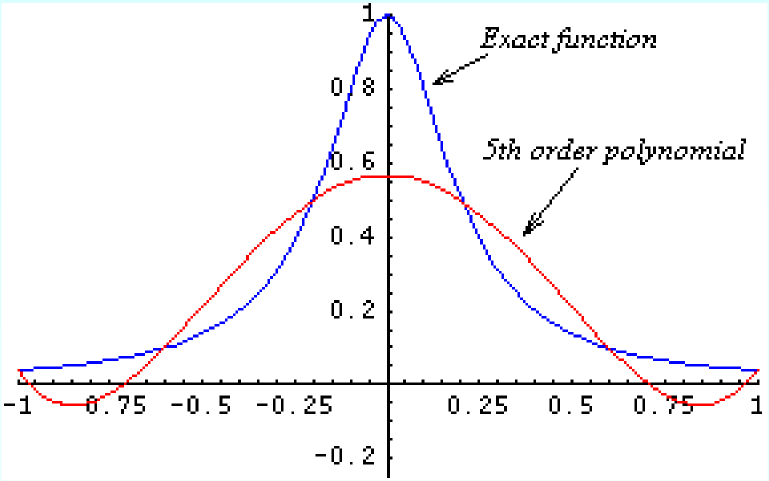


Figure : 5th order polynomial vs. exact function

Why Splines ?

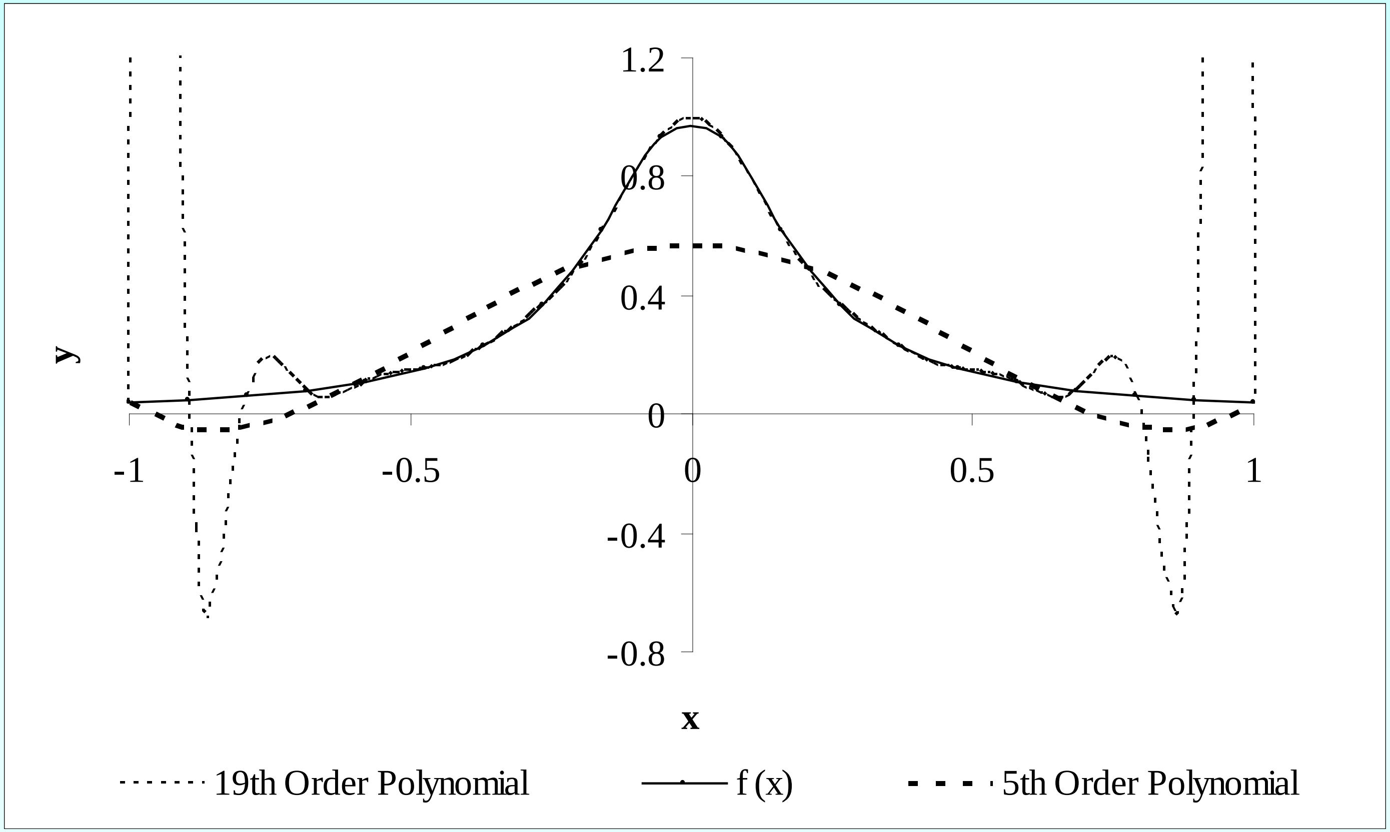
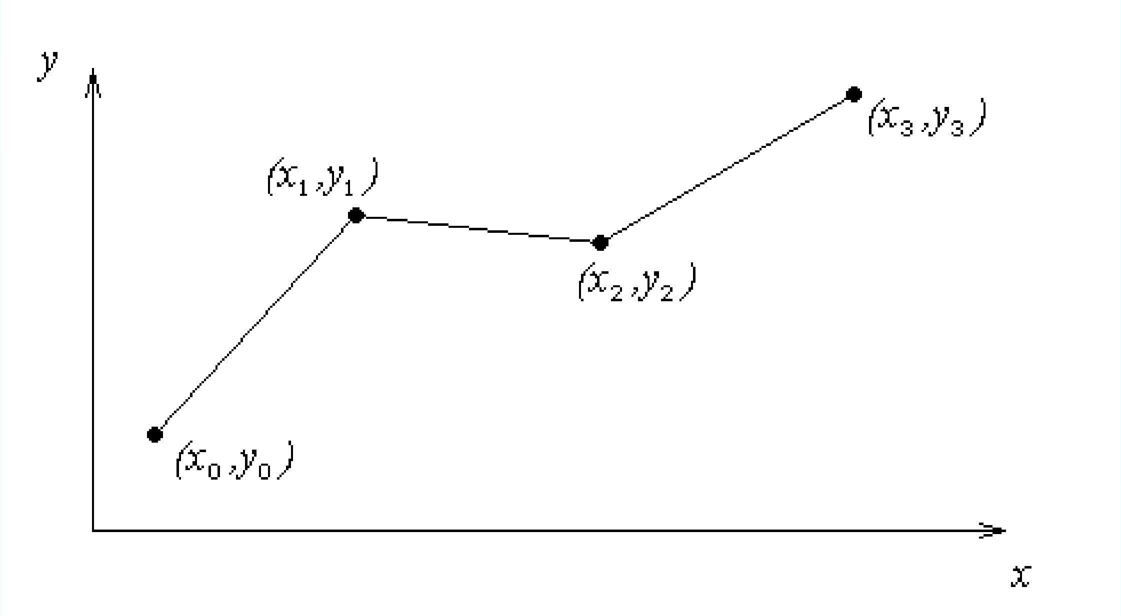


