Chapter 08.02
Euler's Method for Ordinary Differential Equations

After reading this chapter, you should be able to:

1. *develop Euler's Method for solving ordinary differential equations,*
2. *determine how the step size affects the accuracy of a solution,*
3. *derive Euler's formula from Taylor series, and*
4. *use Euler's method to find approximate values of integrals.*

What is Euler's method?

Euler's method is a numerical technique to solve ordinary differential equations of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0 \tag{1}$$

So only first order ordinary differential equations can be solved by using Euler's method. In another chapter we will discuss how Euler's method is used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations. How does one write a first order differential equation in the above form?

Example 1

Rewrite

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y),\ \ y(0) = y_0 \text{ form.}$$

Solution

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x,y)=1.3e^{-x}-2y$$

Example 2

Rewrite

$$e^{y}\frac{dy}{dx}+x^{2}y^{2}=2\sin(3x),\ \ y(0)=5$$

in

$$\frac{dy}{dx}=f(x,y),\ \ y(0)=y_0 \text{ form.}$$

Solution

$$e^{y}\frac{dy}{dx}+x^{2}y^{2}=2\sin(3x),\ \ y(0)=5$$

$$\frac{dy}{dx}=\frac{2\sin(3x)-x^{2}y^{2}}{e^{y}},\ \ y(0)=5$$

In this case

$$f(x,y)=\frac{2\sin(3x)-x^{2}y^{2}}{e^{y}}$$

Derivation of Euler's method

At $x=0$, we are given the value of $y=y_0$. Let us call $x=0$ as x_0. Now since we know the slope of y with respect to x, that is, $f(x,y)$, then at $x=x_0$, the slope is $f(x_0,y_0)$. Both x_0 and y_0 are known from the initial condition $y(x_0)=y_0$.

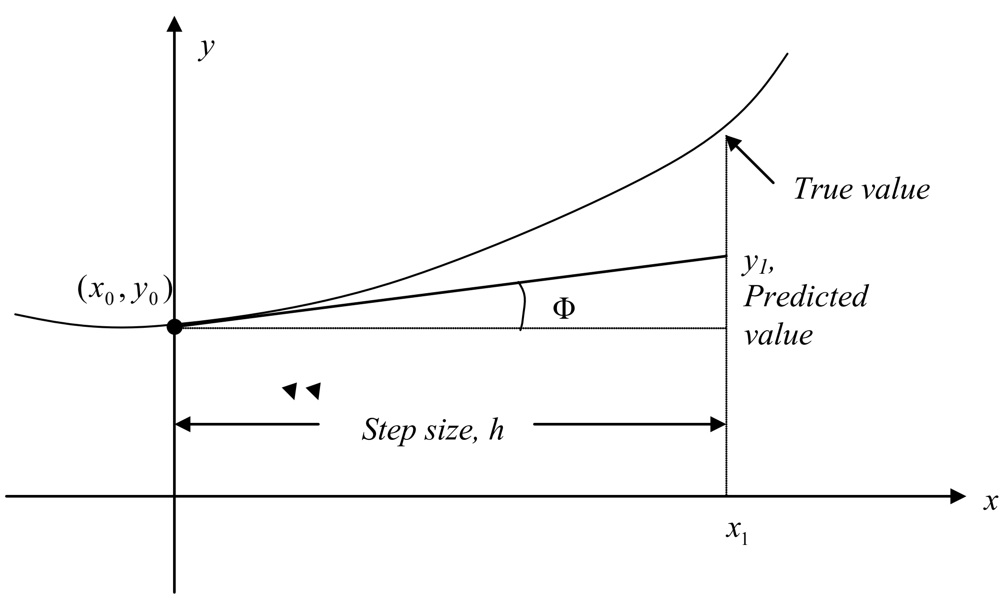

Figure 1 Graphical interpretation of the first step of Euler's method.

