

Chapter 05.04

Lagrangian Interpolation

After reading this chapter, you should be able to:

1. derive Lagrangian method of interpolation,
2. solve problems using Lagrangian method of interpolation, and
3. use Lagrangian interpolants to find derivatives and integrals of discrete functions.

What is interpolation?

Many times, data is given only at discrete points such as (x_0, y_0) , (x_1, y_1) , \dots , (x_{n-1}, y_{n-1}) , (x_n, y_n) . So, how then does one find the value of y at any other value of x ? Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points (Figure 1). Then one can find the value of y at any other value of x . This is called *interpolation*.

Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function $f(x)$ should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

- (A) evaluate,
- (B) differentiate, and
- (C) integrate,

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ data points. One of the methods used to find this polynomial is called the Lagrangian method of interpolation. Other methods include Newton's divided difference polynomial method and the direct method. We discuss the Lagrangian method in this chapter.

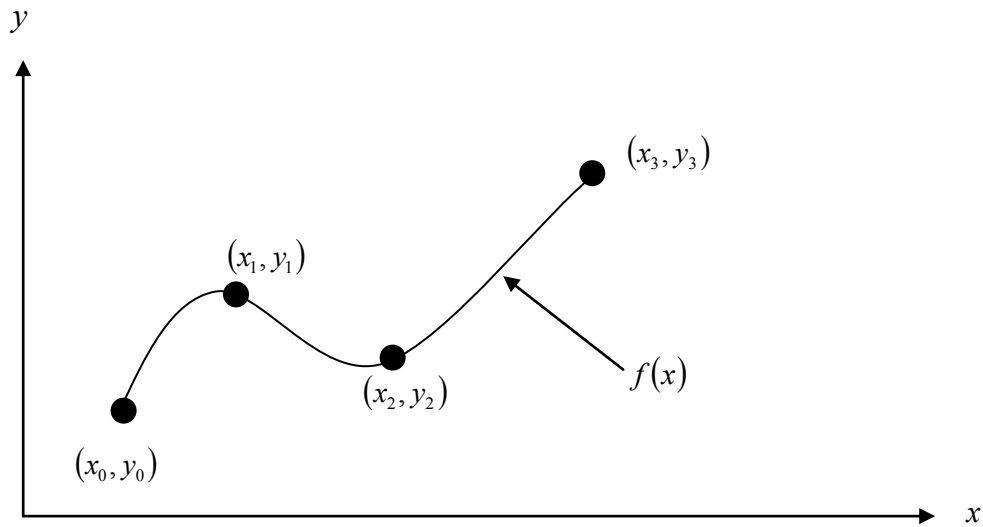


Figure 1 Interpolation of discrete data.

The Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^n L_i(x) f(x_i)$$

where n in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $n+1$ data points as $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}$$

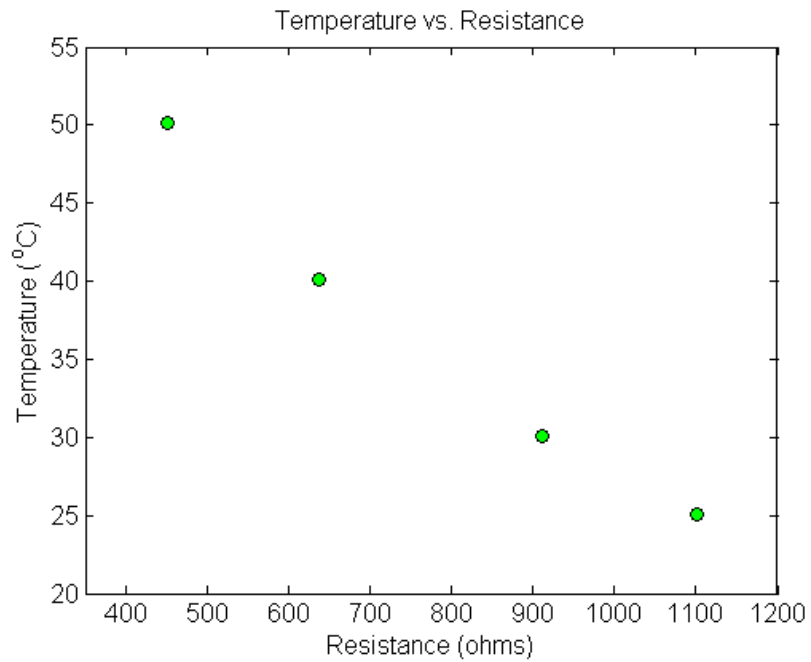
$L_i(x)$ is a weighting function that includes a product of $n-1$ terms with terms of $j=i$ omitted. The application of Lagrangian interpolation will be clarified using an example.

