Euler Method

Mechanical Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

Euler Method

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Euler's Method

$$\frac{dy}{dx} = f(x,y), y(0) = y_0$$

$$\text{Slope} = \frac{Rise}{Run}$$

$$= \frac{y_1 - y_0}{x_1 - x_0}$$

$$= f(x_0, y_0)$$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$= y_0 + f(x_0, y_0)h$$

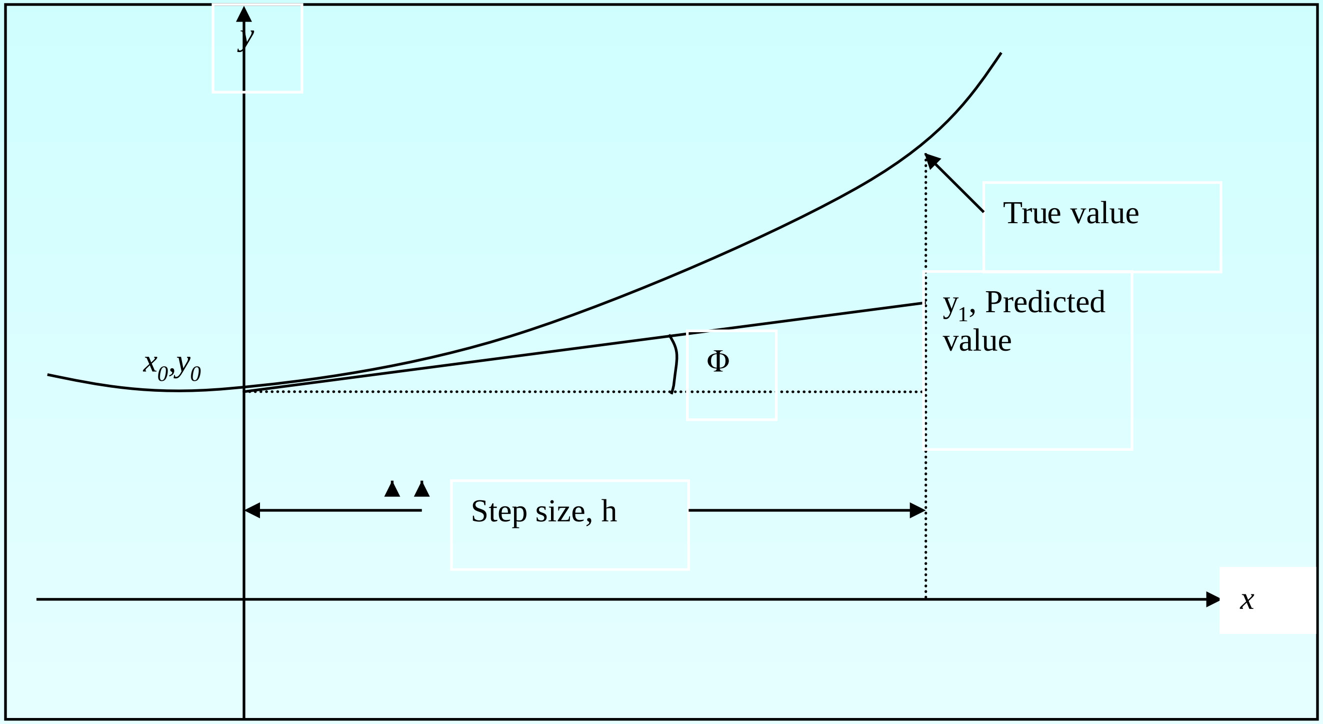

Figure 1 Graphical interpretation of the first step of Euler's method

Euler's Method

$$y_{i+1} = y_i + f\left(x_i, y_i\right)h$$

$$h = x_{i+1} - x_i$$

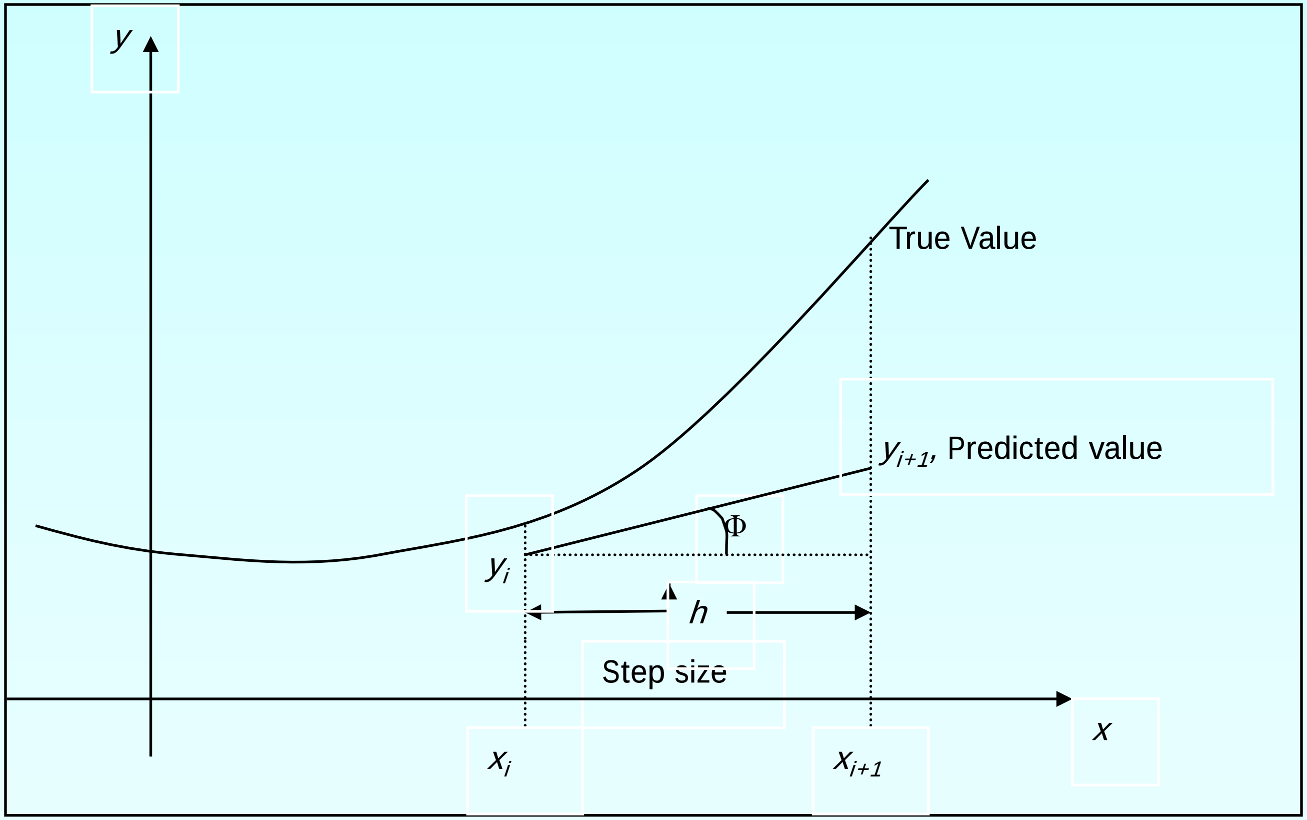

Figure 2. General graphical interpretation of Euler's method

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, \; y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, \; y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A solid steel shaft at room temperature of 27°C is needed to be contracted so that it can be shrunk-fit into a hollow hub. It is placed in a refrigerated chamber that is maintained at −33°C. The rate of change of temperature of the solid shaft θ is given by

$$\frac{d\theta}{dt} = -5.33\times10^{-6}\begin{pmatrix} -3.69\times10^{-6}\theta^4 + 2.33\times10^{-5}\theta^3 + 1.35\times10^{-3}\theta^2 \\ +5.42\times10^{-2}\theta + 5.588 \end{pmatrix}(\theta + 33)$$

$$\theta(0) = 27°\mathrm{C}$$

Using Euler's method, find the temperature of the steel shaft after 86400 seconds. Take a step size of h = 43200 seconds.

