

Backward Difference Approximation of the First Derivative

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Introduction

This worksheet demonstrates the use of Mathcad to illustrate Backward Difference Approximation of the first derivative of continuous functions.

Backward Difference Approximation of the first derivative uses a point h behind of the given value of x at which the derivative of $f(x)$ is to be found.

$$f'(x) \approx \frac{f(x) - f(x - h)}{h}$$

Section 1: Input

The following simulation approximates the first derivative of a function using Backward Difference Approximation. The user inputs are

- function, $f(x)$
- point at which the derivative is to be found, xv
- starting step size, h
- number of times user wants to halve the starting step size, n

The outputs include

- approximate value of the derivative at the point and given initial step size
- exact value
- true error, absolute relative true error, approximate error and absolute relative approximate error, least number of correct significant digits in the solution as a function of step size.

Function $f(x)$

$$f(x) := \exp(2 \cdot x)$$

Value of x at which $f'(x)$ is desired, xv

$$xv := 4.0$$

Starting step size, h

$$h := 0.2$$

Number of times starting step size is halved

n := 12

This is the end of the user section. All the information must be entered before proceeding to the next section.

Section 2: Procedure

The following procedure estimates the solution of first derivate of an equation at a point xv .

$f(x)$ = function

xv = value at which the solution is desired

h = starting step size value

n = number of times starting step size is halved

$$\text{BDD}(f, xv, h) := \begin{array}{l} \text{deriv} \leftarrow \frac{(f(xv) - f(xv - h))}{h} \\ \text{deriv} \end{array}$$

Section 3: Calculation

The exact value EV of the first derivative of the equation:

Given the function

$$f(x) \rightarrow e^{2 \cdot x}$$

First, using the derivative command the solution is found. In the second step, the exact value of the derivative is shown

The solution of the first derivative is

$$f'(x) := \left(\frac{d}{dx} f(x) \right) \quad f'(x) \rightarrow 2 \cdot e^{2 \cdot x}$$

The exact solution of the first derivative is

$$EV := f'(xv)$$

$$EV = 5961.915974$$

The next loop calculates the following:

Av: Approximate value of the first derivative using Backward Difference Approximation by calling the procedure "BDD"

Ev: Exact value of the first derivative

Et: True error

ϵ_t : Absolute relative true percentage error

Ea: Approximate error

ϵ_a : Absolute relative approximate percentage error

n_{sig} : Least number of correct significant digits in an approximation

table1 := for i ∈ 0 .. n - 1

$$\left| \begin{array}{l} N_i \leftarrow 2^i \\ H_i \leftarrow \frac{h}{N_i} \\ AV_i \leftarrow BDD(f, xv, H_i) \\ E_{t_i} \leftarrow EV - AV_i \\ \epsilon_{t_i} \leftarrow \left| \frac{E_{t_i}}{EV} \cdot 100 \right| \\ \text{augment}(H, AV, E_t, \epsilon_t) \end{array} \right.$$

table2 := for $i \in 0..n - 1$

$$\begin{array}{l}
 N_i \leftarrow 2^i \\
 H_i \leftarrow \frac{h}{N_i} \\
 AV_i \leftarrow \text{BDD}(f, xv, H_i) \\
 E_{t_i} \leftarrow EV - AV_i \\
 \varepsilon_{t_i} \leftarrow \left| \frac{E_{t_i}}{EV} \cdot 100 \right| \\
 \text{if } i > 0 \\
 \quad \left| \begin{array}{l}
 E_{a_i} \leftarrow AV_i - AV_{i-1} \\
 \varepsilon_{a_i} \leftarrow \left| \frac{E_{a_i}}{AV_i} \right| \cdot 100 \\
 n_{\text{sig}_i} \leftarrow \text{floor} \left(\left(2 - \log \left(\frac{\varepsilon_{a_i}}{0.5} \right) \right) \right) \text{ if } 0 < \varepsilon_{a_i} < 5 \\
 n_{\text{sig}_i} \leftarrow 0 \text{ otherwise}
 \end{array} \right. \\
 \text{augment}(H, AV, E_a, \varepsilon_a, n_{\text{sig}})
 \end{array}$$

The loop halves the value of the starting step size n times. Each time, the approximate value of the derivative is calculated and saved in a vector. The approximate error is calculated after at least two approximate values of the derivative have been saved. The number of significant digits is calculated. If the number of significant digits calculated is less than zero, it is shown as zero.

Section 4: Table of Values

The next tables show the step size value, approximate value, true error, the absolute relative true percentage error, the approximate error, the absolute relative approximate percentage error and the least number of correct significant digits in an approximation as a function of the step size value.

table1 =

	H	AV	E _t	ε _t
	0	1	2	3
0	0.2	4913.81	1048.11	17.58
1	0.1	5403.56	558.36	9.37
2	0.05	5673.51	288.4	4.84
3	0.03	5815.32	146.59	2.46
4	0.01	5888.01	73.91	1.24
5	0.01	5924.81	37.11	0.62
6	0	5943.32	18.59	0.31
7	0	5952.61	9.31	0.16
8	$7.81 \cdot 10^{-4}$	5957.26	4.66	0.08
9	$3.91 \cdot 10^{-4}$	5959.59	2.33	0.04
10	$1.95 \cdot 10^{-4}$	5960.75	1.16	0.02
11	$9.77 \cdot 10^{-5}$	5961.33	0.58	0.01