So the slope at $x = x_0$ as shown in Figure 1 is

$$\text{Slope} = \frac{Rise}{Run} = \frac{y_1 - y_0}{x_1 - x_0} = f(x_0, y_0)$$

From here

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

Calling $x_1 - x_0$ the step size h, we get

$$y_1 = y_0 + f(x_0, y_0)h \tag{2}$$

One can now use the value of y_1 (an approximate value of y at $x = x_1$) to calculate y_2, and that would be the predicted value at x_2, given by

$$y_2 = y_1 + f(x_1, y_1)h$$
$$x_2 = x_1 + h$$

Based on the above equations, if we now know the value of $y = y_i$ at x_i, then

$$y_{i+1} = y_i + f(x_i, y_i)h \tag{3}$$

This formula is known as Euler's method and is illustrated graphically in Figure 2. In some books, it is also called the Euler-Cauchy method.

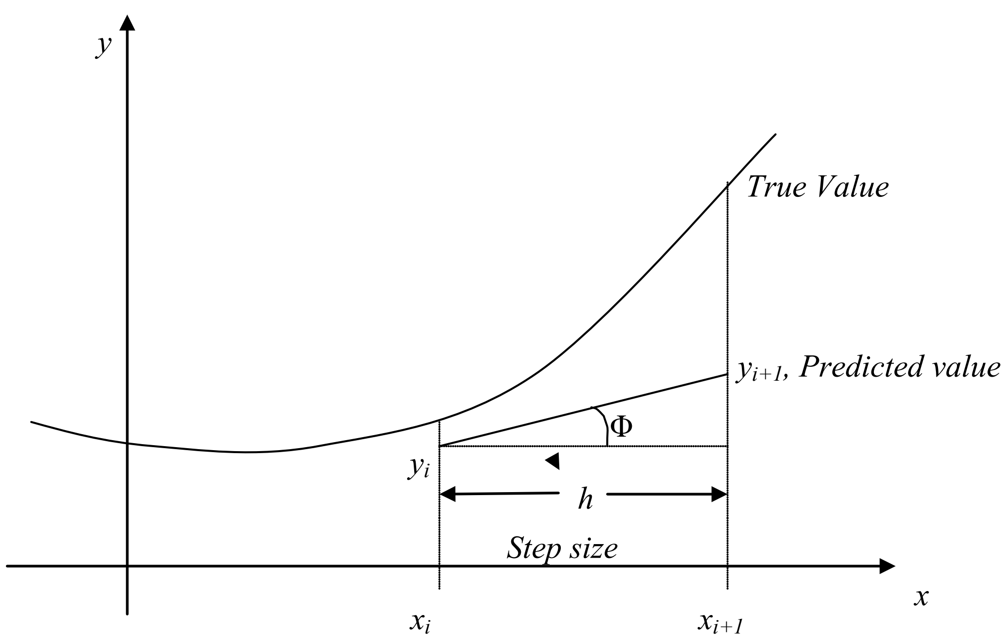

Figure 2 General graphical interpretation of Euler's method.

Example 3

The open loop response, that is, the speed of the motor to a voltage input of 20 V, assuming a system without damping is

$$20 = (0.02)\frac{dw}{dt} + (0.06)w.$$

If the initial speed is zero $(w(0) = 0)$, and using Euler's method, what is the speed at $t = 0.8\,\text{s}$? Assume a step size of $h = 0.4$ s.

Solution

$$\frac{dw}{dt} = 1000 - 3w$$

$$f(t, w) = 1000 - 3w$$

The Euler's method reduces to

$$w_{i+1} = w_i + f(t_i, w_i)h$$

For $i = 0$, $t_0 = 0$, $w_0 = 0$

$$\begin{aligned} w_1 &= w_0 + f(t_0, w_0)h \\ &= 0 + f(0,0) \times 0.4 \\ &= 0 + (1000 - 3 \times (0)) \times 0.4 \\ &= 0 + 1000 \times 0.4 \\ &= 400 \text{ rad/s} \end{aligned}$$

w_1 is the approximate speed of the motor at

$$t = t_1 = t_0 + h = 0 + 0.4 = 0.4\,\text{s}$$

$$w(0.4) \approx w_1 = 400 \text{ rad/s}$$

For $i = 1$, $t_1 = 0.4\,\text{s}$, $w_1 = 400$

$$\begin{aligned} w_2 &= w_1 + f(t_1, w_1)h \\ &= 400.00 + f(0.4, 400) \times 0.4 \\ &= 400.00 + (1000 - 3 \times 400) \times 0.4 \\ &= 400 + (-200) \times 0.4 \\ &= 320 \text{ rad/s} \end{aligned}$$

w_2 is the approximate speed of the motor at

$$t = t_2 = t_1 + h = 0.4 + 0.4 = 0.8\,\text{s}$$

$$w(0.8) \approx w_2 = 320 \text{ rad/s}$$

Figure 1 compares the exact solution with the numerical solution from Euler's method for the step size of $h = 0.4\,\text{s}$.

