Chapter 03.00A

Physical Problem for Nonlinear Equations General Engineering

Problem Statement

You are working for 'DOWN THE TOILET COMPANY' that makes floats for ABC commodes. The ball has a specific gravity of 0.6 and has a radius of 5.5 cm. You are asked to find the depth to which the ball will get submerged when floating in water (see Figure 1).

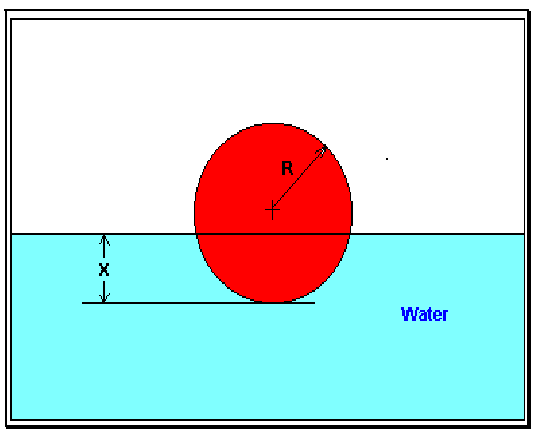

Figure 1 Depth to which the ball is submerged in water

Solution

According to Newton's third law of motion, every action has an equal and opposite reaction. In this case, the weight of the ball is balanced by the buoyancy force (Figure 2).

$$\text{Weight of ball} = \text{Buoyancy force} \tag{1}$$

The weight of the ball is given by

$$\text{Weight of ball} = (\text{Volume of ball}) \times (\text{Density of ball}) \times (\text{Acceleration due to gravity})$$

$$= \left(\frac{4}{3}\pi R^3\right)(\rho_b)(g) \tag{2}$$

where

R = radius of ball (m),

ρ = density of ball $\left(\text{kg/m}^3\right)$,

g = acceleration due to gravity $\left(\text{m/s}^2\right)$.

The buoyancy force[1] is given by

Buoyancy force = Weight of water displaced
= (Volume of ball under water) (Density of water)
(Acceleration due to gravity)

$$=\pi x^2\left(R-\frac{x}{3}\right)\rho_w g \quad (3)$$

where

x = depth to which ball is submerged,

ρ_w = density of water.

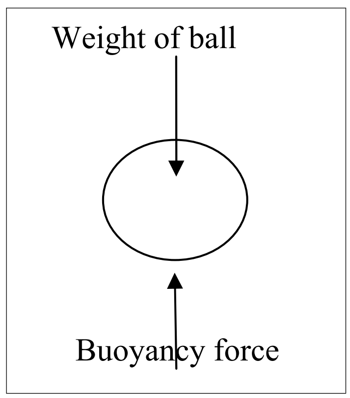

Figure 2 Free Body Diagram showing the forces acting on the ball immersed in water

Now substituting Equations (2) and (3) in Equation (1),

$$\frac{4}{3}\pi R^3\rho_b g=\pi x^2\left(R-\frac{x}{3}\right)\rho_w g$$

$$4R^3\rho_b=3x^2(R-\frac{x}{3})\rho_w$$

$$4R^3\rho_b-3x^2R\rho_w+x^3\rho_w=0$$

$$4R^3\frac{\rho_b}{\rho_w}-3x^2R+x^3=0$$

$$4R^3\gamma_b-3x^2R+x^3=0 \quad (4)$$

where

the specific gravity of the ball, γ_b is given by

$$\gamma_b=\frac{\rho_b}{\rho_w} \quad (5)$$

Given

$R=5.5\,\text{cm}=0.055\,\text{m}$,

$\gamma_b=0.6$, and

substituting in Equation (4), we get

[1] The derivation of the volume of the ball submerged under water is given in the appendix.

$$4(0.055)^3(0.6) - 3x^2(0.055) + x^3 = 0$$

$$3.993\times10^{-4} - 0.165x^2 + x^3 = 0 \qquad (6)$$

The above equation is a nonlinear equation. Solving it would give us the value of 'x', that is, the depth to which the ball is submerged under water.

Appendix A

Derivation of the formula for the volume of a ball submerged under water.
How do you find that the volume of the ball submerged under water as given by

$$V = \frac{\pi h^2(3r-h)}{3} \qquad (7)$$

where

r = radius of the ball,
h = height of the ball to which the ball is submerged.

From calculus,

$$V = \int_{r-h}^{r} A\,dx \qquad (8)$$

where A is the cross-sectioned area at a distance x from the center of the sphere. The lower limit of integration is $x = r - h$ as that is where the water line is

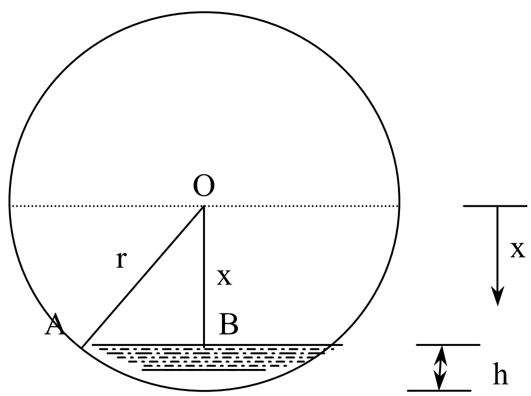

Figure 3 Deriving the equation for volume of ball under water

and the upper limit is r as that is the bottom of the sphere. So, what is the A at any location x.

From Figure 3, for a location x,

$OB = x$,
$OA = r$,

then

$$AB = \sqrt{OA^2 - OB^2}$$
$$= \sqrt{r^2 - x^2}\,. \qquad (9)$$

and AB is the radius of the area at x. So at location B is

$$A = \pi(AB)^2 = \pi\left(r^2 - x^2\right) \tag{10}$$

so

$$V = \int_{r-h}^{r} \pi\left(r^2 - x^2\right)dx$$

$$= \pi\left(r^2 x - \frac{x^3}{3}\right)_{r-h}^{r}$$

$$= \pi\left[\left(r^2 r - \frac{r^3}{3}\right) - \left(r^2(r-h) - \frac{(r-h)^3}{3}\right)\right]$$

$$= \frac{\pi h^2(3r-h)}{3}. \tag{11}$$

NONLINEAR EQUATIONS	
Topic	Physical problem for nonlinear equations for general engineering
Summary	A physical problem of finding the depth to which a ball would float in water is modeled as a nonlinear equation.
Major	General Engineering
Authors	Autar Kaw
Date	December 23, 2009
Web Site	http://numericalmethods.eng.usf.edu

[1] The derivation of the volume of the ball submerged under water is given in the appendix.