Figure : Higher order polynomial interpolation is a bad idea

Linear Interpolation

Given $(x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1})(x_n, y_n)$, fit linear splines to the data. This simply involves forming the consecutive data through straight lines. So if the above data is given in an ascending order, the linear splines are given by $(y_i = f(x_i))$

Figure : Linear splines



Linear Interpolation (contd)

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0), \qquad x_0 \le x \le x_1$$

$$= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1), \qquad x_1 \le x \le x_2$$

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$$= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}), \quad x_{n-1} \le x \le x_n$$

Note the terms of

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

in the above function are simply slopes between x_{i-1} and x_i.

Example

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a rectangular plate. The hole centers in the plate that describe the path the arm needs to take are given below.

If the laser is traversing from x = 2 to x = 4.25 in a linear path,

Find: the value of y at x = 4 using linear splines, the path of the robot if it follows linear splines, the length of that path.

x (m)	y (m)
2	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.2	3.5
10.6	5.0

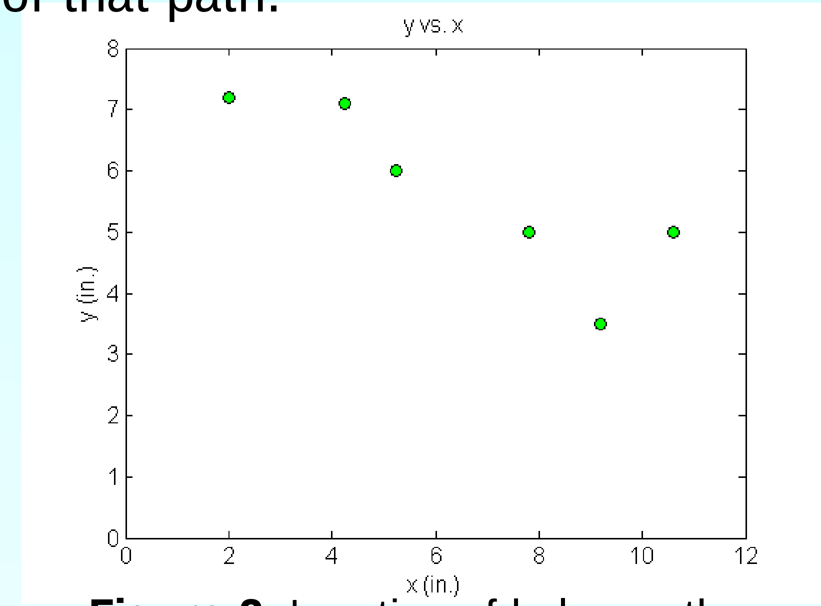
