Concepts of Approximate Error: Absolute Approximate Error, Relative Approximate Error, and Absolute Relative Approximate Error

Sean Rodby, Luke Snyder, Autar Kaw
University of South Florida
United States of America
kaw@eng.usf.edu
Website: http://numericalmethods.eng.usf.edu
Version: Mathcad 14; Last Revised: August 28, 2008

Introduction

The following worksheet demonstrates how to calculate different definitions related to approximate error, such as approximate error, absolute approximate error, relative approximate error, absolute relative approximate error, and the least number of significant digits associated with these calculations. The concept is demonstrated using an example of a Maclaurin series. The user will choose which function to perform the calculation for in the *Input* section of the program. The choices are given as 1 for e^x, 2 for *sin(x)*, and 3 for *cos(x)*. The *true value* of these functions will be assumed as given by the Mathcad commands for these functions.

Section 1: Input

This is the only section where the user interacts with the program.

- Pick the function of your desire by choosing an integer: 1 for e^x; 2 for *sin(x)*; 3 for *cos(x)*

      funcchoice := 1

- Maximum number of terms used in the Maclaurin series

      n := 15

- Value of x at which the function is calculated

      xv := 1.6

Section 2: Procedure

First, determine which function will be used in the calculations, based on the user input. Once the function is determined, the value is calculated using a Maclaurin series in a repetitive loop.

$$f(x) := \begin{array}{|l} y \leftarrow e^x \ \ \text{if} \ \ \text{funcchoice} = 1 \\ y \leftarrow \sin(x) \ \ \text{if} \ \ \text{funcchoice} = 2 \\ y \leftarrow \cos(x) \ \ \text{if} \ \ \text{funcchoice} = 3 \\ \text{return} \ y \end{array}$$

$$\text{sumprevious} := 0$$

$$\text{sumpresent}(n) := \begin{array}{|l} \text{for} \ i \in 1 .. n \\ \qquad \begin{array}{|l} Y \leftarrow \text{sumprevious} + \dfrac{xv^{i-1}}{(i-1)!} \ \ \text{if} \ \ \text{funcchoice} = 1 \\[2ex] Y \leftarrow \text{sumprevious} + \dfrac{(-1)^{i-1} xv^{2 \cdot i - 1}}{(2 \cdot i - 1)!} \ \ \text{if} \ \ \text{funcchoice} = 2 \\[2ex] Y \leftarrow \text{sumprevious} + \dfrac{(-1)^{i+1} xv^{2 \cdot i - 2}}{(2 \cdot i - 1)!} \ \ \text{if} \ \ \text{funcchoice} = 3 \\[2ex] \text{sumprevious} \leftarrow Y \end{array} \\ Y \end{array}$$

Section 3: Calculation

This loop here calculates the following
N = number of terms used
A = approximate value
E_a = approximate error
$|E_a|$ = absolute approximate error
ε_a = percentage relative approximate error
$|\varepsilon_a|$ = percentage absolute relative approximate error
n_{sig} = least number of significant digits correct in an approximation

$$\text{table1} := \text{ for } i \in 0..n$$
$$\left|
\begin{array}{l}
N_i \leftarrow i + 1 \\[2mm]
A_i \leftarrow \text{sumpresent}(i + 1) \\[2mm]
\text{if } i > 0 \\[2mm]
\quad \left|
\begin{array}{l}
E_{a_i} \leftarrow A_i - \text{sumpresent}(i) \\[2mm]
E_{aa_i} \leftarrow \left| A_i - \text{sumpresent}(i) \right|
\end{array}
\right. \\[4mm]
\text{augment}\left(N, A, E_a, E_{aa}\right)
\end{array}
\right.$$

$\text{table2} := $ for $i \in 0..n$

$\quad \left| \begin{array}{l} N_i \leftarrow i + 1 \\[4pt] A_i \leftarrow \text{sumpresent}(i + 1) \\[4pt] \text{if } i > 0 \\[4pt] \quad \left| \begin{array}{l} E_{a_i} \leftarrow A_i - \text{sumpresent}(i) \\[10pt] \varepsilon_{a_i} \leftarrow \dfrac{E_{a_i}}{A_i} \cdot 100 \\[14pt] \varepsilon_{aa_i} \leftarrow \left| \dfrac{E_{a_i}}{A_i} \right| \cdot 100 \\[16pt] \text{sig}_i \leftarrow \text{floor}\left(2 - \log\left(\dfrac{\varepsilon_{aa_i}}{100 \cdot 0.5} \right) \right) \\[16pt] n_{\text{sig}_i} \leftarrow \text{sig}_i \quad \text{if } \text{sig}_{i>0} \\[8pt] n_{\text{sig}_i} \leftarrow 0 \quad \text{otherwise} \end{array} \right. \\[8pt] \text{augment}\left(N, A, \varepsilon_a, \varepsilon_{aa}, n_{\text{sig}} \right) \end{array} \right.$