$$\frac{d\theta}{dt} = -5.33\times10^{-6}\left(-3.69\times10^{-6}\theta^4 + 2.33\times10^{-5}\theta^3 + 1.35\times10^{-3}\theta^2 + 5.42\times10^{-2}\theta + 5.588\right)(\theta + 33)$$

$$f(t,\theta) = -5.33\times10^{-6}\left(-3.69\times10^{-6}\theta^4 + 2.33\times10^{-5}\theta^3 + 1.35\times10^{-3}\theta^2 + 5.42\times10^{-2}\theta + 5.588\right)(\theta + 33)$$

$$\theta_{i+1} = \theta_i + f(t_i, \theta_i)h$$

Solution

Step 1: $i = 0,\ t_0 = 0,\ \theta_0 = 27$

$$\theta_1 = \theta_0 + f\left(t_0, \theta_0\right)h$$
$$= 27 + f\left(0,\ 27\right)43200$$
$$= 27 + \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}\left(27\right)^4 + 2.33\times10^{-5}\left(27\right)^3 \\ +1.35\times10^{-3}\left(27\right)^2 + 5.42\times10^{-2}\left(27\right) + 5.588\end{pmatrix}\left(27+33\right)\right)43200$$
$$= 27 + \left(-0.0020893\right)43200$$
$$= -63.258°\text{C}$$

θ_1 is the approximate temperature at

$$t = t_1 = t_0 + h = 0 + 43200 = 43200\,\text{s}$$

$$\theta\left(43200\right) \approx \theta_1 = -63.258°C$$

Solution Cont

Step 2: $i = 1, \quad t_1 = 43200, \quad \theta_1 = -63.258$

$$
\begin{aligned}
\theta_2 &= \theta_1 + f(t_1, \theta_1)h \\
&= -63.258 + f(43200, -63.258)43200 \\
&= -63.258 + \left(-5.33\times10^{-6} \begin{pmatrix} -3.69\times10^{-6}(-63.258)^4 + 2.33\times10^{-5}(-63.258)^3 \\ +1.35\times10^{-3}(-63.258)^2 + 5.42\times10^{-2}(-63.258) + 5.588 \end{pmatrix} (-63.258 + 33) \right) 43200 \\
&= -63.258 + (-0.0092607)43200 \\
&= -463.32^\circ\text{C}
\end{aligned}
$$

