CONCEPTS OF ERROR

The Quadratic Formula as a Way to Show the Subtraction of Small Numbers

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Introduction

The following worksheet illustrates the use of a quadratic equation solution for showing the effect of significant digits on round-off errors. The user will enter the *a*, *b* and *c* values as given by the equation for the standard form of a quadratic equation: $ax^2 + bx + c = 0$, as well as the number of significant digits to be displayed in a table that will be created at the end of the program. Two variations of the quadratic equation solution will be used :

(A)
$$x1 = \frac{-b + \sqrt{b^2 - 4a \cdot c}}{2a}$$
$$x2 = \frac{-b - \sqrt{b^2 - 4a \cdot c}}{2a}$$

(B)
$$x1 = \frac{2c}{-b - \sqrt{b^2 - 4a \cdot c}}$$
$$x2 = \frac{2c}{-b + \sqrt{b^2 - 4a \cdot c}}$$

Section 1: Input

This is the only section where the user interacts with the program.

The quadratic formula is derived from the standard form of a quadratic equation: $ax^2 + bx + c = 0$.

- Enter coefficient *a*

  $a := 0.001$

- Enter coefficient *b*

  $b := -4.946268$

- Enter coefficient *c*

  $c := 0.002$

- Enter range of significant digits to be used.

  sig_low := 7

  sig_high := 10

Section 2: Significant Digit Arithmetic Functions

The following functions modify standard arithmetic operators allowing computation with the appropriate number of significant digits.
NOTE: To ensure proper representation of significant digits, please insert the number of decimal digits displayed as a number greater than the range you are inputting by going to toolbar ->Format -> Result -> Number Format tab.

$$sdscale(sd,x) := \left| \begin{array}{l} \text{return } 0 \text{ if } x = 0 \\ n \leftarrow sd - (floor(log(|x|)) + 1) \\ x \leftarrow x \cdot 10^{n} \\ round(x) \cdot 10^{-n} \end{array} \right.$$

$add(a,b) := a + b$

$mul(a,b) := a \cdot b$

$sub(a,b) := a - b$

$div(a,b) := a \div b$

$$sd\_dyadic(op,sd,x,y) := \left| \begin{array}{l} z \leftarrow op(sdscale(sd,x),sdscale(sd,y)) \\ sdscale(sd,z) \end{array} \right.$$

$sdadd(sd,x,y) := sd\_dyadic(add,sd,x,y)$

$sdsub(sd,x,y) := sd\_dyadic(sub,sd,x,y)$

$sdmul(sd,x,y) := sd\_dyadic(mul,sd,x,y)$

$sddiv(sd,x,y) := sd\_dyadic(div,sd,x,y)$

Section 3: Calculation

The following calculations will be performed inside a loop so that the number of significant digits used can be varied as specified by the user.

Variation 1:

$$x1a = \frac{-b + \sqrt{b^2 - 4a \cdot c}}{2a}$$

$$x2a = \frac{-b - \sqrt{b^2 - 4a \cdot c}}{2a}$$

$$x1a(dig) := \left| \begin{array}{l} ROOT \leftarrow sdsub(dig, sdmul(dig, b, b), sdmul(dig, 4a, c)) \\ TOP \leftarrow sdadd\left(dig, -b, \sqrt{ROOT}\right) \\ FIN \leftarrow sddiv(dig, TOP, 2 \cdot a) \\ \text{return } FIN \end{array} \right.$$

$$x2a(dig) := \left| \begin{array}{l} ROOT \leftarrow sdsub(dig, sdmul(dig, b, b), sdmul(dig, 4a, c)) \\ TOP \leftarrow sdsub\left(dig, -b, \sqrt{ROOT}\right) \\ FIN \leftarrow sddiv(dig, TOP, 2 \cdot a) \\ \text{return } FIN \end{array} \right.$$