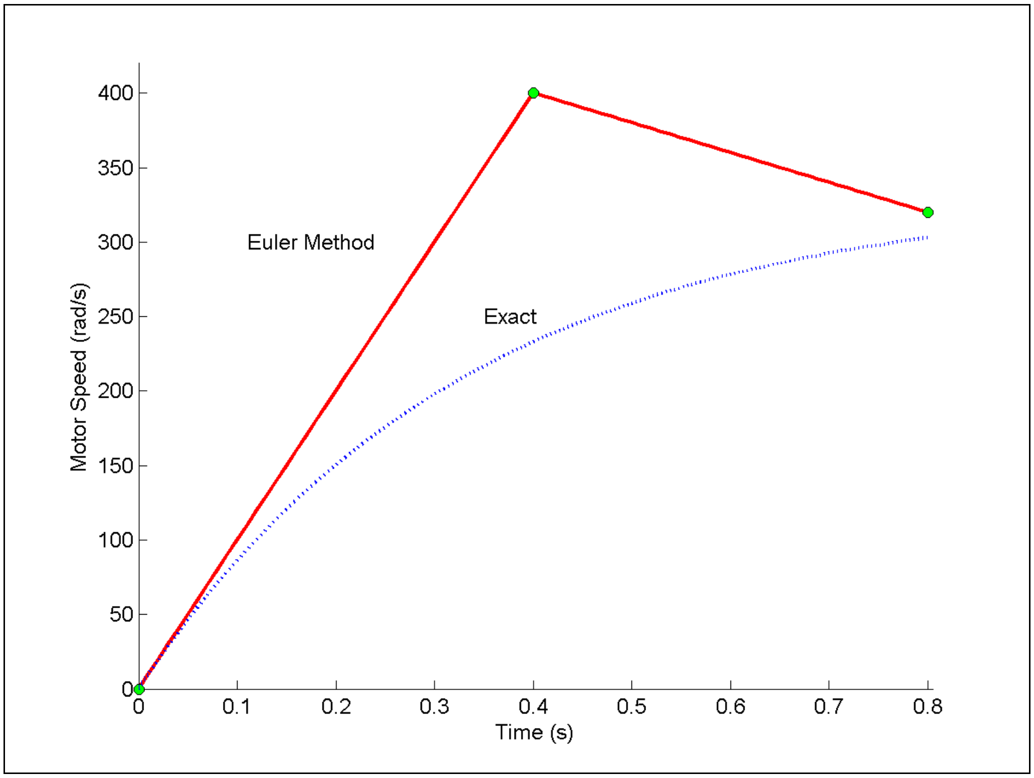

Figure 3 Comparing exact and Euler's method.

The problem was solved again using smaller step sizes. The results are given below in Table 1.

Table 1 Speed of motor at 0.8 seconds as a function of step size, h.

Step size, h	$w(0.8)$	E_t	$\lvert \in_t \rvert \%$
0.8	800	−496.91	163.95
0.4	320	−16.906	5.5778
0.2	324.8	−21.706	7.1615
0.1	314.18	−11.023	3.6370
0.05	308.58	−5.4890	1.8110

Figure 4 shows how the speed of the motor varies as a function of time for different step sizes.