θ_2 is the approximate temperature at

$$t = t_2 = t_1 + h = 43200 + 43200 = 86400 \text{ s}$$

$$\theta(86400) \approx \theta_2 = -463.32^\circ C$$

Solution Cont

The solution to this nonlinear equation at *t=86400s* is

$$\theta(86400) = -26.099°C$$

Comparison of Exact and Numerical Solutions

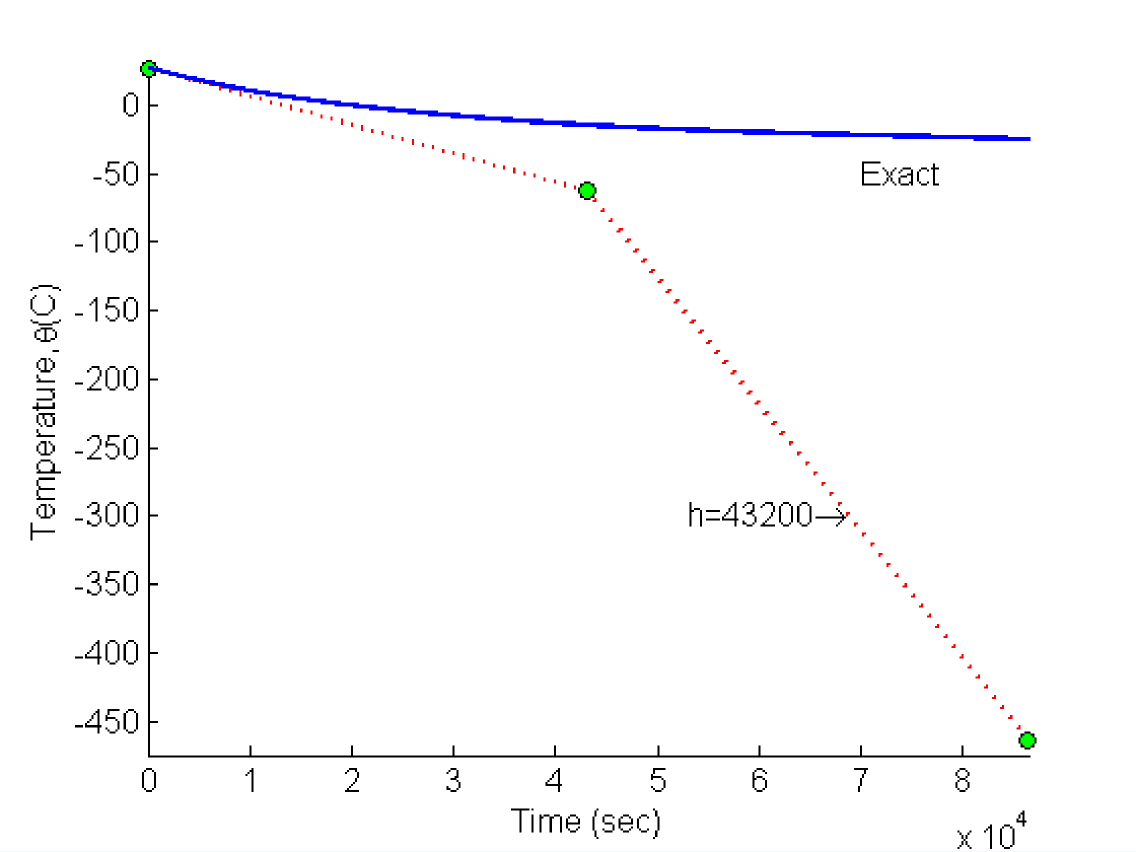

Figure 3. Comparing exact and Euler's method

Effect of step size

Table 1 Temperature at 86400 seconds as a function of step size, h

Step size, h	$\theta(86400)$	E_t	$\lvert\in_t\rvert$ %
86400	−153.52	127.42	488.21
43200	−463.32	437.22	1675.2
21600	−29.542	3.4421	14.189
10800	−27.795	1.6962	6.4988
5400	−26.958	0.85870	3.2902

Comparison with exact results

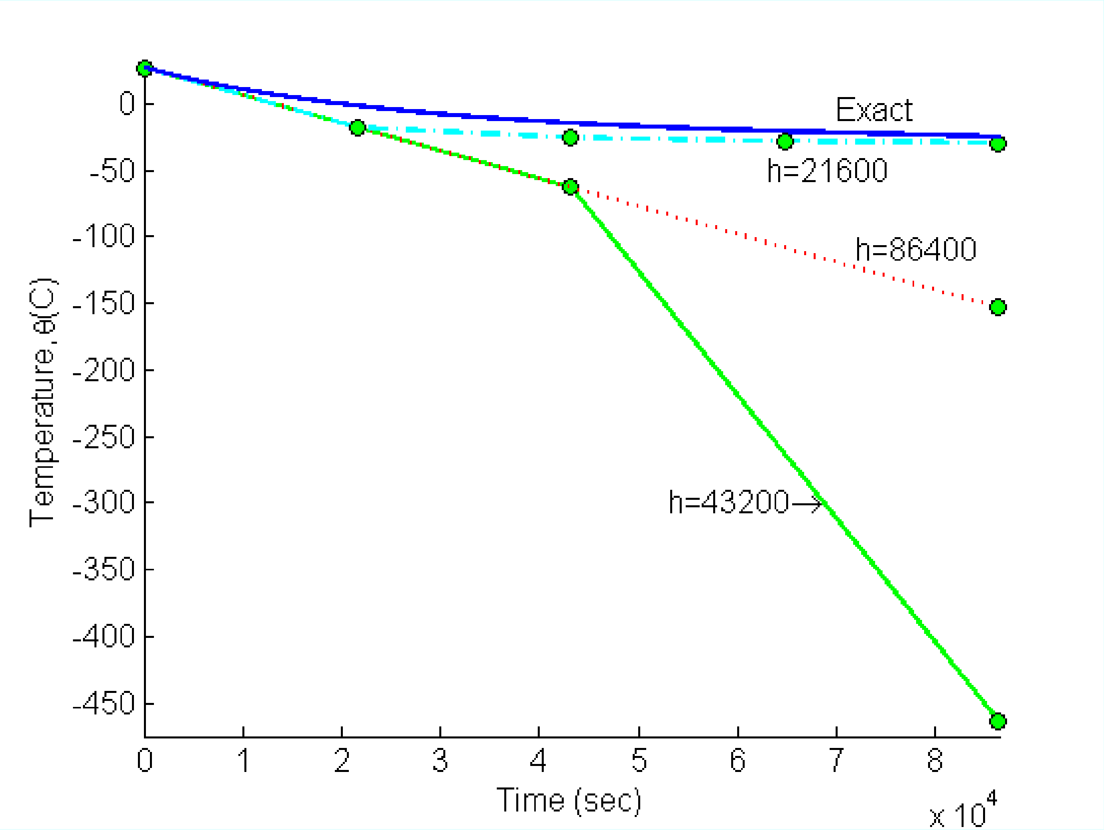

Figure 4 Comparison of Euler's method with exact solution for different step sizes