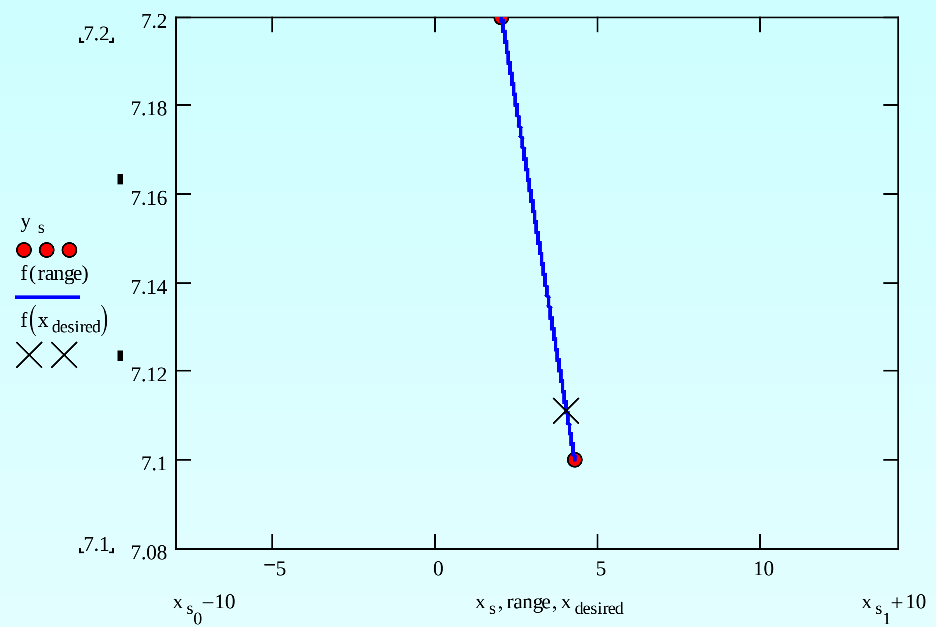


Figure 2 Location of holes on the rectangular plate.

Linear Interpolation



$$x_0 = 2.00, \qquad y(x_0) = 7.2$$

$$x_1 = 4.25, \qquad y(x_1) = 7.1$$

$$y(x) = y(x_0) + \frac{y(x_1) - y(x_0)}{x_1 - x_0}(x - x_0)$$

$$= 7.2 + \frac{7.1 - 7.2}{4.25 - 2.00}(x - 2.00)$$

$$y(x) = 7.2 - 0.044444(x - 2.00)\ ,\ 2.00 \le x \le 4.25$$

At $x = 4,$

$$y(4.00) = 7.2 - 0.044444(4.00 - 2.00)$$

$$= 7.1111 \text{ in.}$$

Linear Interpolation (contd)

Find the path of the robot if it follows linear splines.

The linear spline connecting x = 2.00 and x = 4.25.

$$y(x) = 7.2 - 0.044444(x - 2.00), \qquad 2.00 \le x \le 4.25$$

Similarly

$$y(x) = 7.1 - 1.1(x - 4.25), \qquad 4.25 \le x \le 5.25$$

$$y(x) = 6.0 - 0.39063(x - 5.25), \qquad 5.25 \le x \le 7.81$$

$$y(x) = 5.0 - 1.0791(x - 7.81), \qquad 7.81 \le x \le 9.20$$

$$y(x) = 3.5 + 1.0714(x - 9.20), \qquad 9.20 \le x \le 10.60$$

Linear Interpolation (contd)

Find the length of the path traversed by the robot following linear splines.

The length of the robot's path can be found by simply adding the length of the line segments together. The length of a straight line from one point (x_0, y_0) to another point (x_1, y_1) is given by $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$.

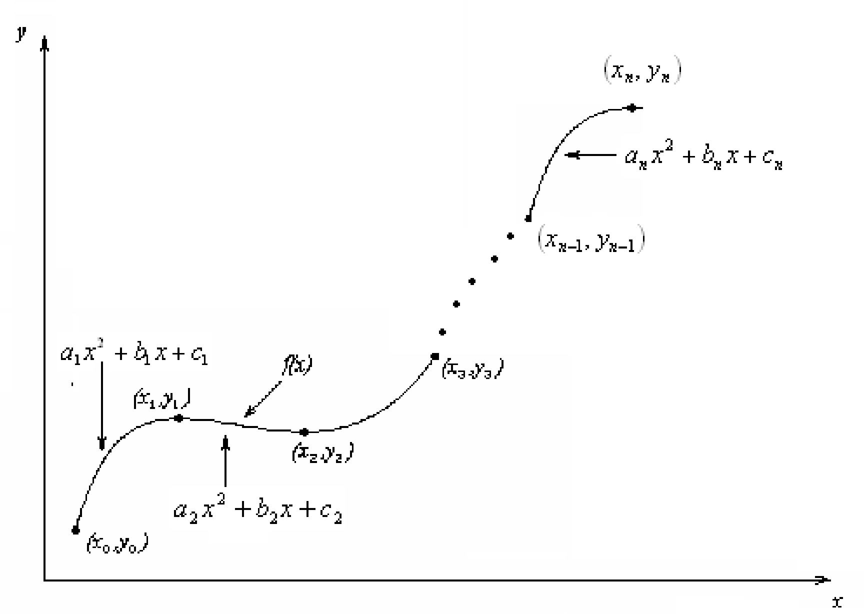
Hence, the length of the linear splines from x = 2.00 to x = 10.60 is