Variation 2:

$$x1b = \frac{2c}{-b - \sqrt{b^2 - 4a \cdot c}}$$

$$x2b = \frac{2c}{-b + \sqrt{b^2 - 4a \cdot c}}$$

$$x1b(dig) := \left| \begin{array}{l} ROOT \leftarrow sdsub(dig, sdmul(dig, b, b), sdmul(dig, 4a, c)) \\ BOTT \leftarrow sdsub\left(dig, -b, \sqrt{ROOT}\right) \\ FIN \leftarrow sddiv(dig, 2c, BOTT) \;\; \text{if } BOTT \neq 0 \\ FIN \leftarrow undefined \;\; \text{otherwise} \\ \text{return } FIN \end{array} \right.$$

$$x2b(dig) := \left| \begin{array}{l} ROOT \leftarrow sdsub(dig, sdmul(dig, b, b), sdmul(dig, 4a, c)) \\ BOTT \leftarrow sdadd\left(dig, -b, \sqrt{ROOT}\right) \\ FIN \leftarrow sddiv(dig, 2c, BOTT) \;\; \text{if } BOTT \neq 0 \\ FIN \leftarrow undefined \;\; \text{otherwise} \\ \text{return } FIN \end{array} \right.$$

$$table := \text{for } i \in 0 .. sig\_high - sig\_low$$

$$\left| \begin{array}{l} k_i \leftarrow sig\_low + i \\ X1a_i \leftarrow x1a\left(k_i\right) \\ X2a_i \leftarrow x2a\left(k_i\right) \\ X1b_i \leftarrow x1b\left(k_i\right) \\ X2b_i \leftarrow x2b\left(k_i\right) \\ augment(k, X1a, X1b, X2a, X2b) \end{array} \right.$$

