Chapter 08.04
Runge-Kutta 4th Order Method for Ordinary Differential Equations

After reading this chapter, you should be able to

1. *develop Runge-Kutta 4th order method for solving ordinary differential equations,*
2. *find the effect size of step size has on the solution,*
3. *know the formulas for other versions of the Runge-Kutta 4th order method*

What is the Runge-Kutta 4th order method?

Runge-Kutta 4$^{\text{th}}$ order method is a numerical technique used to solve ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

So only first order ordinary differential equations can be solved by using the Runge-Kutta 4$^{\text{th}}$ order method. In other sections, we have discussed how Euler and Runge-Kutta methods are used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations.

How does one write a first order differential equation in the above form?

Example 1

Rewrite

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y), \; y(0) = y_0 \text{ form.}$$

Solution

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example 2

Rewrite

$$e^{y}\frac{dy}{dx} + x^2y^2 = 2\sin(3x),\ y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y),\ y(0) = y_0 \text{ form.}$$

Solution

$$e^{y}\frac{dy}{dx} + x^2y^2 = 2\sin(3x),\ y(0) = 5$$

$$\frac{dy}{dx} = \frac{2\sin(3x) - x^2y^2}{e^y},\ y(0) = 5$$

In this case

$$f(x, y) = \frac{2\sin(3x) - x^2y^2}{e^y}$$

The Runge-Kutta 4th order method is based on the following

$$y_{i+1} = y_i + (a_1k_1 + a_2k_2 + a_3k_3 + a_4k_4)h \tag{1}$$

where knowing the value of $y = y_i$ at x_i, we can find the value of $y = y_{i+1}$ at x_{i+1}, and

$$h = x_{i+1} - x_i$$

Equation (1) is equated to the first five terms of Taylor series

$$y_{i+1} = y_i + \frac{dy}{dx}\Big|_{x_i,y_i}(x_{i+1} - x_i) + \frac{1}{2!}\frac{d^2y}{dx^2}\Big|_{x_i,y_i}(x_{i+1} - x_i)^2 + \frac{1}{3!}\frac{d^3y}{dx^3}\Big|_{x_i,y_i}(x_{i+1} - x_i)^3 + \frac{1}{4!}\frac{d^4y}{dx^4}\Big|_{x_i,y_i}(x_{i+1} - x_i)^4 \tag{2}$$

Knowing that $\frac{dy}{dx} = f(x, y)$ and $x_{i+1} - x_i = h$

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!}f'(x_i, y_i)h^2 + \frac{1}{3!}f''(x_i, y_i)h^3 + \frac{1}{4!}f'''(x_i, y_i)h^4 \tag{3}$$

Based on equating Equation (2) and Equation (3), one of the popular solutions used is

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \tag{4}$$

$$k_1 = f(x_i, y_i) \tag{5a}$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right) \tag{5b}$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right) \tag{5c}$$

$$k_4 = f(x_i + h, y_i + k_3 h) \tag{5d}$$

Example 3

The open loop response, that is, the speed of the motor to a voltage input of 20V, assuming a system without damping is

$$20 = (0.02)\frac{dw}{dt} + (0.06)w.$$

If the initial speed is zero $(w(0) = 0)$,and using the Runge-Kutta 4th order method, what is the speed at $t = 0.8\,\text{s}$? Assume a step size of $h = 0.4\,\text{s}$.

Solution

$$\frac{dw}{dt} = 1000 - 3w$$

$$f(t,w) = 1000 - 3w$$

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

For $i = 0$, $t_0 = 0$, $w_0 = 0$

$$\begin{aligned}
k_1 &= f(t_0, w_0) \\
&= f(0,0) \\
&= 1000 - 3\times 0 \\
&= 1000
\end{aligned}$$

$$\begin{aligned}
k_2 &= f\left(t_0 + \frac{1}{2}h, w_0 + \frac{1}{2}k_1 h\right) \\
&= f\left(0 + \left(\frac{1}{2}\times 0.4\right), 0 + \left(\frac{1}{2}(1000)\times 0.4\right)\right) \\
&= f(0.2, 200) \\
&= 1000 - 3\times 200 \\
&= 400
\end{aligned}$$