$$
\begin{aligned}
L &= \sqrt{(4.25-2.00)^2 + (7.1-7.2)^2} + \sqrt{(5.25-4.25)^2 + (6.0-7.1)^2} \\
&+ \sqrt{(7.81-5.25)^2 + (5.0-6.0)^2} + \sqrt{(9.20-7.81)^2 + (3.5-5.0)^2} \\
&+ \sqrt{(10.60-9.20)^2 + (5.0-3.5)^2} \\
&= 10.584 \text{ in.}
\end{aligned}
$$

Quadratic Interpolation

Given $(x_0, y_0),(x_1, y_1),......,(x_{n-1}, y_{n-1}),(x_n, y_n)$, fit quadratic splines through the data. The splines are given by

$$
\begin{aligned}
f(x) &= a_1x^2 + b_1x + c_1, & x_0 \le x \le x_1 \\
&= a_2x^2 + b_2x + c_2, & x_1 \le x \le x_2 \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&= a_nx^2 + b_nx + c_n, & x_{n-1} \le x \le x_n
\end{aligned}
$$



Find a_i, b_i, c_i, $i = 1, 2, \ldots, n$

Quadratic Interpolation (contd)

Each quadratic spline goes through two consecutive data points

$$a_1 x_0^2 + b_1 x_0 + c_1 = f(x_0)$$

$$a_1 x_1^2 + b_1 x_1 + c_1 = f(x_1)$$

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$$a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1})$$

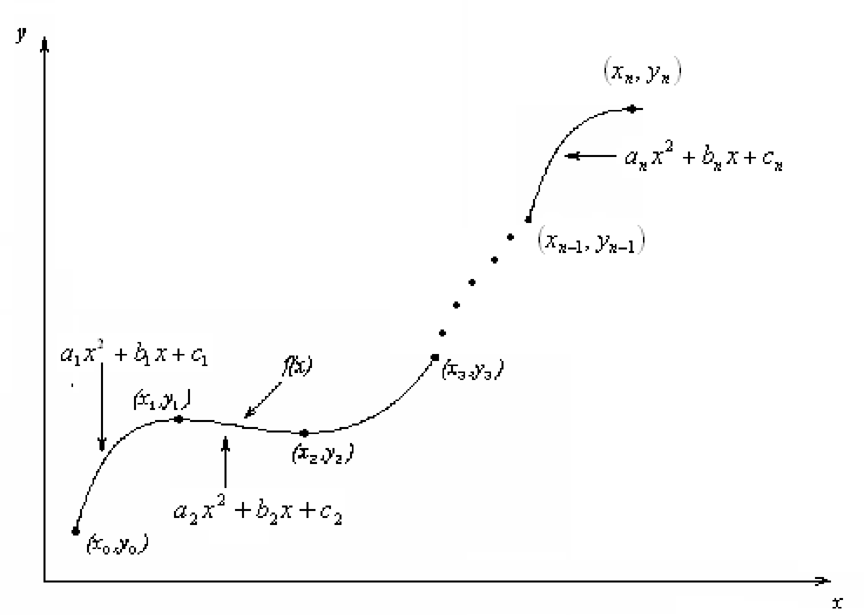
$$a_i x_i^2 + b_i x_i + c_i = f(x_i)$$

.

.

$$a_n x_{n-1}^2 + b_n x_{n-1} + c_n = f(x_{n-1})$$

$$a_n x_n^2 + b_n x_n + c_n = f(x_n)$$



This condition gives 2n equations

Quadratic Splines (contd)

The first derivatives of two quadratic splines are continuous at the interior points.

