Runge 4th Order Method

Mechanical Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

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Runge-Kutta 4th Order Method

For $\frac{dy}{dx} = f(x, y), y(0) = y_0$

Runge Kutta 4th order method is given by

$$y_{i+1} = y_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f\left(x_i + h, y_i + k_3 h\right)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, \; y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, \; y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A solid steel shaft at room temperature of 27°C is needed to be contracted so that it can be shrunk-fit into a hollow hub. It is placed in a refrigerated chamber that is maintained at −33°C. The rate of change of temperature of the solid shaft θ is given by

$$\frac{d\theta}{dt} = -5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}\theta^4 + 2.33\times10^{-5}\theta^3 + 1.35\times10^{-3}\theta^2 \\ +5.42\times10^{-2}\theta + 5.588\end{pmatrix}(\theta+33)$$

$$\theta(0) = 27\,^\circ C$$

Find the temperature of the steel shaft after 24 hours. Take a step size of h=43200 seconds.

$$\frac{d\theta}{dt} = -5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}\theta^4 + 2.33\times10^{-5}\theta^3 + 1.35\times10^{-3}\theta^2 \\ +5.42\times10^{-2}\theta + 5.588\end{pmatrix}(\theta+33)$$

$$f(t,\theta) = -5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}\theta^4 + 2.33\times10^{-5}\theta^3 + 1.35\times10^{-3}\theta^2 \\ +5.42\times10^{-2}\theta + 5.588\end{pmatrix}(\theta+33)$$

$$\theta_{i+1} = \theta_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

Solution

Step 1: $i=0,\ t_0=0,\ \theta_0=27°C$

$$k_1 = f\left(t_0,\theta_0\right)= f\left(0,\ 27\right)=\left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}\left(27\right)^4+2.33\times10^{-5}\left(27\right)^3\\+1.35\times10^{-3}\left(27\right)^2+5.42\times10^{-2}\left(27\right)+5.588\end{pmatrix}\left(27+33\right)\right)=-0.0020893$$

$$k_2 = f\left(t_0+\frac{1}{2}h,\theta_0+\frac{1}{2}k_1h\right)= f\left(0+\frac{1}{2}43200,\ 27+\frac{1}{2}\left(-0.0020893\right)43200\right)= f\left(21600,-18.129\right)$$

$$=\left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}\left(-18.129\right)^4+2.33\times10^{-5}\left(-18.129\right)^3\\+1.35\times10^{-3}\left(-18.129\right)^2+5.42\times10^{-2}\left(-18.129\right)+5.588\end{pmatrix}\left(-18.129+33\right)\right)=-0.00035761$$

$$k_3 = f\left(t_0+\frac{1}{2}h,\ \theta_0+\frac{1}{2}k_2h\right)= f\left(0+\frac{1}{2}\left(43200\right),\ 27+\frac{1}{2}\left(-0.00035761\right)43200\right)= f\left(21600,\ 19.276\right)$$

$$=\left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}\left(19.276\right)^4+2.33\times10^{-5}\left(19.276\right)^3\\+1.35\times10^{-3}\left(19.276\right)^2+5.42\times10^{-2}\left(19.276\right)+5.588\end{pmatrix}\left(19.276+33\right)\right)=-0.0018924$$

$$k_4 = f\left(t_0+h,\ \theta_0+k_3h\right)= f\left(0+43200,\ 27+\frac{1}{2}\left(-0.0018924\right)43200\right)= f\left(43200,-54.751\right)$$

$$=\left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}\left(-54.751\right)^4+2.33\times10^{-5}\left(-54.751\right)^3\\+1.35\times10^{-3}\left(-54.751\right)^2+5.42\times10^{-2}\left(-54.751\right)+5.588\end{pmatrix}\left(-54.751+33\right)\right)=-0.0035147$$

Solution Cont

$$\theta_1 = \theta_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 27 + \frac{1}{6}\left(-0.0020893 + 2\left(-0.00035761\right) + 2\left(-0.0018924\right) + \left(-0.0035147\right)\right)43200$$

$$= 27 + \frac{1}{6}\left(-0.010104\right)43200$$

$$= -45.749°C$$

θ_1 is the approximate temperature at

$$t = t_1 = t_0 + h = 0 + 43200 = 43200s$$

$$\theta\left(43200\right) \approx \theta_1 = -45.749°C$$

Solution Cont

Step 2: $i=1, t_1 = 43200, \theta_1 = -45.749$

$$k_1 = f(t_1,\theta_1) = f(43200,\ -45.749) = \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(-45.749)^4+2.33\times10^{-5}(-45.749)^3\\+1.35\times10^{-3}(-45.749)^2+5.42\times10^{-2}(-45.749)+5.588\end{pmatrix}(-45.749+33)\right)$$