Example 1

Thermistors are used to measure the temperature of bodies. Thermistors are based on materials' change in resistance with temperature. To measure temperature, manufacturers provide you with a temperature vs. resistance calibration curve. If you measure resistance, you can find the temperature. A manufacturer of thermistors makes several observations with a thermistor, which are given in Table 1.

Table 1 Temperature as a function of resistance.

R (ohm)	T ($^{\circ}\text{C}$)
1101.0	25.113
911.3	30.131
636.0	40.120
451.1	50.128

**Figure 2** Resistance vs. temperature.

Determine the temperature corresponding to 754.8 ohms using a first order Lagrange polynomial.

Solution

For first order Lagrange polynomial interpolation (also called linear interpolation), the temperature is given by

$$\begin{aligned}
 T(R) &= \sum_{i=0}^1 L_i(R)T(R_i) \\
 &= L_0(R)T(R_0) + L_1(R)T(R_1)
 \end{aligned}$$

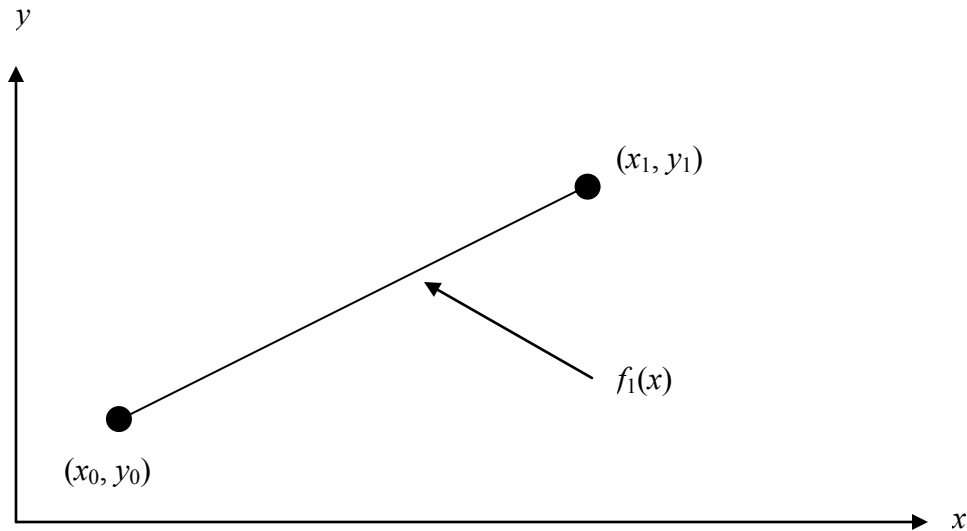


Figure 3 Linear interpolation.

Since we want to find the temperature at $R = 754.8$, we need to choose the two data points that are closest to $R = 754.8$ that also bracket $R = 754.8$ to evaluate it. The two points are $R_0 = 911.3$ and $R_1 = 636.0$.