For example, the derivative of the first spline

$$a_1x^2 + b_1x + c_1 \quad \text{is} \quad 2a_1x + b_1$$

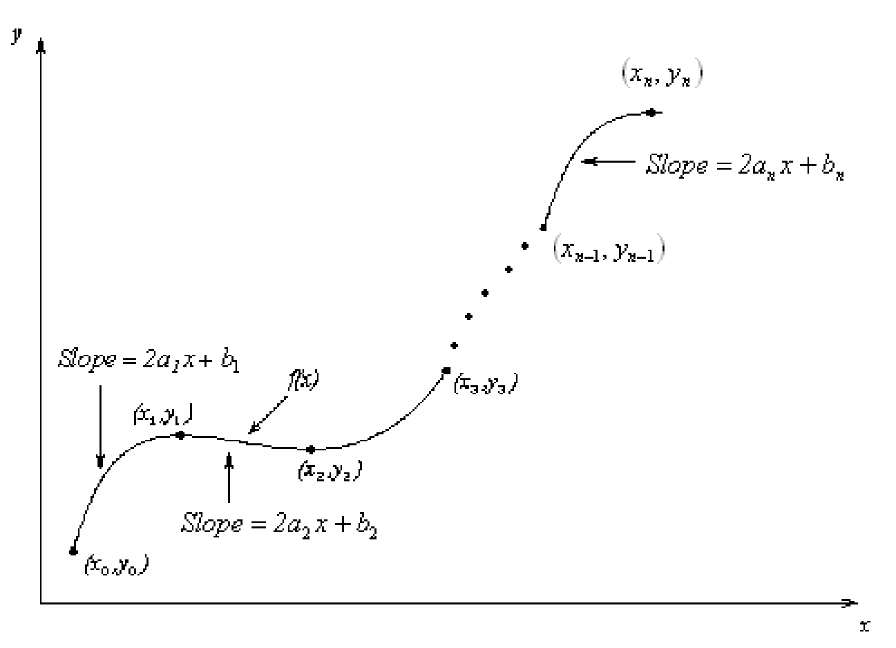
The derivative of the second spline

$$a_2x^2 + b_2x + c_2 \quad \text{is} \quad 2a_2x + b_2$$

and the two are equal at $x = x_1$ giving

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2$$

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0$$



Quadratic Splines (contd)

Similarly at the other interior points,

$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$

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$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

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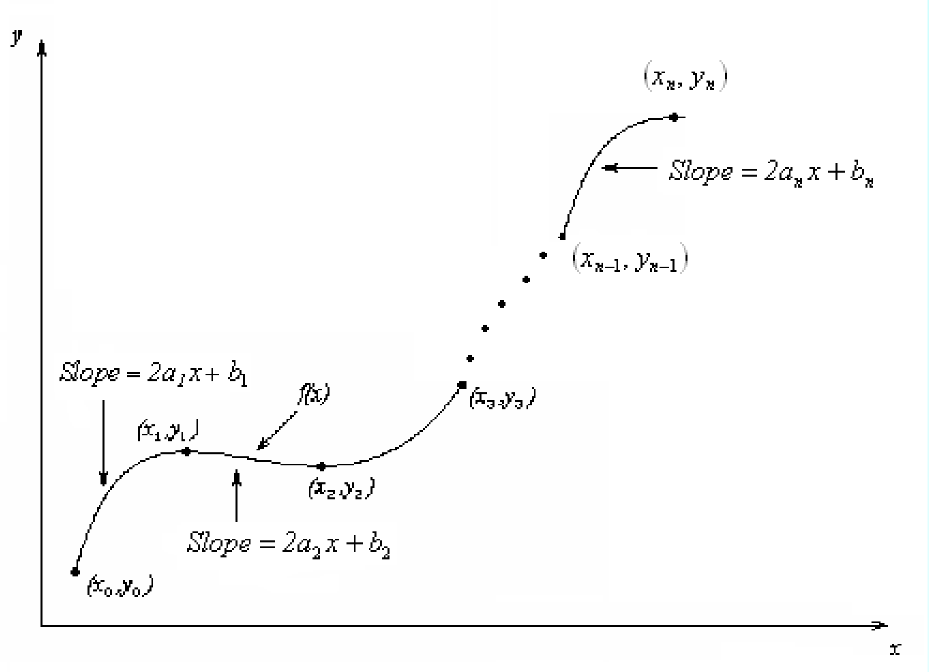
$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$



We have (n-1) such equations. The total number of equations is $(2n) + (n-1) = (3n-1)$.

We can assume that the first spline is linear, that is $a_1 = 0$

Quadratic Splines (contd)

This gives us '3n' equations and '3n' unknowns. Once we find the '3n' constants, we can find the function at any value of 'x' using the splines,

$$f(x) = a_1x^2 + b_1x + c_1, \qquad x_0 \le x \le x_1$$

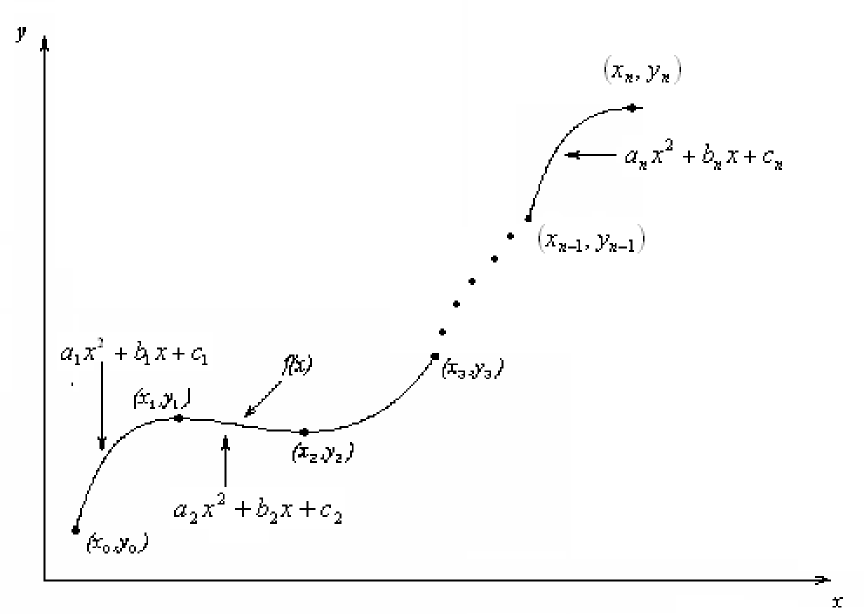
$$= a_2x^2 + b_2x + c_2, \qquad x_1 \le x \le x_2$$

.