$$= -0.00084673$$

$$k_2 = f\left(t_1+\frac{1}{2}h, \theta_1+\frac{1}{2}k_1h\right) = f\left(43200+\frac{1}{2}(43200),\ -45.749+\frac{1}{2}(-0.00084673)43200\right) = f(64800, -64.038)$$

$$= \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(-64.038)^4+2.33\times10^{-5}(-64.038)^3\\+1.35\times10^{-3}(-64.038)^2+5.42\times10^{-2}(-64.038)+5.588\end{pmatrix}(-64.038+33)\right) = -0.010012$$

$$k_3 = f\left(t_1+\frac{1}{2}h, \theta_1+\frac{1}{2}k_2h\right) = f\left(43200+\frac{1}{2}(43200),\ -45.749+\frac{1}{2}(-0.010012)43200\right) = f(64800, -262.01)$$

$$= \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(-262.01)^4+2.33\times10^{-5}(-262.01)^3\\+1.35\times10^{-3}(-262.01)^2+5.42\times10^{-2}(-262.01)+5.588\end{pmatrix}(-22.588+33)\right) = -21.636$$

$$k_4 = f(t_1+h,\ \theta_1+k_3h) = f(43200+43200,\ -45.749+(-21.636)43200) = f(86400, -9.3474\times10^5)$$

$$= \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(-9.3474\times10^5)^4+2.33\times10^{-5}(-9.3474\times10^5)^3\\+1.35\times10^{-3}(-9.3474\times10^5)^2+5.42\times10^{-2}(-9.3474\times10^5)+5.588\end{pmatrix}(-9.3474\times10^5+33)\right) = -1.4035\times10^{19}$$

Solution Cont

$$\theta_2 = \theta_1 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

$$= -45.749 + \frac{1}{6}\left(-0.00084673 + 2\left(-0.010012\right) + 2\left(-21.636\right) + \left(-1.4035\times10^{19}\right)\right)43200$$