Then

$$R_0 = 911.3, T(R_0) = 30.131$$

$$R_1 = 636.0, T(R_1) = 40.120$$

gives

$$\begin{aligned} L_0(R) &= \prod_{\substack{j=0 \\ j \neq 0}}^1 \frac{R - R_j}{R_0 - R_j} \\ &= \frac{R - R_1}{R_0 - R_1} \end{aligned}$$

$$\begin{aligned} L_1(R) &= \prod_{\substack{j=0 \\ j \neq 1}}^1 \frac{R - R_j}{R_1 - R_j} \\ &= \frac{R - R_0}{R_1 - R_0} \end{aligned}$$

Hence

$$\begin{aligned} T(R) &= \frac{R - R_1}{R_0 - R_1} T(R_0) + \frac{R - R_0}{R_1 - R_0} T(R_1) \\ &= \frac{R - 636.0}{911.3 - 636.0} (30.131) + \frac{R - 911.3}{636.0 - 911.3} (40.120), \quad 636.0 \leq R \leq 911.3 \\ T(754.8) &= \frac{754.8 - 636.0}{911.3 - 636.0} (30.131) + \frac{754.8 - 911.3}{636.0 - 911.3} (40.120) \end{aligned}$$

$$\begin{aligned}
 &= 0.43153(30.131) + 0.56847(40.120) \\
 &= 35.809^\circ\text{C}
 \end{aligned}$$

You can see that $L_0(R) = 0.43153$ and $L_1(R) = 0.56847$ are like weightages given to the temperatures at $R_0 = 911.3$ and $R_1 = 636.0$ to calculate the temperature at $R = 754.8$.

Example 2

Thermistors are used to measure the temperature of bodies. Thermistors are based on materials' change in resistance with temperature. To measure temperature, manufacturers provide you with a temperature vs. resistance calibration curve. If you measure resistance, you can find the temperature. A manufacturer of thermistors makes several observations with a thermistor, which are given in Table 2.

Table 2 Temperature as a function of resistance.

R (ohm)	T ($^\circ\text{C}$)
1101.0	25.113
911.3	30.131
636.0	40.120
451.1	50.128

Determine the temperature corresponding to 754.8 ohms using a second order Lagrange polynomial. Find the absolute relative approximate error for the second order polynomial approximation.

Solution

For second order Lagrange polynomial interpolation (also called quadratic interpolation), the temperature is given by

$$\begin{aligned}
 T(R) &= \sum_{i=0}^2 L_i(R)T(R_i) \\
 &= L_0(R)T(R_0) + L_1(R)T(R_1) + L_2(R)T(R_2)
 \end{aligned}$$

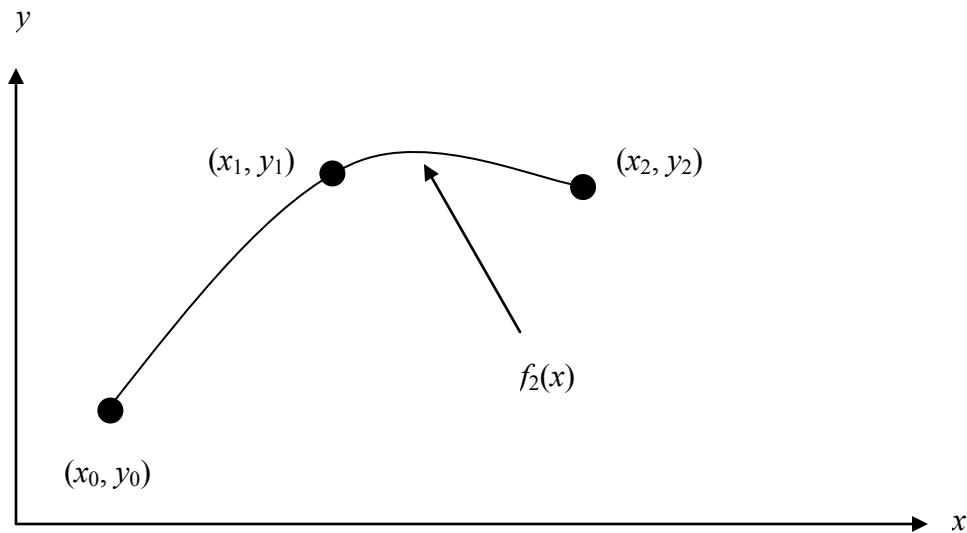


Figure 4 Quadratic interpolation.