Section 4: Table of Values

$$table1 =$$

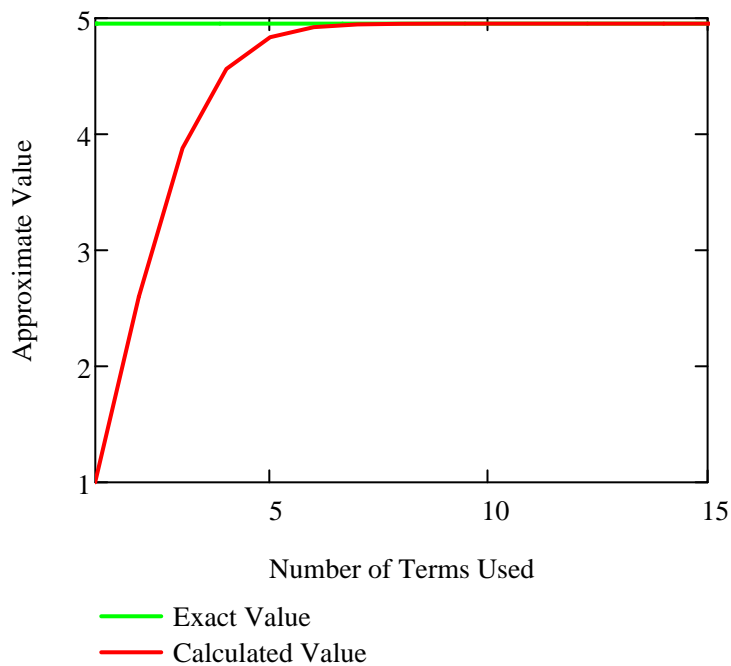|	N	A	E_a	$\left|E_a\right|$
	0	1	2	3
0	1	1	0	0
1	2	2.6	1.6	1.6
2	3	3.88	1.28	1.28
3	4	4.563	0.683	0.683
4	5	4.836	0.273	0.273
5	6	4.923	0.087	0.087
6	7	4.946	0.023	0.023
7	8	4.952	$5.326 \cdot 10^{-3}$	$5.326 \cdot 10^{-3}$
8	9	4.953	$1.065 \cdot 10^{-3}$	$1.065 \cdot 10^{-3}$
9	10	4.953	$1.894 \cdot 10^{-4}$	$1.894 \cdot 10^{-4}$
10	11	4.953	$3.03 \cdot 10^{-5}$	$3.03 \cdot 10^{-5}$
11	12	4.953	$4.407 \cdot 10^{-6}$	$4.407 \cdot 10^{-6}$
12	13	4.953	$5.876 \cdot 10^{-7}$	$5.876 \cdot 10^{-7}$
13	14	4.953	$7.232 \cdot 10^{-8}$	$7.232 \cdot 10^{-8}$
14	15	4.953	$8.266 \cdot 10^{-9}$	$8.266 \cdot 10^{-9}$
15	16	4.953	$8.817 \cdot 10^{-10}$	$8.817 \cdot 10^{-10}$

$$table2 =$$

|	N	A	ε_a	$\left|\varepsilon_a\right|$	n_{sig}
	0	1	2	3	4
0	1	1	0	0	0
1	2	2.6	61.538	61.538	1
2	3	3.88	32.99	32.99	2
3	4	4.563	14.962	14.962	2
4	5	4.836	5.647	5.647	2
5	6	4.923	1.775	1.775	3
6	7	4.946	0.471	0.471	4
7	8	4.952	0.108	0.108	4
8	9	4.953	0.022	0.022	5
9	10	4.953	$3.823 \cdot 10^{-3}$	$3.823 \cdot 10^{-3}$	6
10	11	4.953	$6.117 \cdot 10^{-4}$	$6.117 \cdot 10^{-4}$	6
11	12	4.953	$8.898 \cdot 10^{-5}$	$8.898 \cdot 10^{-5}$	7
12	13	4.953	$1.186 \cdot 10^{-5}$	$1.186 \cdot 10^{-5}$	8
13	14	4.953	$1.46 \cdot 10^{-6}$	$1.46 \cdot 10^{-6}$	9
14	15	4.953	$1.669 \cdot 10^{-7}$	$1.669 \cdot 10^{-7}$	10
15	16	4.953	$1.78 \cdot 10^{-8}$	$1.78 \cdot 10^{-8}$	11

