Chapter 08.00G

Physical Problem for Mechanical Engineering Ordinary Differential Equations

Problem Statement

To make the fulcrum (Figure 1) of a bascule bridge, a long hollow steel shaft called the trunnion is shrink fit into a steel hub. The resulting steel trunnion-hub assembly is then shrink fit into the girder of the bridge.

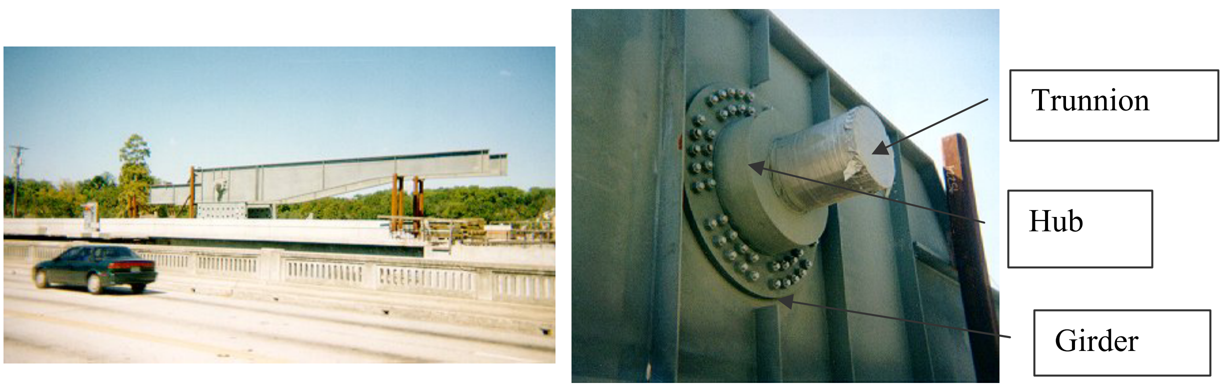

Figure 1 Trunnion-Hub-Girder (THG) assembly.

This is done by first immersing the trunnion in a cold medium such as liquid nitrogen. After the trunnion reaches the steady state temperature of the cold medium, the trunnion outer diameter contracts. The trunnion is taken out of the medium and slid though the hole of the hub (Figure 1).

When the trunnion heats up, it expands and creates an interference fit with the hub. Because of the large thermal shock that the trunnion undergoes, it is suggested that the trunnion be cooled in stages. First, put the trunnion in a refrigerated chamber, then dip it in dry-ice/alcohol mixture and then immerse it in a bath of liquid nitrogen. However, this approach will take more time. One is mainly concerned about the cool down time in the refrigerated chamber. Assuming that the room temperature is 27°C and the refrigerated chamber is set is -33°C, how much time would it take for the trunnion to cool down to -33°C.

Let us do a simplified problem of a trunnion that is solid and assume the trunnion can be treated as a lumped-mass system. What does a lumped system mean[1]?

Let us develop the mathematical model for the above problem. When the trunnion is placed in the refrigerated chamber, the trunnion loses heat to its surroundings by convection.

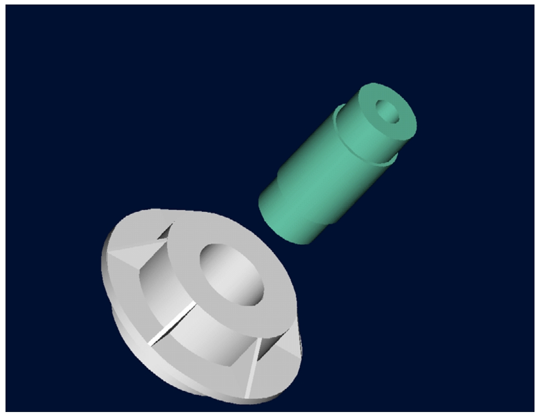

Figure 2 Trunnion slide through the hub after contracting

Rate of heat lost due to convection $= h(\theta)A(\theta - \theta_a)$. (1)

where

$h(\theta)$ = the convective cooling coefficient, $\text{W/m}^2\text{.K}$ and is a function of temperature