.

.

$$= a_nx^2 + b_nx + c_n, \qquad x_{n-1} \le x \le x_n$$



Example

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a rectangular plate. The hole centers in the plate that describe the path the arm needs to take are given below.

If the laser is traversing from x = 2 to x = 4.25 in a linear path,

Find: the length of the path traversed by the robot using quadratic splines and compare the answer to the linear spline and a fifth order polynomial result.

x (m)	y (m)
2	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.2	3.5
10.6	5.0

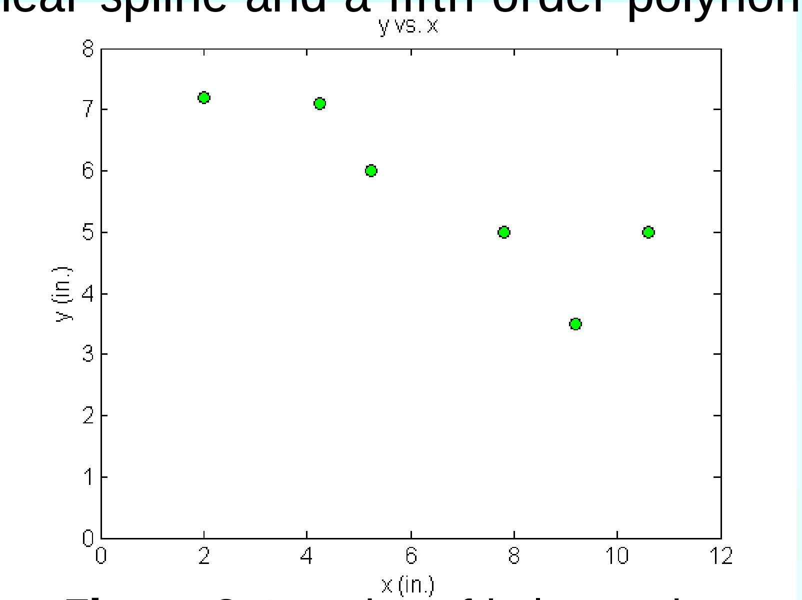


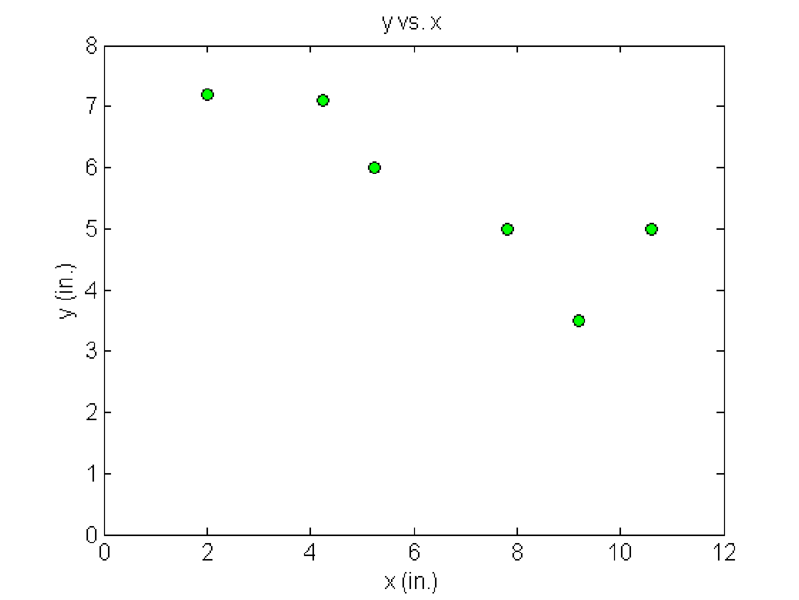
Figure 2 Location of holes on the rectangular plate.