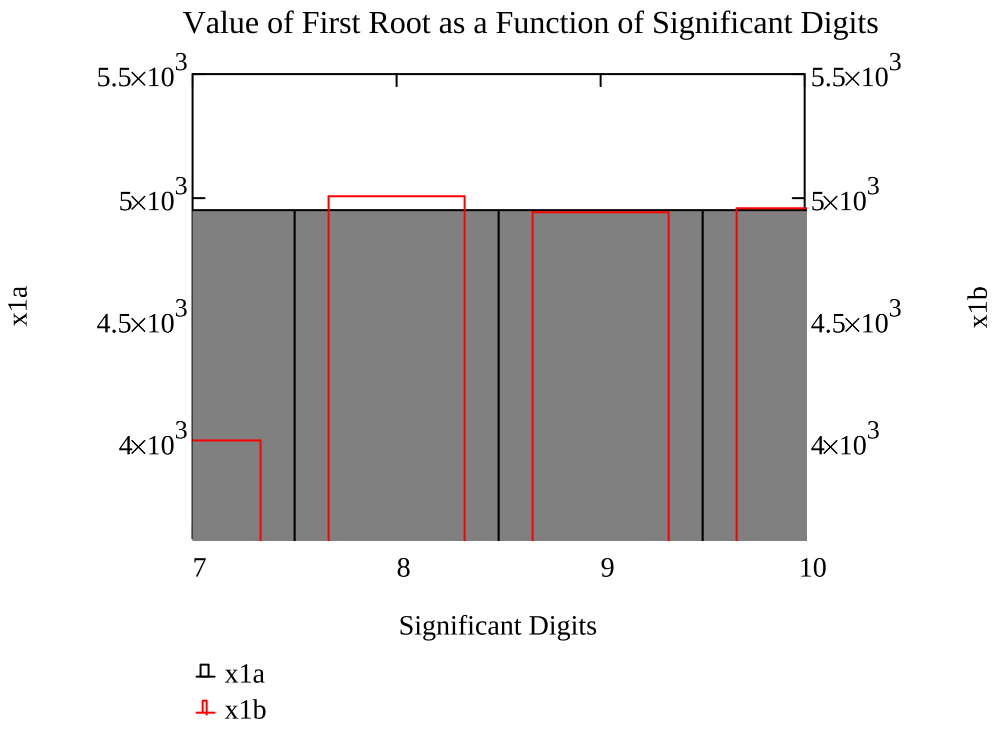
Section 4: Table of Values

table =

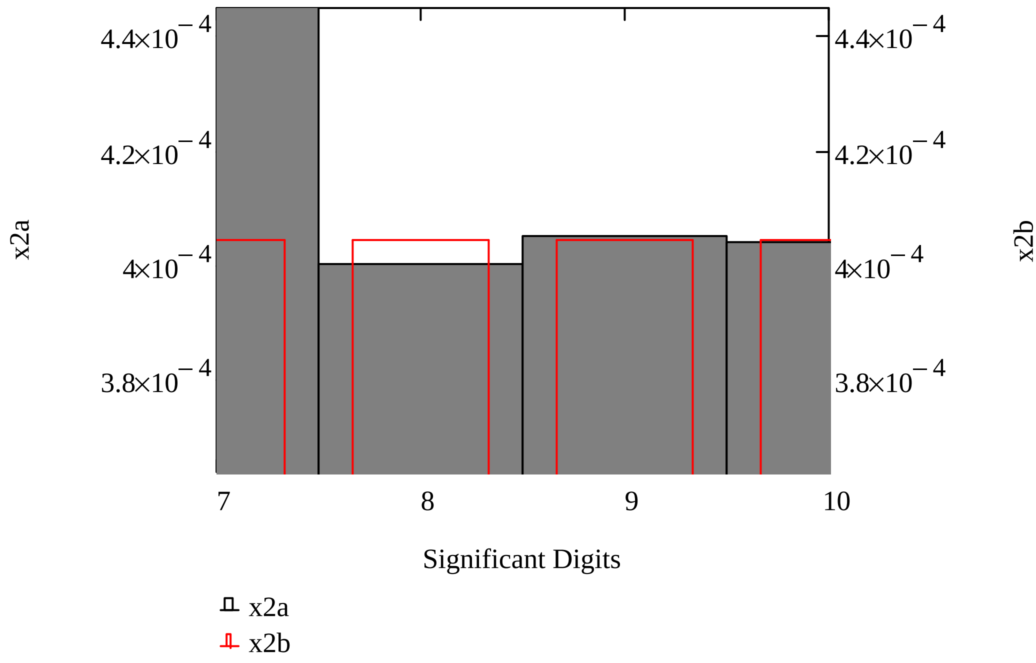
	Digits	x1a	x1b	x2a	x2b
	0	1	2	3	4
0	7	$4.946 \cdot 10^3$	$4 \cdot 10^3$	$5 \cdot 10^{-4}$	$4.043 \cdot 10^{-4}$
1	8	$4.946 \cdot 10^3$	$5 \cdot 10^3$	$4 \cdot 10^{-4}$	$4.043 \cdot 10^{-4}$
2	9	$4.946 \cdot 10^3$	$4.938 \cdot 10^3$	$4.05 \cdot 10^{-4}$	$4.043 \cdot 10^{-4}$
3	10	$4.946 \cdot 10^3$	$4.95 \cdot 10^3$	$4.04 \cdot 10^{-4}$	$4.043 \cdot 10^{-4}$

Section 5: Graphs

These bar graphs will show the values of *x*1 and *x*2 for both variations of the quadratic function.



Value of Second Root as a Function of Significant Digits



Conclusion

Subtraction of numbers that are nearly equal can result in unwanted inaccuracies. The number of significant digits used in calculations plays a large role in the creation of these inaccuracies and the magnitude of the round-off errors. Hence, when the accuracy of calculations is critical, it is necessary to understand possible sources of error and how they are best avoided.

References

Sources of Error. See:
http://numericalmethods.eng.usf.edu/mcd/gen/01aae/mcd_gen_aae_txt_sourcesoferror.pdf
Propagation of Errors. See:
http://numericalmethods.eng.usf.edu/mcd/gen/01aae/mcd_gen_aae_txt_propagationoferrors.pdf