$A =$ surface area

$\theta_a =$ ambient temperature of refrigerated chamber

The energy stored in the mass is given by

Energy stored by mass = $mC\theta$ (2)

where

m = mass of the trunnion, kg

C = specific heat of the trunnion, J/(kg-K)

From an energy balance,

Rate at which heat is gained - Rate at which heat is lost =

= Rate at which heat is stored

gives

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[1] It implies that the internal conduction in the trunnion is large enough that the temperature throughout the trunnion is uniform. This allows us to make the assumption that the temperature is only a function of time and not of the location in the trunnion. This means that if a differential equation governs this physical problem, it would be an ordinary differential equation for a lumped system and a partial differential equation for a non-lumped system. In your heat transfer course, you will learn when a system can be considered lumped or non-lumped. In simplistic terms, this distinction is based on the material, geometry, and heat exchange factors of the ball with its surroundings. *We will prove later that the trunnion can be treated as a lumped system.*

$$-h(\theta)A(\theta-\theta_a)=mC\frac{d\theta}{dt} \qquad (3)$$

Note that the convective cooling coefficient is a function of temperature. Other material parameters such as density (affecting mass), specific heat and thermal conductivity (affecting whether the system can be considered lumped or not) of the trunnion material are functions of temperature as well, but within our temperature range of 27°C to -33°C these vary by only 10% from the room temperature values. So we will assume these parameters to be constant. Let us now determine the constants needed for the above ordinary differential equation.

<u>Convection coefficient of air</u>

The convection coefficient of air, $h(\theta)$ is not a constant function of temperature and is given by

$$h=\frac{Nu\times k}{D} \qquad (4)$$

where

Nu is the Nusselt number,

k is the thermal conductivity of air,

D is the characteristic diameter and in this case is taken as the outer diameter of the trunnion.

The Nusselt number for a vertical cylinder is given by the empirical formula [1]:

$$Nu=\left(0.825+\frac{0.387Ra^{\frac{1}{6}}}{\left(1+\left(\frac{0.492}{\Pr}\right)^{\frac{9}{16}}\right)^{\frac{8}{27}}}\right)^{2} \qquad (5)$$

where

Pr is the Prandtl number given by

$$\Pr=\frac{\nu_k}{\alpha} \qquad (6)$$

ν_k is the kinematic viscosity of the fluid,

α is the thermal diffusivity,

R_a is the Rayleigh number given by:

$$R_a=G_r\times \Pr \qquad (7)$$

G_r is the Grashoff number given by:

$$Gr=\frac{g\beta(T_{wall}-T_{fluid})D^3}{\nu_k^{\ 2}} \qquad (8)$$

g is the gravitational constant,

β is the volumetric thermal expansion coefficient,

T_{wall} is the temperature of the wall,
T_{fluid} is the temperature of the fluid.

To calculate the convection coefficient, we use the following value for kinematics viscosity ,ν thermal conductivity , k and thermal diffusivity ,α and volumetric coefficient, β of air as a function of temperature, T [Ref. 1] .

Table 1 Properties of air as a function of temperature.

T_{lookup}	ν	k	α	β
°C	m^2/s	W/(m-K)	m^2/s	1/K
-173.15	2.00×10^{-6}	9.34×10^{-3}	2.54×10^{-6}	1.00×10^{-2}
-123.15	4.43×10^{-6}	1.38×10^{-2}	5.84×10^{-6}	6.67×10^{-3}
-73.15	7.59×10^{-6}	1.81×10^{-2}	1.03×10^{-5}	5.00×10^{-3}
-23.15	1.14×10^{-5}	2.23×10^{-2}	1.59×10^{-5}	4.00×10^{-3}
26.85	1.59×10^{-5}	2.63×10^{-2}	2.25×10^{-5}	3.33×10^{-3}

Other constants needed are:

$$D = 0.25\,\text{m}$$

$$g = 9.8\,\text{m/s}^2$$

$$T_{fluid} = -33^o\,\text{C}$$

Table 2 Convection coefficient of air as a function of temperature.