Effects of step size on Euler's Method

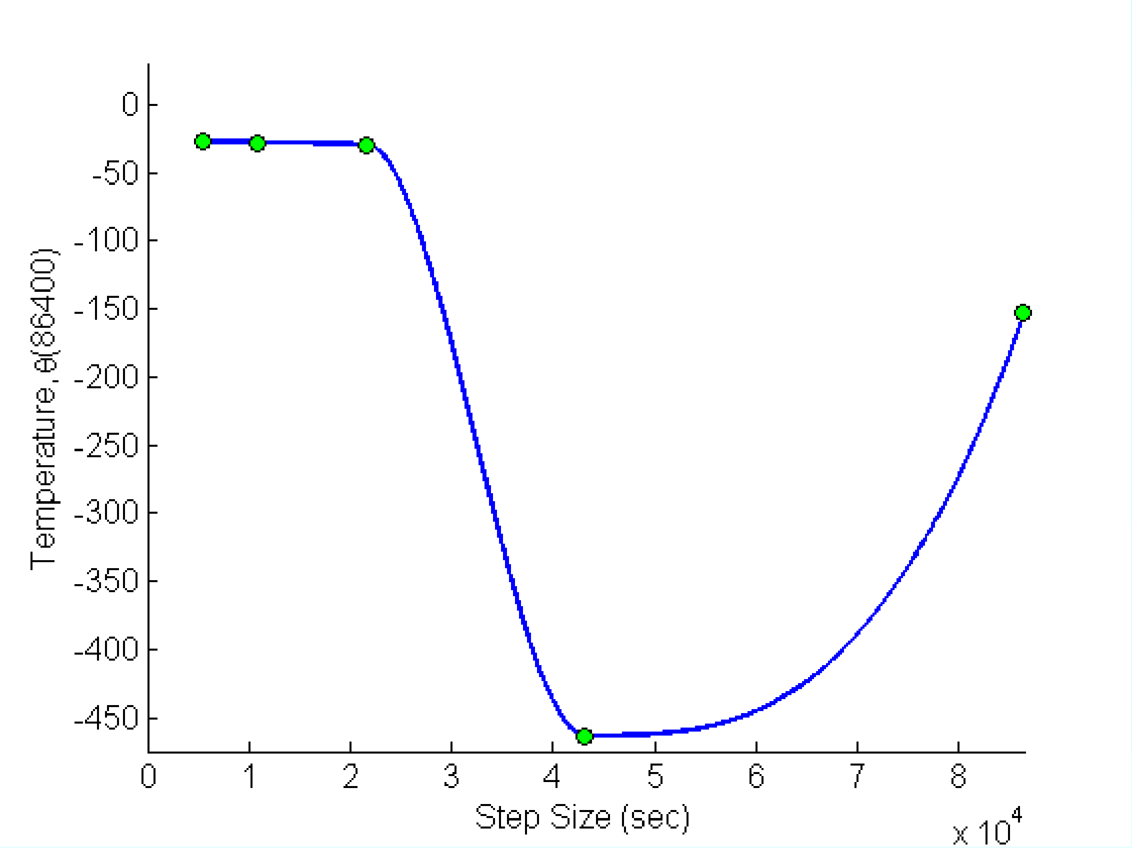

Figure 5. Effect of step size in Euler's method.

Errors in Euler's Method

It can be seen that Euler's method has large errors. This can be illustrated using Taylor series.

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i, y_i} (x_{i+1} - x_i) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_i, y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x_i, y_i} (x_{i+1} - x_i)^3 + \ldots$$

$$y_{i+1} = y_i + f(x_i, y_i)(x_{i+1} - x_i) + \frac{1}{2!} f'(x_i, y_i)(x_{i+1} - x_i)^2 + \frac{1}{3!} f''(x_i, y_i)(x_{i+1} - x_i)^3 + \ldots$$

As you can see the first two terms of the Taylor series

$y_{i+1} = y_i + f(x_i, y_i)h$ are the Euler's method.

The true error in the approximation is given by

$$E_t = \frac{f'(x_i, y_i)}{2!} h^2 + \frac{f''(x_i, y_i)}{3!} h^3 + \ldots \qquad E_t \propto h^2$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/euler_method.html

THE END

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