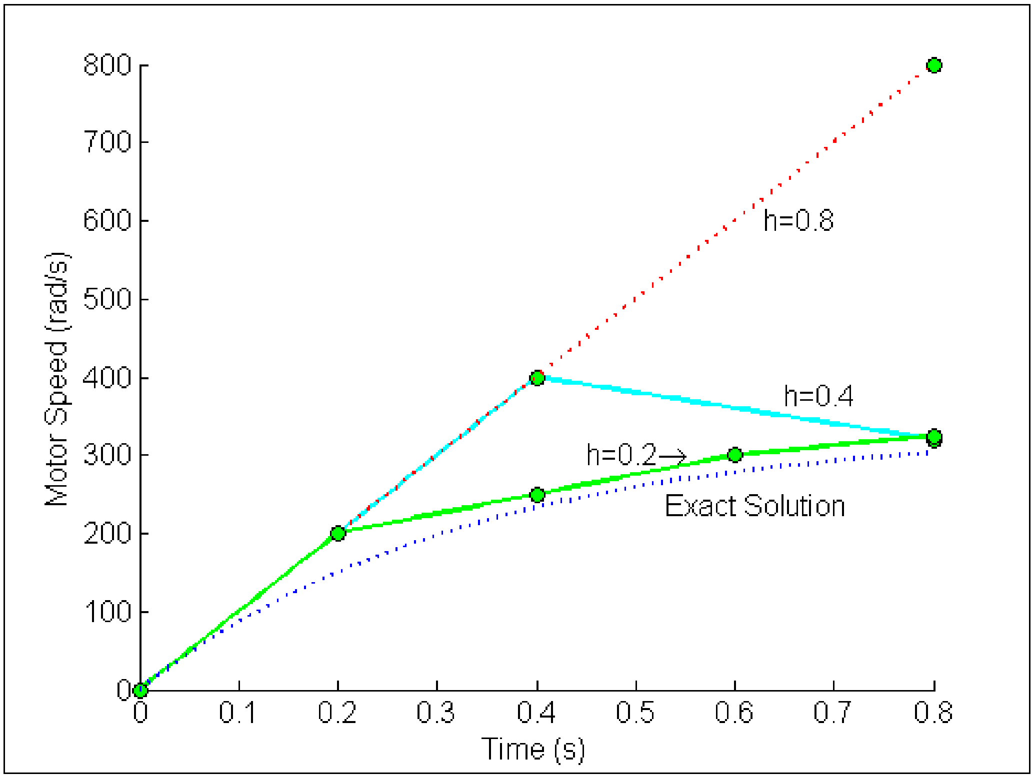

Figure 4 Comparison of Euler's method with exact solution for different step sizes.

The values of the calculated speed of the motor at $t = 0.8$ s as a function of step size are plotted in Figure 5.