Since we want to find the temperature at $R = 754.8$, we need to choose the three data points that are closest to $R = 754.8$ that also bracket $R = 754.8$ to evaluate it. The three points are $R_0 = 911.3$, $R_1 = 636.0$ and $R_2 = 451.1$.

Then

$$R_0 = 911.3, \quad T(R_0) = 30.131$$

$$R_1 = 636.0, \quad T(R_1) = 40.120$$

$$R_2 = 451.1, \quad T(R_2) = 50.128$$

gives

$$\begin{aligned} L_0(R) &= \prod_{\substack{j=0 \\ j \neq 0}}^2 \frac{R - R_j}{R_0 - R_j} \\ &= \left(\frac{R - R_1}{R_0 - R_1} \right) \left(\frac{R - R_2}{R_0 - R_2} \right) \end{aligned}$$

$$\begin{aligned} L_1(R) &= \prod_{\substack{j=0 \\ j \neq 1}}^2 \frac{R - R_j}{R_1 - R_j} \\ &= \left(\frac{R - R_0}{R_1 - R_0} \right) \left(\frac{R - R_2}{R_1 - R_2} \right) \end{aligned}$$

$$\begin{aligned} L_2(R) &= \prod_{\substack{j=0 \\ j \neq 2}}^2 \frac{R - R_j}{R_2 - R_j} \\ &= \left(\frac{R - R_0}{R_2 - R_0} \right) \left(\frac{R - R_1}{R_2 - R_1} \right) \end{aligned}$$

Hence

$$\begin{aligned}
 T(R) &= \left(\frac{R - R_1}{R_0 - R_1} \right) \left(\frac{R - R_2}{R_0 - R_2} \right) T(R_0) + \left(\frac{R - R_0}{R_1 - R_0} \right) \left(\frac{R - R_2}{R_1 - R_2} \right) T(R_1) \\
 &\quad + \left(\frac{R - R_0}{R_2 - R_0} \right) \left(\frac{R - R_1}{R_2 - R_1} \right) T(R_2), \quad R_0 \leq R \leq R_2 \\
 T(754.8) &= \frac{(754.8 - 636.0)(754.8 - 451.1)}{(911.3 - 636.0)(911.3 - 451.1)} (30.131) \\
 &\quad + \frac{(754.8 - 911.3)(754.8 - 451.1)}{(636.0 - 911.3)(636.0 - 451.1)} (40.120) \\
 &\quad + \frac{(754.8 - 911.3)(754.8 - 636.0)}{(451.1 - 911.3)(451.1 - 636.0)} (50.128) \\
 &= (0.28478)(30.131) + (0.93372)(40.120) + (-0.21850)(50.128) \\
 &= 35.089^\circ\text{C}
 \end{aligned}$$

The absolute relative approximate error $|\epsilon_a|$ obtained between the results from the first and second order polynomial is

$$\begin{aligned}
 |\epsilon_a| &= \left| \frac{35.089 - 35.809}{35.089} \right| \times 100 \\
 &= 2.0543\%
 \end{aligned}$$

Example 3

Thermistors are used to measure the temperature of bodies. Thermistors are based on materials' change in resistance with temperature. To measure temperature, manufacturers provide you with a temperature vs. resistance calibration curve. If you measure resistance, you can find the temperature. A manufacturer of thermistors makes several observations with a thermistor, which are given in Table 3.

Table 3 Temperature as a function of resistance.

R (ohm)	T ($^\circ\text{C}$)
1101.0	25.113
911.3	30.131
636.0	40.120
451.1	50.128

- Determine the temperature corresponding to 754.8 ohms using a third order Lagrange polynomial. Find the absolute relative approximate error for the third order polynomial approximation.
- The actual calibration curve used by industry is given by

$$\frac{1}{T} = \sum_{i=0}^3 L_i(\ln R) \frac{1}{T(\ln R_i)}$$

$$= L_0(\ln R) \frac{1}{T(\ln R_0)} + L_1(\ln R) \frac{1}{T(\ln R_1)} + L_2(\ln R) \frac{1}{T(\ln R_2)} + L_3(\ln R) \frac{1}{T(\ln R_3)}$$

substituting $y = \frac{1}{T}$ and $x = \ln R$, the calibration curve is given by

$$y(x) = L_0(x)y(x_0) + L_1(x)y(x_1) + L_2(x)y(x_2) + L_3(x)y(x_3)$$

Table 4 Manipulation for the given data.