Solution

Since there are six data points,

five quadratic splines pass through them.

$$
\begin{aligned}
y(x) &= a_1x^2 + b_1x + c_1, && 2.00 \le x \le 4.25 \\
&= a_2x^2 + b_2x + c_2, && 4.25 \le x \le 5.25 \\
&= a_3x^2 + b_3x + c_3, && 5.25 \le x \le 7.81 \\
&= a_4x^2 + b_4x + c_4, && 7.81 \le x \le 9.20 \\
&= a_5x^2 + b_5x + c_5, && 9.20 \le x \le 10.60
\end{aligned}
$$



Solution (contd)

Setting up the equations

Each quadratic spline passes through two consecutive data points giving

$a_1x^2 + b_1x + c_1$ passes through x = 2.00 and x = 4.25,

$$a_1(2.00)^2 + b_1(2.00) + c_1 = 7.2 \qquad (1)$$

Similarly,

$$a_1(4.25)^2 + b_1(4.25) + c_1 = 7.1 \qquad (2)$$

$$a_2(4.25)^2 + b_2(4.25) + c_2 = 7.1 \qquad (3)$$

$$a_2(5.25)^2 + b_2(5.25) + c_2 = 6.0 \qquad (4)$$

$$a_3(5.25)^2 + b_3(5.25) + c_3 = 6.0 \qquad (5)$$

$$a_3(7.81)^2 + b_3(7.81) + c_3 = 5.0 \qquad (6)$$

$$a_4(7.81)^2 + b_4(7.81) + c_4 = 5.0 \qquad (7)$$

$$a_4(9.20)^2 + b_4(9.20) + c_4 = 3.5 \qquad (8)$$

$$a_5(9.20)^2 + b_5(9.20) + c_5 = 3.5 \qquad (9)$$

$$a_5(10.60)^2 + b_5(10.60) + c_5 = 5.0 \qquad (10)$$

Solution (contd)

Quadratic splines have continuous derivatives at the interior data points

At x = 4.25

$$2a_1(4.25) + b_1 - 2a_2(4.25) - b_2 = 0 \quad (11)$$

At x = 5.25

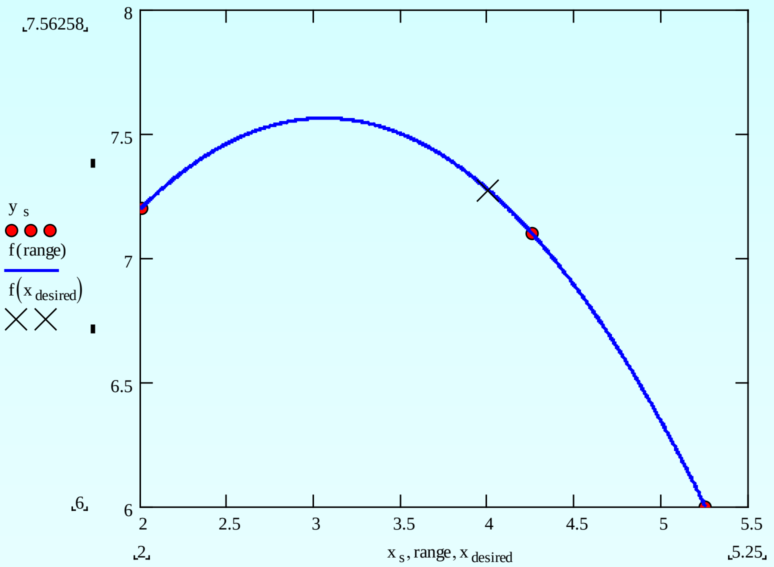
$$2a_2(5.25) + b_2 - 2a_3(5.25) - b_3 = 0 \quad (12)$$

At x = 7.81

$$2a_3(7.81) + b_3 - 2a_4(7.81) - b_4 = 0 \quad (13)$$

At x = 9.20

$$2a_4(9.20) + b_4 - 2a_5(9.20) - b_5 = 0 \quad (14)$$



Assuming the first spline $a_1x^2 + b_1x + c_1$ is linear,

$$a_1 = 0 \quad (15)$$

Solution (contd)

$$
\begin{bmatrix}
4 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
18.063 & 4.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 18.063 & 4.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 27.563 & 5.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 27.563 & 5.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 60.996 & 7.81 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 60.996 & 7.81 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 84.64 & 9.2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 84.64 & 9.2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 112.36 & 10.6 & 1 \\
8.5 & 1 & 0 & -8.5 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 10.5 & 1 & 0 & -10.5 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 15.62 & 1 & 0 & -15.62 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 18.4 & 1 & 0 & -18.4 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \\ a_5 \\ b_5 \\ c_5 \end{bmatrix}
=
\begin{bmatrix} 7.2 \\ 7.1 \\ 7.1 \\ 6.0 \\ 6.0 \\ 5.0 \\ 5.0 \\ 3.5 \\ 3.5 \\ 5.0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Solution (contd)

Solving the above 15 equations gives the 15 unknowns as

i	a_i	a_i	a_i
1	0	−0.044444	7.2889
2	−1.0556	8.9278	−11.777
3	0.68943	−9.3945	36.319
4	−1.7651	28.945	−113.40
5	3.2886	−64.042	314.34

Solution (contd)