T_{wall}	$T_{average}$ [= ½(T fluid + T wall)]	h
ºC	ºC	W/(m².K)
-33	-16.50	5.846E-02
-18	-9.00	4.527E+00
-8	-4.00	5.214E+00
2	1.00	5.702E+00
27	13.50	6.533E+00

The values in Table 2 are interpolated from Table 1 where temperatures are chosen as the average value for the wall and ambient temperature. The above data is interpolated as:

$$h(\theta) = -3.69\times10^{-6}\theta^4 + 2.33\times10^{-5}\theta^3 + 1.35\times10^{-3}\theta^2 + 5.42\times10^{-2}\theta + 5.59 \tag{9}$$

Area, A

Outer radius of trunnion, $a = 0.125\,\text{m}$.

Length, of trunnion, $L = 1.36\,\text{m}$.

gives

$$\begin{aligned} A &= (2\pi a)L + 2\pi a^2 \\ &= 2\pi(.125)(1.36) + 2\pi(0.125)^2 \\ &= 1.166\ m^2 \end{aligned} \tag{10}$$

Mass, m

Length of the trunnion, $L = 1.36\,\text{m}$

Density of trunnion material, ρ = 7800 kg/m^3

gives

$$\begin{aligned} m &= \rho V \\ &= \rho(\pi r^2 L) \\ &= 7800\pi(0.125)^2 1.36 \\ &= 520.72 \text{ kg} \end{aligned} \tag{11}$$

Other constants

Specific heat, C = 420 J/ (kg-K)

Initial temperature of the trunnion, $T(0)$ = 27° C,

Ambient temperature, $T_a = -33°\text{C}$

The first order ordinary differential equation:

$$-h(\theta)A(\theta-\theta_a) = mC\frac{d\theta}{dt}$$

is given by

$$-\left(-3.69\times10^{-6}\theta^4 + 2.33\times10^{-5}\theta^3 + 1.35\times10^{-3}\theta^2 + 5.42\times10^{-2}\theta + 5.588\right)\times$$
$$\times(1.166)\times(\theta+33) = (520.72)\times(420)\times\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = -5.331\times10^{-6}(-3.69\times10^{-6}\theta^4 + 2.33\times10^{-5}\theta^3 + 1.35\times10^{-3}\theta^2 + 5.42\times10^{-2}\theta + 5.588)(\theta+33) \tag{12}$$

$$\theta(0) = 27\,°\text{C}$$

Is the assumption of the trunnion considered as a lumped system correct?

To determine whether a system is lumped, we calculate the Biot number which defined as

$$\text{Bi} = \frac{hL}{k_s} \tag{13}$$

where

h = average surface conductance,

L = significant length dimension (volume of body/surface area),

k_s = thermal conductivity of solid body.

If $B_i < 0.1$, the temperature in the body is uniform within 5% error.

In our case:

$$h = 4.407\frac{W}{m^2.K}$$

$$L = 1.36\text{m}$$

$$k_s = 81\frac{W}{m.K} \qquad \text{[Ref. 2]}$$

$$Bi = \frac{4.407\times1.36}{81}$$

$= 0.074 < 0.1$

This gives us a Biot number that is less than 0.1. One can hence assume the trunnion to be a lumped mass system.

References

[1]. F. B. Incropera, D. P. Dewitt, Introduction to Heat Transfer, 3rd Ed, John Wiley & Sons, Inc., New York, NY, 2000, Chap. 9.
[2]. F. Kreith, M. S. Bohn, Principle of Heat Transfer, 4th Ed, Harper & Row, Publishers, New York, NY, 1993, Appendix 2.

ORDINARY DIFFERENTIAL EQUATION	
Topic	Ordinary Differential Equations
Summary	A physical problem of finding how much time it would take a trunnion to cool down in a refrigerated chamber. To find the time, the problem would be modeled as a ordinary differential equation.
Major	Mechanical Engineering
Authors	Autar Kaw
Date	December 23, 2009
Web Site	http://numericalmethods.eng.usf.edu