The following graphs show the calculated value of f(x) using Maclaurin series as a function, approximate error, absolute approximate error, absolute relative approximate error, relative approximate error, and least number of significant digits as a function of step size. Each graph displays the error results of each of the methods of approximation.
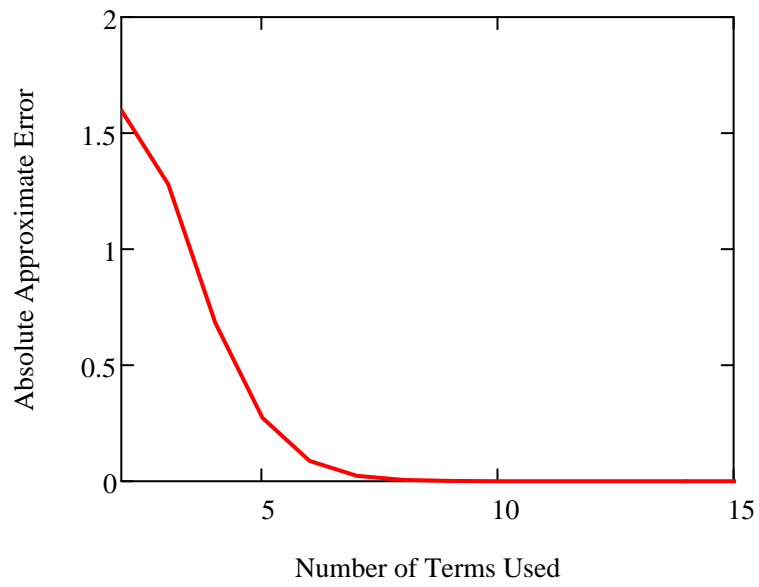
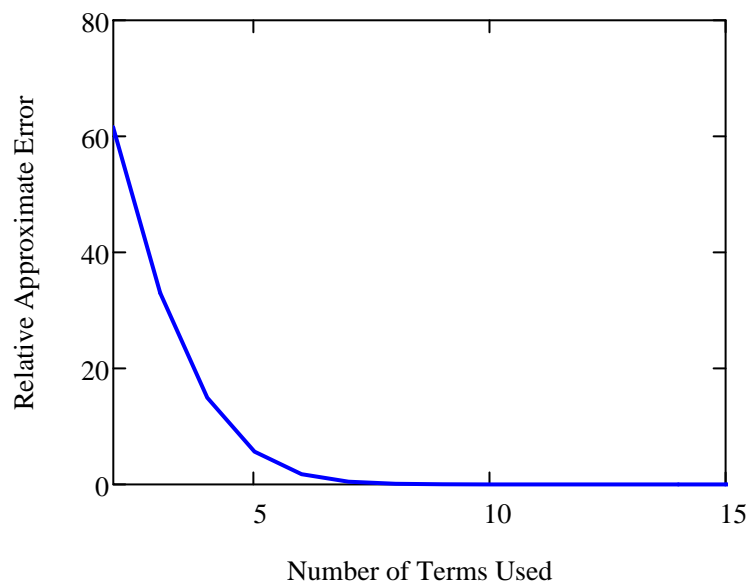Calculated Value of f(x) Using Maclaurin Series vs. Number of Terms
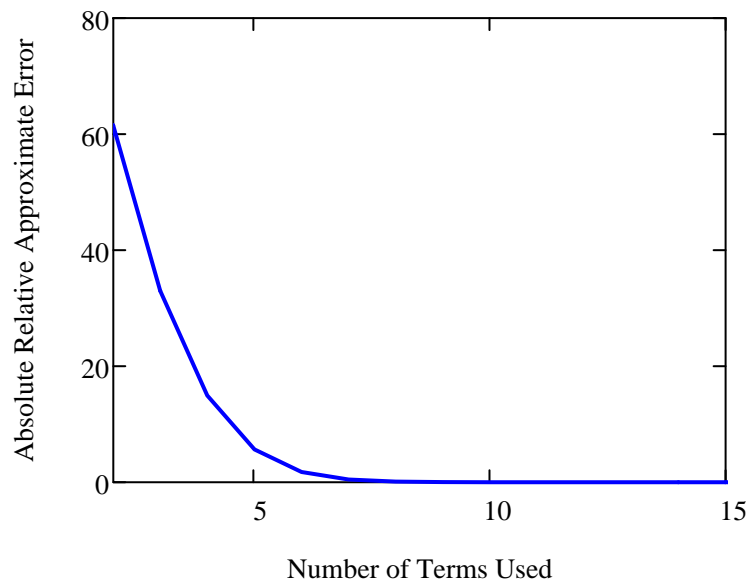