$$= -45.749 + \frac{1}{6}\left(-1.4035\times10^{19}\right)43200$$

$$= -1.0105\times10^{23}\,{}^{\circ}C$$

θ_2 is the approximate temperature at

$$t_2 = t_1 + h = 43200 + 43200 = 86400$$

$$\theta(86400) \approx \theta_2 = -1.1015\times10^{23}\,{}^{\circ}C$$

Solution Cont

The solution to this nonlinear equation at *t=86400s* is

$$\theta(86400) = -26.099°C$$

Comparison with exact results

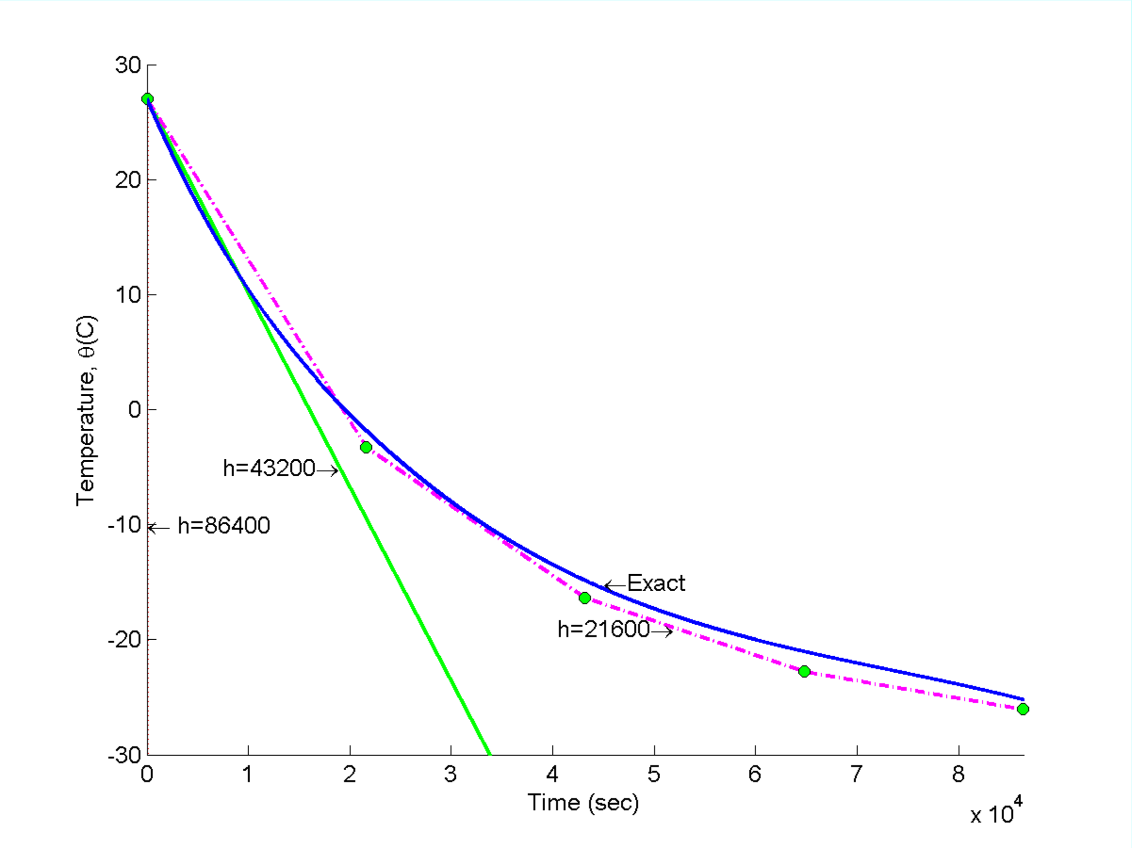

Figure 1. Comparison of Runge-Kutta 4th order method with exact solution

Effect of step size

Table 1. Value of temperature at 86400 seconds for different step sizes

Step Size, h	$\theta(86400)$	E_t	$\lvert\in_t\rvert\,\%$
86400	$-5.3468\quad 10^{28}$	$-5.2468\quad 10^{28}$	$2.0487\quad 10^{29}$
43200	$-1.0105\quad 10^{23}$	$-0.034808\quad 10^{23}$	$3.8718\quad 10^{23}$
21600	-26.061	-0.038680	0.14820
10800	-26.094	-0.0050630	0.019400
5400	-26.097	-0.0015763	0.0060402

$\theta(86400) = -26.099^{\circ}C$ (exact)