R (ohm)	T ($^{\circ}\text{C}$)	x ($\ln R$)	y ($\frac{1}{T}$)
1101.0	25.113	7.0040	0.039820
911.3	30.131	6.8149	0.033188
636.0	40.120	6.4552	0.024925
451.1	50.128	6.1117	0.019949

Find the calibration curve and find the temperature corresponding to 754.8 ohms. What is the difference between the results from part (a)? Is the difference larger using the results from part (a) or part (b), if the actual measured value at 754.8 ohms is 35.285°C ?

Solution

a) For third order Lagrange polynomial interpolation (also called cubic interpolation), we choose the temperature given by

$$\begin{aligned} T(R) &= \sum_{i=0}^3 L_i(R)T(R_i) \\ &= L_0(R)T(R_0) + L_1(R)T(R_1) + L_2(R)T(R_2) + L_3(R)T(R_3) \end{aligned}$$

Since we want to find the temperature at $R = 754.8$, we need to choose the four data points that are closest to $R = 754.8$ that also bracket $R = 754.8$ to evaluate it. The four data points are $R_0 = 1101.0$, $R_1 = 911.3$, $R_2 = 636.0$ and $R_3 = 451.1$.

Then

$$R_0 = 1101.0, \quad T(R_0) = 25.113$$

$$R_1 = 911.3, \quad T(R_1) = 30.131$$

$$R_2 = 636.0, \quad T(R_2) = 40.120$$

$$R_3 = 451.1, \quad T(R_3) = 50.128$$

gives

$$\begin{aligned} L_0(R) &= \prod_{\substack{j=0 \\ j \neq 0}}^3 \frac{R - R_j}{R_0 - R_j} \\ &= \left(\frac{R - R_1}{R_0 - R_1} \right) \left(\frac{R - R_2}{R_0 - R_2} \right) \left(\frac{R - R_3}{R_0 - R_3} \right) \end{aligned}$$

$$\begin{aligned}
L_1(R) &= \prod_{\substack{j=0 \\ j \neq 1}}^3 \frac{R - R_j}{R_1 - R_j} \\
&= \left(\frac{R - R_0}{R_1 - R_0} \right) \left(\frac{R - R_2}{R_1 - R_2} \right) \left(\frac{R - R_3}{R_1 - R_3} \right) \\
L_2(R) &= \prod_{\substack{j=0 \\ j \neq 2}}^3 \frac{R - R_j}{R_2 - R_j} \\
&= \left(\frac{R - R_0}{R_2 - R_0} \right) \left(\frac{R - R_1}{R_2 - R_1} \right) \left(\frac{R - R_3}{R_2 - R_3} \right) \\
L_3(R) &= \prod_{\substack{j=0 \\ j \neq 3}}^3 \frac{R - R_j}{R_3 - R_j} \\
&= \left(\frac{R - R_0}{R_3 - R_0} \right) \left(\frac{R - R_1}{R_3 - R_1} \right) \left(\frac{R - R_2}{R_3 - R_2} \right)
\end{aligned}$$

Hence

$$\begin{aligned}
T(R) &= \left(\frac{R - R_1}{R_0 - R_1} \right) \left(\frac{R - R_2}{R_0 - R_2} \right) \left(\frac{R - R_3}{R_0 - R_3} \right) T(R_0) + \left(\frac{R - R_0}{R_1 - R_0} \right) \left(\frac{R - R_2}{R_1 - R_2} \right) \left(\frac{R - R_3}{R_1 - R_3} \right) T(R_1) \\
&\quad + \left(\frac{R - R_0}{R_2 - R_0} \right) \left(\frac{R - R_1}{R_2 - R_1} \right) \left(\frac{R - R_3}{R_2 - R_3} \right) T(R_2) + \left(\frac{R - R_0}{R_3 - R_0} \right) \left(\frac{R - R_1}{R_3 - R_1} \right) \left(\frac{R - R_2}{R_3 - R_2} \right) T(R_3)
\end{aligned}$$