$$\begin{aligned}
k_3 &= f\left(t_0 + \frac{1}{2}h, w_0 + \frac{1}{2}k_2 h\right) \\
&= f\left(0 + \left(\frac{1}{2}\times 0.4\right), 0 + \left(\frac{1}{2}(400)\times 0.4\right)\right) \\
&= f(0.2, 80) \\
&= 1000 - 3\times 80
\end{aligned}$$

$$\begin{aligned}
&= 760\\
k_4 &= f(t_0 + h, w_0 + k_3 h)\\
&= f(0 + (0.4), 0 + ((760)\times 0.4))\\
&= f(0.4, 304)\\
&= 1000 - 3\times 304\\
&= 88\\
w_1 &= w_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h\\
&= 0 + \frac{1}{6}(1000 + 2\times(400) + 2\times(760) + (88))\times 0.4\\
&= 0 + \frac{1}{6}(3408)\times 0.4\\
&= 227.2 \text{ rad/s}
\end{aligned}$$

w_1 is the approximate speed of the motor at

$$t = t_1 = t_0 + h = 0 + 0.4 = 0.4\,\text{s}$$
$$w(0.4) \approx w_1 = 227.2 \text{ rad/s}$$

For $i = 1$, $t_1 = 0.4$, $w_1 = 227.2$

$$\begin{aligned}
k_1 &= f(t_1, w_1)\\
&= f(0.4, 227.2)\\
&= 1000 - 3\times 227.2\\
&= 318.4\\
k_2 &= f\left(t_1 + \frac{1}{2}h, w_1 + \frac{1}{2}k_1 h\right)\\
&= f\left(0.4 + \left(\frac{1}{2}\times 0.4\right), 227.2 + \left(\frac{1}{2}(318.4)\times 0.4\right)\right)\\
&= f(0.6, 290.88)\\
&= 1000 - 3\times 290.88\\
&= 127.36\\
k_3 &= f\left(t_1 + \frac{1}{2}h, w_1 + \frac{1}{2}k_2 h\right)\\
&= f\left(0.4 + \left(\frac{1}{2}\times 0.4\right), 227.2 + \left(\frac{1}{2}(127.36)\times 0.4\right)\right)\\
&= f(0.6, 252.67)\\
&= 1000 - 3\times 252.67\\
&= 241.98\\
k_4 &= f(t_1 + h, w_1 + k_3 h)\\
&= f(0.4 + 0.4, 227.2 + (241.98\times 0.4))\\
&= f(0.8, 323.99)\\
&= 1000 - 3\times 323.99\\
&= 28.019
\end{aligned}$$

$$w_2 = w_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$
$$= 227.2 + \frac{1}{6}(318.4 + 2\times(127.36) + 2\times(241.98) + 28.019)\times 0.4$$
$$= 227.2 + \frac{1}{6}(1085.1)\times 0.4$$
$$= 299.54 \text{ rad/s}$$

w_2 is the approximate speed of the motor at

$$t = t_2 = t_1 + h = 0.4 + 0.4 = 0.8\, s$$
$$w(0.8) \approx w_2 = 299.54 \text{ rad/s}$$

The exact solution of the ordinary differential equation is given by

$$w(t) = \left(\frac{1000}{3}\right) - \left(\frac{1000}{3}\right)e^{-3t}$$

The solution to this nonlinear equation at $t = 0.8\text{s}$ is

$$w(0.8) = 303.09 \text{ rad/s}$$

Figure 1 compares the exact solution with the numerical solution using the Runge-Kutta 4th order method using different step sizes.