Effects of step size on Runge-Kutta 4th Order Method

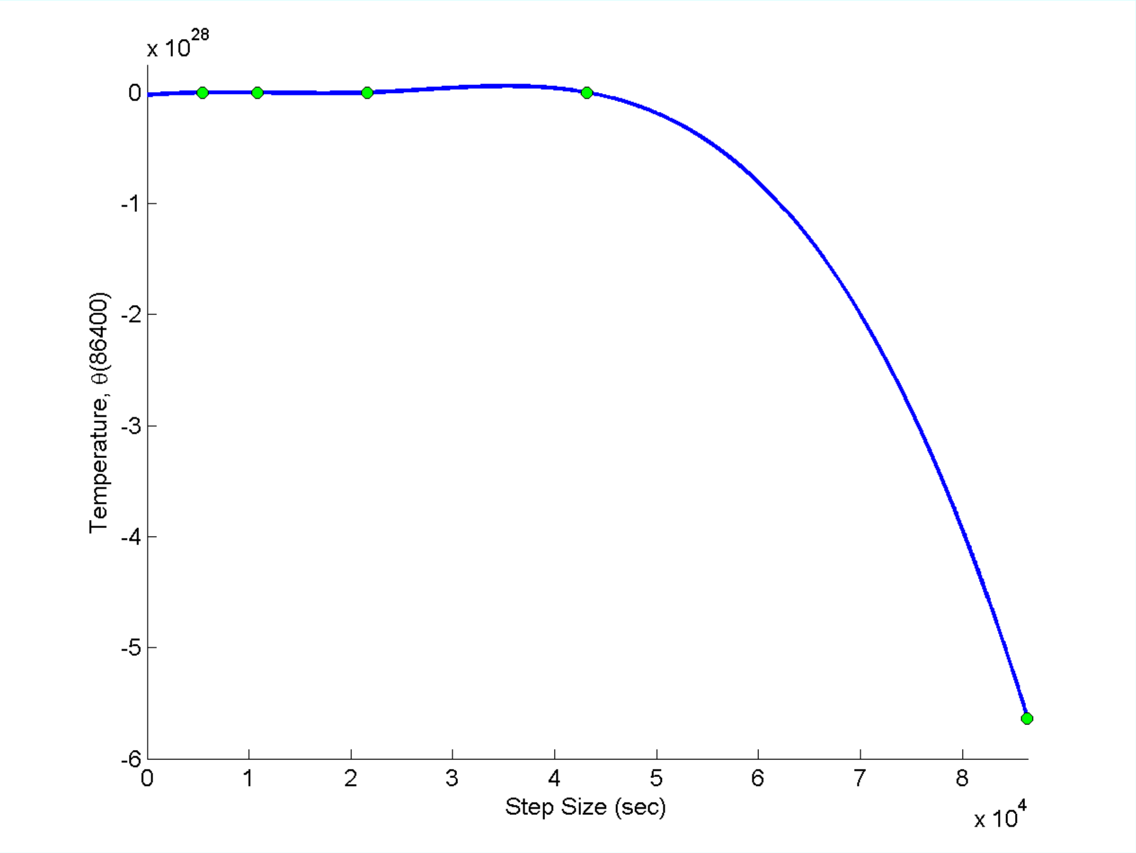

Figure 2. Effect of step size in Runge-Kutta 4th order method

Comparison of Euler and Runge-Kutta Methods

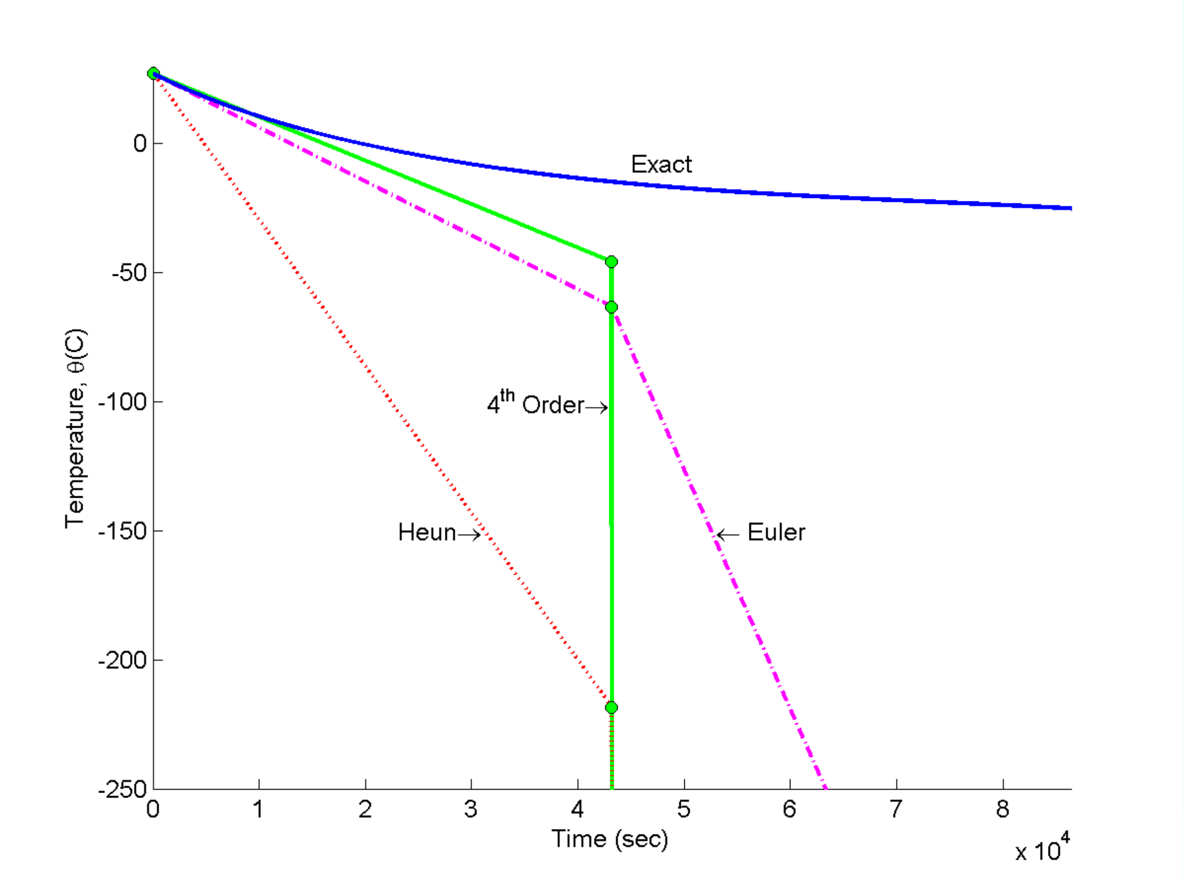

Figure 3. Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/runge_kutta_4th_method.html

THE END

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