$R_0 \leq R \leq R_3$

$$\begin{aligned}
T(754.8) &= \frac{(754.8 - 911.3)(754.8 - 636.0)(754.8 - 451.1)}{(1101.0 - 911.3)(1101.0 - 636.0)(1101.0 - 451.1)} (25.113) \\
&\quad + \frac{(754.8 - 1101.0)(754.8 - 636.0)(754.8 - 451.1)}{(911.3 - 1101.0)(911.3 - 636.0)(911.3 - 451.1)} (30.131) \\
&\quad + \frac{(754.8 - 1101.0)(754.8 - 911.3)(754.8 - 451.1)}{(636.0 - 1101.0)(636.0 - 911.3)(636.0 - 451.1)} (40.120) \\
&\quad + \frac{(754.8 - 1101.0)(754.8 - 911.3)(754.8 - 636.0)}{(451.1 - 1101.0)(451.1 - 911.3)(451.1 - 636.0)} (50.128) \\
&= (-0.098494)(25.113) + (0.51972)(30.131) + (0.69517)(40.120) \\
&\quad + (-0.11639)(50.128) \\
&= 35.242^\circ\text{C}
\end{aligned}$$

The absolute relative approximate error $|\epsilon_a|$ obtained between the results from the second and third order polynomial is

$$\begin{aligned}
|\epsilon_a| &= \left| \frac{35.242 - 35.089}{35.242} \right| \times 100 \\
&= 0.43458\%
\end{aligned}$$

b) Finding the cubic interpolant using the Lagrangian method of interpolation for

$$y(x) = L_0(x)y(x_0) + L_1(x)y(x_1) + L_2(x)y(x_2) + L_3(x)y(x_3)$$

requires that we first calculate the new values of x and y .

x ($\ln R$)	y $\left(\frac{1}{T}\right)$
7.0040	0.039820
6.8149	0.033188
6.4552	0.024925
6.1117	0.019949

Then

$$x_0 = 7.0040, \quad y(x_0) = 0.039820$$

$$x_1 = 6.8149, \quad y(x_1) = 0.033188$$

$$x_2 = 6.4552, \quad y(x_2) = 0.024925$$

$$x_3 = 6.1117, \quad y(x_3) = 0.019949$$

gives

$$\begin{aligned} L_0(x) &= \prod_{\substack{j=0 \\ j \neq 0}}^3 \frac{x - x_j}{x_0 - x_j} \\ &= \left(\frac{x - x_1}{x_0 - x_1} \right) \left(\frac{x - x_2}{x_0 - x_2} \right) \left(\frac{x - x_3}{x_0 - x_3} \right) \end{aligned}$$

$$\begin{aligned} L_1(x) &= \prod_{\substack{j=0 \\ j \neq 1}}^3 \frac{x - x_j}{x_1 - x_j} \\ &= \left(\frac{x - x_0}{x_1 - x_0} \right) \left(\frac{x - x_2}{x_1 - x_2} \right) \left(\frac{x - x_3}{x_1 - x_3} \right) \end{aligned}$$

$$\begin{aligned} L_2(x) &= \prod_{\substack{j=0 \\ j \neq 2}}^3 \frac{x - x_j}{x_2 - x_j} \\ &= \left(\frac{x - x_0}{x_2 - x_0} \right) \left(\frac{x - x_1}{x_2 - x_1} \right) \left(\frac{x - x_3}{x_2 - x_3} \right) \end{aligned}$$

$$\begin{aligned} L_3(x) &= \prod_{\substack{j=0 \\ j \neq 3}}^3 \frac{x - x_j}{x_3 - x_j} \\ &= \left(\frac{x - x_0}{x_3 - x_0} \right) \left(\frac{x - x_1}{x_3 - x_1} \right) \left(\frac{x - x_2}{x_3 - x_2} \right) \end{aligned}$$