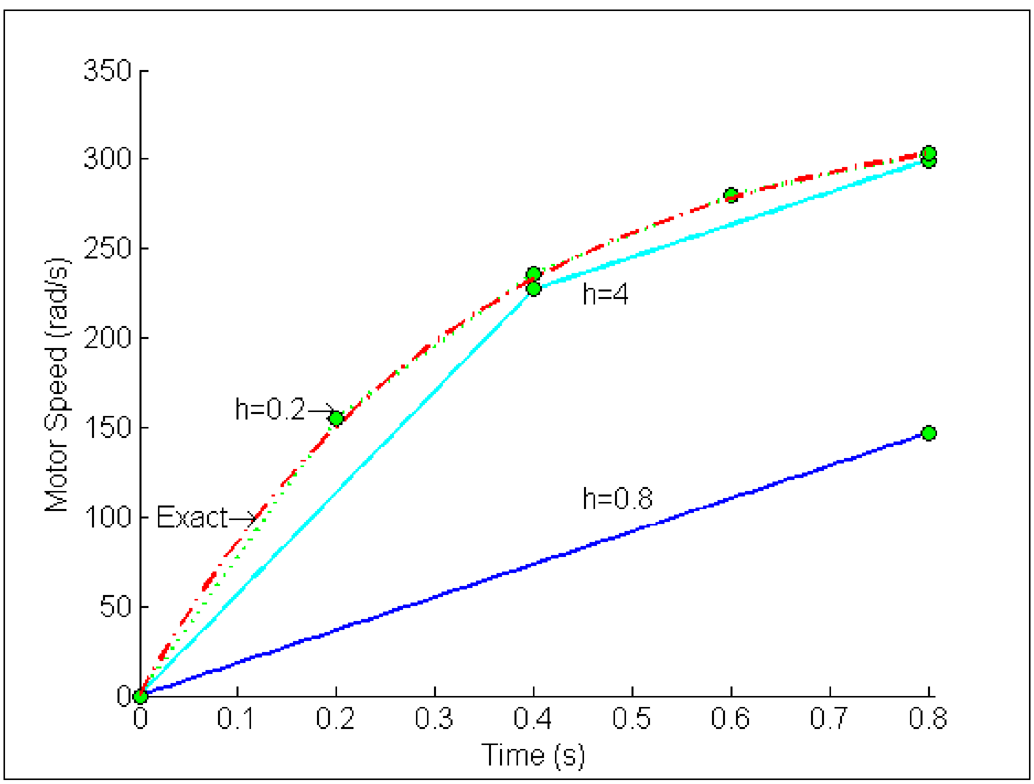

Figure 1 Comparison of Runge-Kutta 4th order method with exact solution for different step sizes.

Table 1 and Figure 2 show the effect of step size on the value of the calculated speed of the motor at $t = 0.8\,\text{s}$.

Table 1 Values of speed of the motor at 0.8 seconds for different step sizes.

Step size, h	$w(0.8)$	E_t	$\lvert\in_t\rvert\,\%$
0.8	147.20	155.89	51.434
0.4	299.54	3.5535	1.1724
0.2	302.96	0.12988	0.042852
0.1	303.09	0.0062962	0.0020773
0.05	303.09	0.00034702	0.00011449