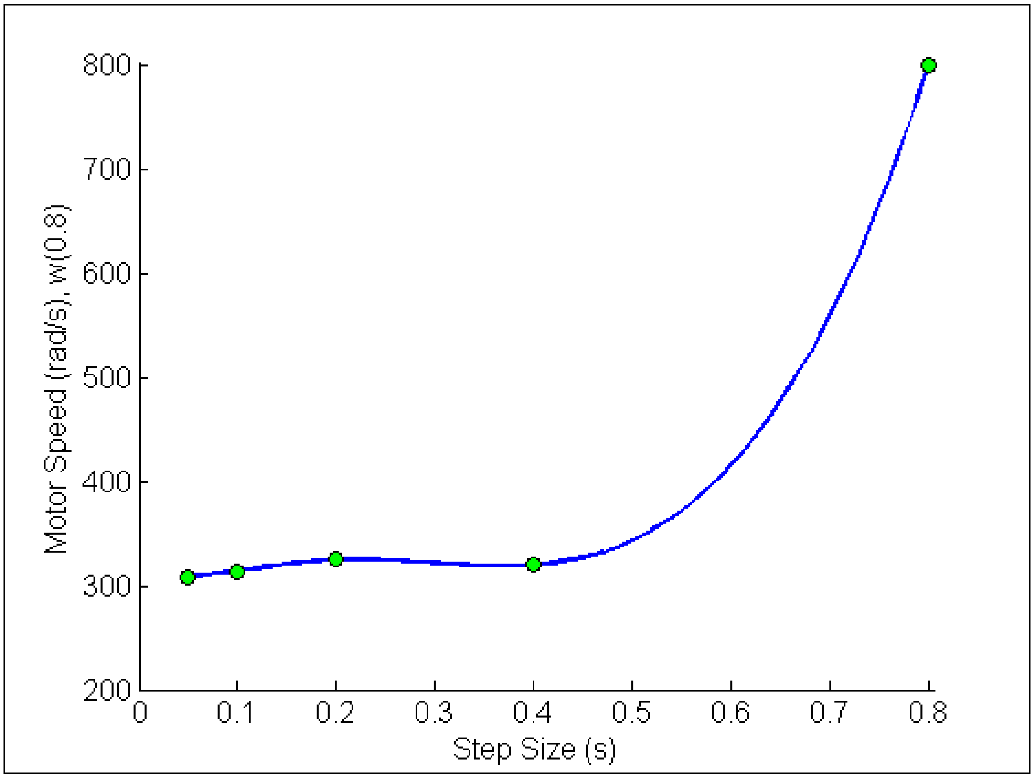

Figure 5 Effect of step size in Euler's method.

The exact solution of the ordinary differential equation is given by

$$w(t) = \left(\frac{1000}{3}\right) - \left(\frac{1000}{3}\right)e^{-3t}$$

The solution to this nonlinear equation at $t = 0.8\text{s}$ is

$$w(0.8) = 303.09 \text{ rad/s}$$

Can one solve a definite integral using numerical methods such as Euler's method of solving ordinary differential equations?

Let us suppose you want to find the integral of a function $f(x)$

$$I = \int_a^b f(x)dx .$$

Both fundamental theorems of calculus would be used to set up the problem so as to solve it as an ordinary differential equation.

The first fundamental theorem of calculus states that if f is a continuous function in the interval [a,b], and F is the antiderivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

The second fundamental theorem of calculus states that if f is a continuous function in the open interval D, and a is a point in the interval D, and if

$$F(x)=\int_a^x f(t)dt$$

then

$$F'(x)=f(x)$$

at each point in D.

Asked to find $\int_a^b f(x)dx$, we can rewrite the integral as the solution of an ordinary differential equation (here is where we are using the second fundamental theorem of calculus)

$$\frac{dy}{dx}=f(x),\ y(a)=0,$$

where then $y(b)$ (here is where we are using the first fundamental theorem of calculus) will give the value of the integral $\int_a^b f(x)dx$.

Example 4

Find an approximate value of

$$\int_5^8 6x^3dx$$

using Euler's method of solving an ordinary differential equation. Use a step size of $h=1.5$.

Solution

Given $\int_5^8 6x^3dx$, we can rewrite the integral as the solution of an ordinary differential equation

$$\frac{dy}{dx}=6x^3,\ y(5)=0$$

where $y(8)$ will give the value of the integral $\int_5^8 6x^3dx$.

$$\frac{dy}{dx}=6x^3=f(x,y),\ y(5)=0$$

The Euler's method equation is

$$y_{i+1}=y_i+f(x_i,y_i)h$$

<u>Step 1</u>

$$i=0,\ x_0=5,\ y_0=0$$

$$h=1.5$$

$$\begin{aligned}x_1&=x_0+h\\&=5+1.5\\&=6.5\end{aligned}$$

$$y_1=y_0+f(x_0,y_0)h$$

$$
\begin{aligned}
&= 0 + f(5,0) \times 1.5 \\
&= 0 + (6 \times 5^3) \times 1.5 \\
&= 1125 \\
&\approx y(6.5)
\end{aligned}
$$

Step 2

$$i = 1,\ x_1 = 6.5,\ y_1 = 1125$$

$$
\begin{aligned}
x_2 &= x_1 + h \\
&= 6.5 + 1.5 \\
&= 8
\end{aligned}
$$

$$
\begin{aligned}
y_2 &= y_1 + f(x_1, y_1)h \\
&= 1125 + f(6.5,1125) \times 1.5 \\
&= 1125 + (6 \times 6.5^3) \times 1.5 \\
&= 3596.625 \\
&\approx y(8)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_5^8 6x^3 dx &= y(8) - y(5) \\
&\approx 3596.625 - 0 \\
&= 3596.625
\end{aligned}
$$

ORDINARY DIFFERENTIAL EQUATIONS	
Topic	Euler's Method for ordinary differential equations
Summary	Textbook notes on Euler's method for solving ordinary differential equations
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