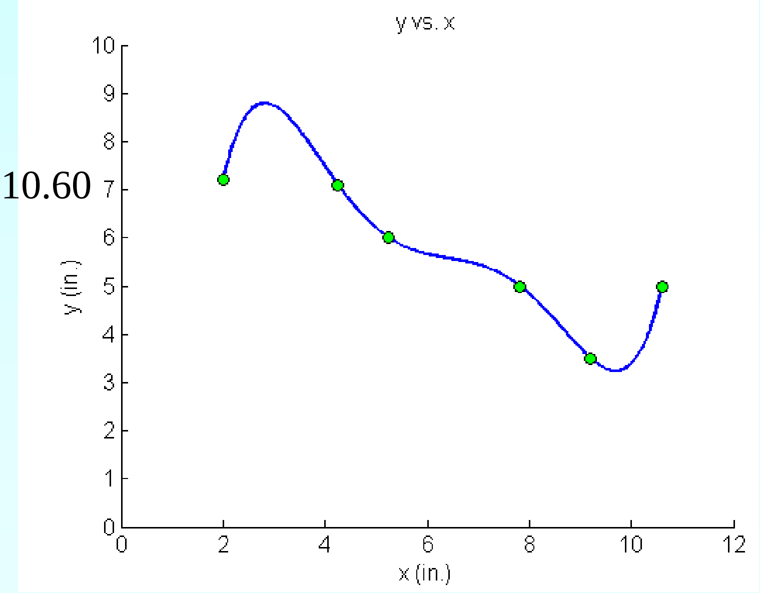
Therefore, the splines are given by

$$
\begin{aligned}
y(x) &= -0.04444x + 7.2889, & 2.00 \le x \le 4.25 \\
&= -1.0556x^2 + 8.9278x - 11.777, & 4.25 \le x \le 5.25 \\
&= 0.68943x^2 - 9.3945x + 36.319, & 5.25 \le x \le 7.81 \\
&= -1.7651x^2 + 28.945x - 113.40, & 7.81 \le x \le 9.20 \\
&= 3.2886x^2 - 64.042x + 314.34, & 9.20 \le x \le 10.60
\end{aligned}
$$

Solution (contd)

The length of a curve of a function $y = f(x)$ from 'a' to 'b' is given by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



In this case, f(x) is defined by five separate functions a = 2.00 to b = 10.60

$$\frac{dy}{dx} = \frac{d}{dx}(-0.044444x + 7.2889), \qquad 2.00 \le x \le 4.25$$

$$= \frac{d}{dx}(-1.0556x^2 + 8.9278x - 11.777), \qquad 4.25 \le x \le 5.25$$

$$= \frac{d}{dx}(0.68943x^2 - 9.3945x + 36.319), \qquad 5.25 \le x \le 7.81$$

$$= \frac{d}{dx}(-1.7651x^2 + 28.945x - 113.40), \qquad 7.81 \le x \le 9.20$$

$$= \frac{d}{dx}(3.2886x^2 - 64.042x + 314.34), \qquad 9.20 \le x \le 10.60$$

Solution (contd)

$$L = \int_{2.00}^{4.25} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx + \int_{4.25}^{5.25} \sqrt{1+\left(\frac{dy}{dx}\right)^2} + \int_{5.25}^{7.81} \sqrt{1+\left(\frac{dy}{dx}\right)^2} + \int_{7.81}^{9.20} \sqrt{1+\left(\frac{dy}{dx}\right)^2} + \int_{9.20}^{10.60} \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= \int_{2.00}^{4.25} \sqrt{1+(-0.044444)^2}\,dx + \int_{4.25}^{5.25} \sqrt{1+(-2.1111x+8.9278)^2}\,dx + \int_{5.25}^{7.81} \sqrt{1+(1.3788x-9.3945)^2}\,dx$$

$$+ \int_{7.81}^{9.20} \sqrt{1+(-3.5302x+28.945)^2}\,dx + \int_{9.20}^{10.60} \sqrt{1+(6.5772x-64.042)^2}\,dx$$

$$= 2.2522 + 1.5500 + 3.6596 + 2.6065 + 3.8077$$

$$= 13.876$$

Comparison

Compare the answer from part (a) to linear spline result and fifth order polynomial result.

We can find the length of the fifth order polynomial result in a similar fashion to the quadratic splines without breaking the integrals into five intervals. The fifth order polynomial result through the six points is given by

$$y(x) = -30.898 + 41.344x - 15.855x^2 + 2.7862x^3 - 0.23091x^4 + 0.0072923x^5, \quad 2 \le x \le 10.6$$

Therefore,

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{2.00}^{10.60} \sqrt{1 + \left(41.344 - 31.710x + 8.3586x^2 - 0.92364x^3 + 0.036461x^4\right)}dx$$

$$= 13.123$$

Comparison

The absolute relative approximate error obtained between the results from the linear and quadratic spline is

$$\left|\in_a\right| = \left|\frac{13.876-10.584}{13.876}\right|\times 100$$

$$= 0.23724\%$$

The absolute relative approximate error obtained between the results from the fifth order polynomial and quadratic spline is

$$\left|\in_a\right| = \left|\frac{13.876-13.123}{13.876}\right|\times 100$$

$$= 0.054239\%$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/spline_method.html

THE END

http://numericalmethods.eng.usf.edu