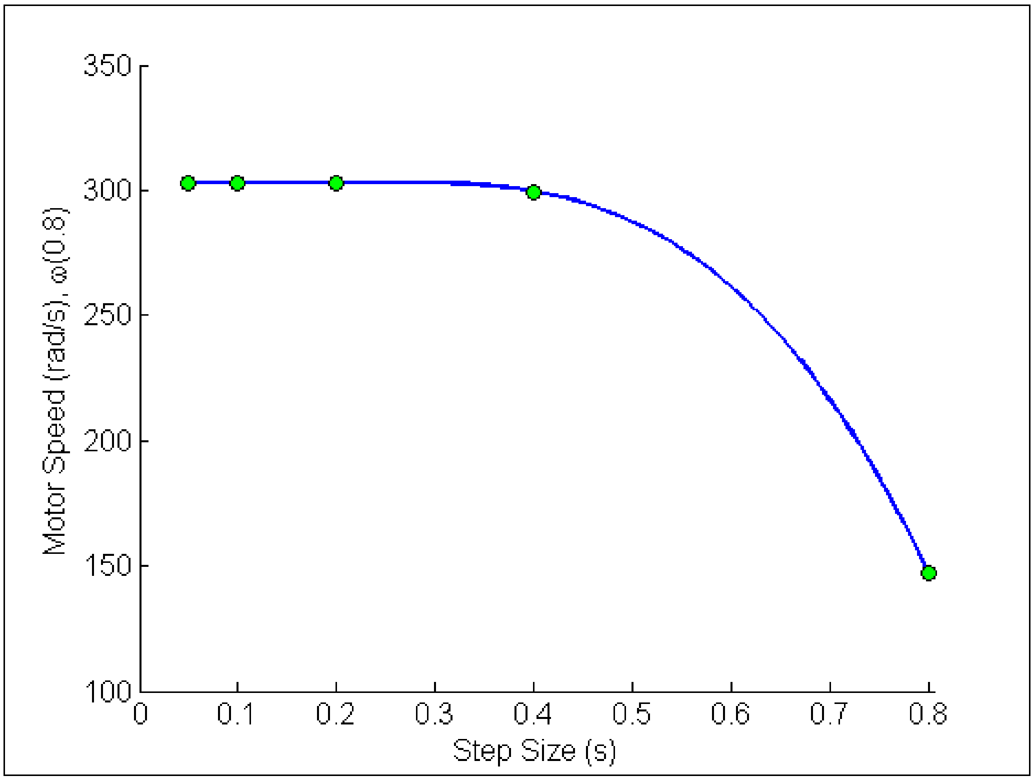

Figure 2 Effect of step size in Runge-Kutta 4th order method.

In Figure 3, we are comparing the exact results with Euler's method (Runge-Kutta 1st order method), Heun's method (Runge-Kutta 2nd order method) and the Runge-Kutta 4th order method.

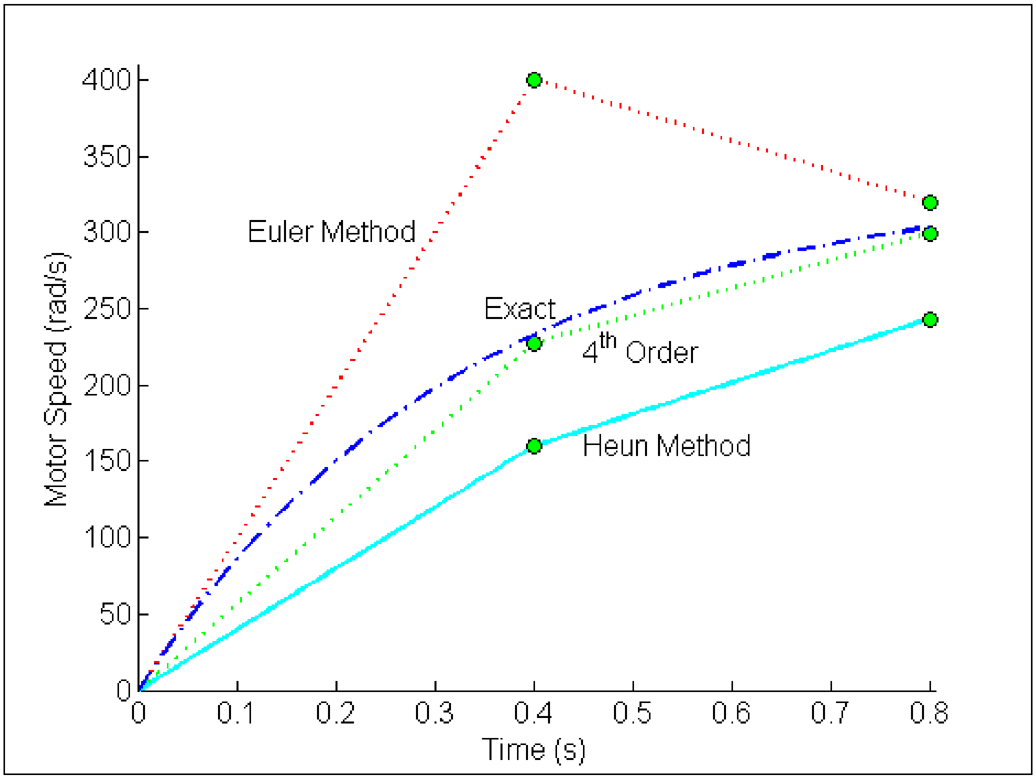

Figure 3 Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.

ORDINARY DIFFERENTIAL EQUATIONS	
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