Approximate Error vs. Number of Terms

6
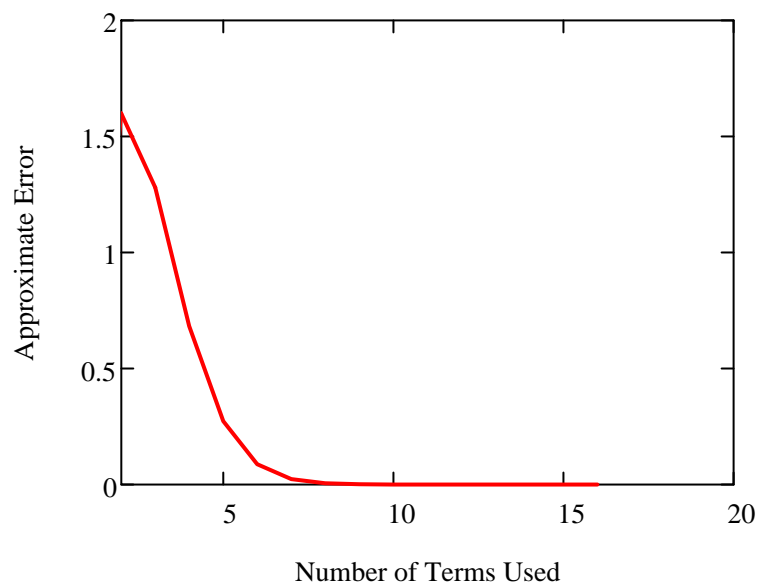
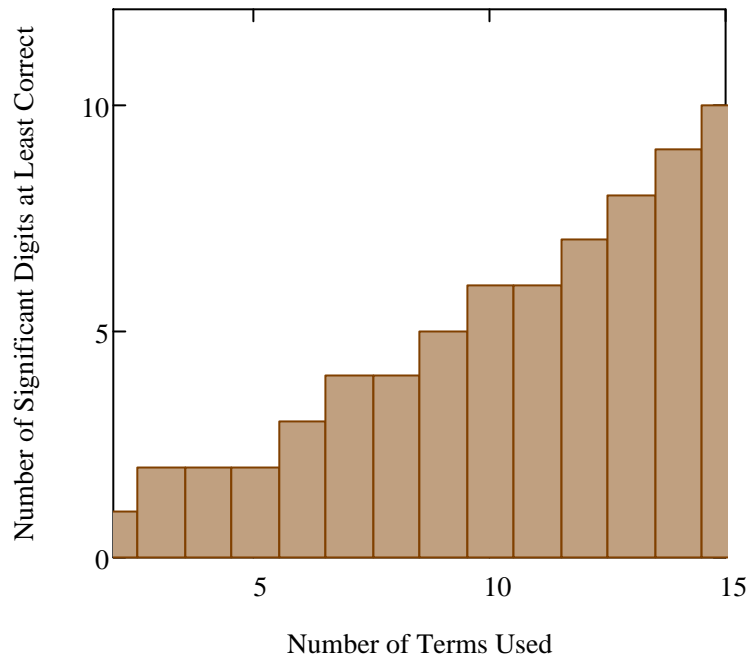## Absolute Approximate Error vs. Number of Terms

Relative Approximate Error vs. Number of Terms

Absolute Relative Approximate Error vs. Number of Terms

Number of Significant Digits at Least Correct vs. Number of Terms

Conclusion

This worksheet shows how the number of terms taken in a Maclaurin series affects the accuracy of the calculated answer through the analysis of error. Note that though approximate error shows the magnitude of the error, it does not indicate how bad the error really is. Hence, relative approximate error is used here to give a more complete picture of the state of error.

References

Measuring Errors.
See:
http://numericalmethods.eng.usf.edu/mcd/gen/01aae/mcd_gen_aae_txt_measuringerror.pdf

Legal Notice: The copyright for this application is owned by the author(s). Neither PTC nor the author(s) are responsible for any errors contained within and are not liable for any damages resulting from the use of this material. This application is intended for non-commercial, non-profit use only. Contact the author for permission if you wish to use this application in for-profit activities.