Hence

$$y(x) = \left(\frac{x-x_1}{x_0-x_1} \right) \left(\frac{x-x_2}{x_0-x_2} \right) \left(\frac{x-x_3}{x_0-x_3} \right) y(x_0) + \left(\frac{x-x_0}{x_1-x_0} \right) \left(\frac{x-x_2}{x_1-x_2} \right) \left(\frac{x-x_3}{x_1-x_3} \right) y(x_1) \\ + \left(\frac{x-x_0}{x_2-x_0} \right) \left(\frac{x-x_1}{x_2-x_1} \right) \left(\frac{x-x_3}{x_2-x_3} \right) y(x_2) + \left(\frac{x-x_0}{x_3-x_0} \right) \left(\frac{x-x_1}{x_3-x_1} \right) \left(\frac{x-x_2}{x_3-x_2} \right) y(x_3)$$

$x_0 \leq x \leq x_3$

Since we're looking for the temperature at $R = 754.8$, we will be using

$$x = \ln(754.8) \\ = 6.6265$$

At $x = 6.6265$,

$$y(6.6265) = \frac{(6.6265 - 6.8149)(6.6265 - 6.4552)(6.6265 - 6.1117)}{(7.0040 - 6.8149)(7.0040 - 6.4552)(7.0040 - 6.1117)} (0.039820) \\ + \frac{(6.6265 - 7.0040)(6.6265 - 6.4552)(6.6265 - 6.1117)}{(6.8149 - 7.0040)(6.8149 - 6.4552)(6.8149 - 6.1117)} (0.033188) \\ + \frac{(6.6265 - 7.0040)(6.6265 - 6.8149)(6.6265 - 6.1117)}{(6.4552 - 7.0040)(6.4552 - 6.8149)(6.4552 - 6.1117)} (0.024925) \\ + \frac{(6.6265 - 7.0040)(6.6265 - 6.8149)(6.6265 - 6.4552)}{(6.1117 - 7.0040)(6.1117 - 6.8149)(6.1117 - 6.4552)} (0.019949) \\ = (-0.17938)(0.039820) + (0.69585)(0.033188) \\ + (0.54005)(0.024925) + (-0.056519)(0.019949) \\ = 0.028285$$

Finally, since $y = \frac{1}{T}$,

$$T = \frac{1}{y} \\ = \frac{1}{0.028285} \\ = 35.355^\circ\text{C}$$

Since the actual measured value at 754.8 ohms is 35.285°C , the absolute relative true error between the value used for part (a) is

$$|\epsilon_t| = \left| \frac{35.285 - 35.242}{35.285} \right| \times 100 \\ = 0.12253\%$$

and for part (b) is

$$|\epsilon_t| = \left| \frac{35.285 - 35.355}{35.285} \right| \times 100 \\ = 0.19825\%$$

Therefore, a cubic polynomial interpolant given by the Lagrangian method of interpolation, that is,

$$T(R) = \sum_{i=0}^3 L_i(R)T(R_i)$$

$$= L_0(R)T(R_0) + L_1(R)T(R_1) + L_2(R)T(R_2) + L_3(R)T(R_3)$$

obtained more accurate results than the calibration curve of

$$\frac{1}{T} = \sum_{i=0}^3 L_i(\ln R) \frac{1}{T(\ln R_i)}$$

$$= L_0(\ln R) \frac{1}{T(\ln R_0)} + L_1(\ln R) \frac{1}{T(\ln R_1)} + L_2(\ln R) \frac{1}{T(\ln R_2)} + L_3(\ln R) \frac{1}{T(\ln R_3)}$$

INTERPOLATION

Topic	Lagrange Interpolation
Summary	Textbook notes on the Lagrangian method of interpolation
Major	Electrical Engineering
Authors	Autar Kaw, Michael Keteltas
Last Revised	November 12, 2012
Web Site	http://numericalmethods.eng.usf.edu