table2 =

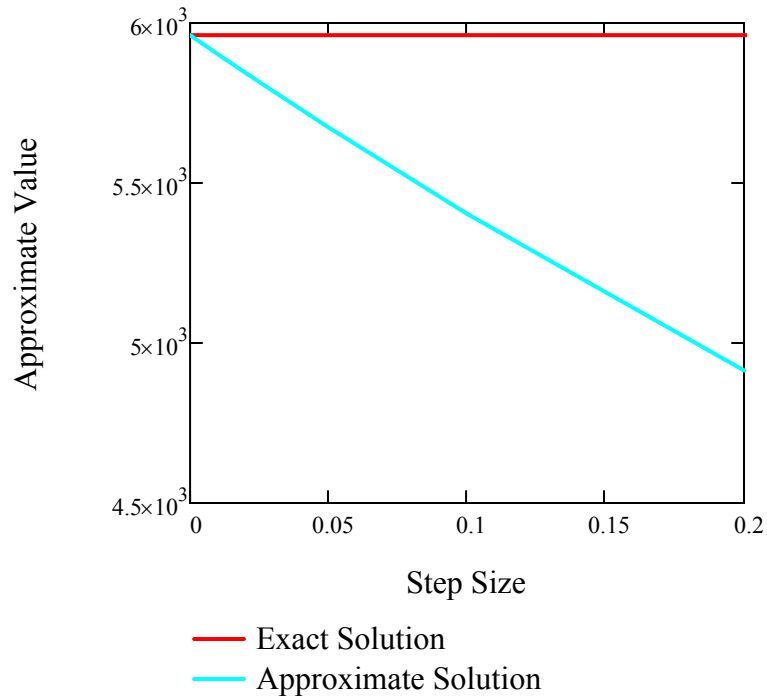
	H	AV	E _a	ε _a	n _{sig}
	0	1	2	3	4
0	0.2	4913.81	0	0	0
1	0.1	5403.56	489.75	9.06	0
2	0.05	5673.51	269.95	4.76	1
3	0.03	5815.32	141.81	2.44	1
4	0.01	5888.01	72.69	1.23	1
5	0.01	5924.81	36.8	0.62	1
6	0	5943.32	18.51	0.31	2
7	0	5952.61	9.29	0.16	2
8	$7.81 \cdot 10^{-4}$	5957.26	4.65	0.08	2
9	$3.91 \cdot 10^{-4}$	5959.59	2.33	0.04	3
10	$1.95 \cdot 10^{-4}$	5960.75	1.16	0.02	3
11	$9.77 \cdot 10^{-5}$	5961.33	0.58	0.01	3

Section 5: Graphs

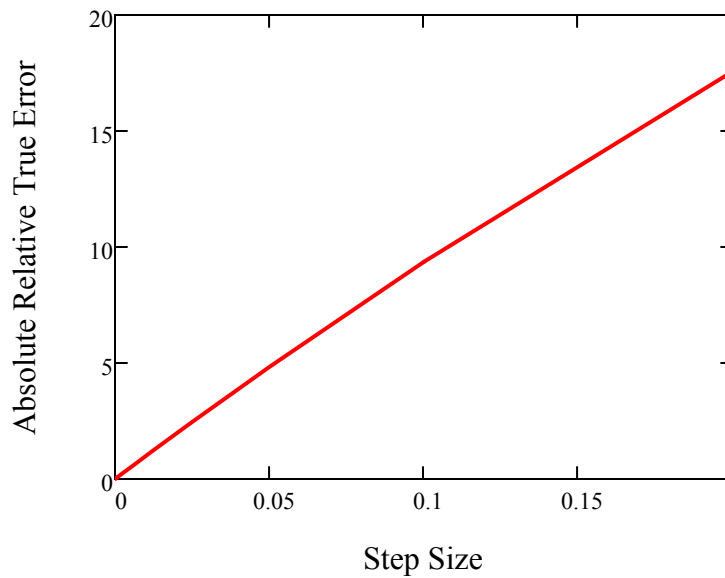
The following graphs show the approximate solution, absolute relative true error, absolute relative approximate error and least number of significant digits as a function of step size.

$x := 0 \dots h + 1$

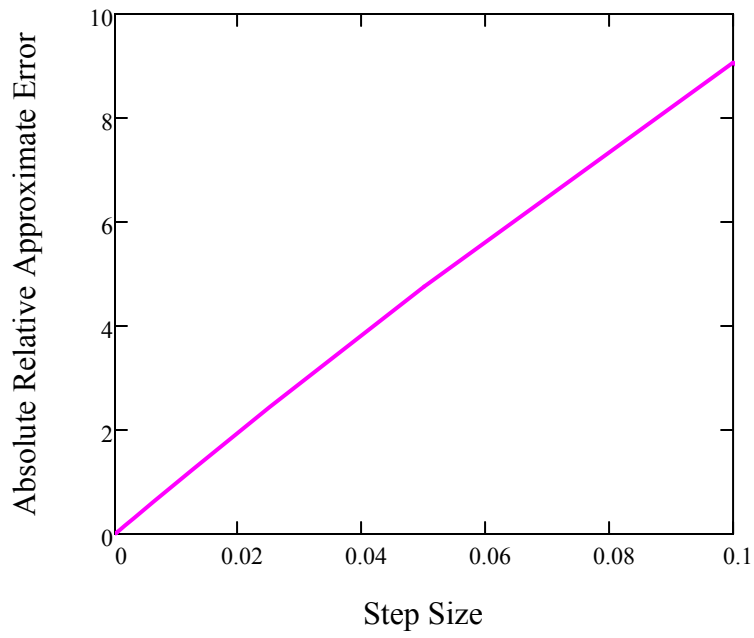
Approximate Solution of the First Derivative using Backward Difference Approximation as a Function of Step Size



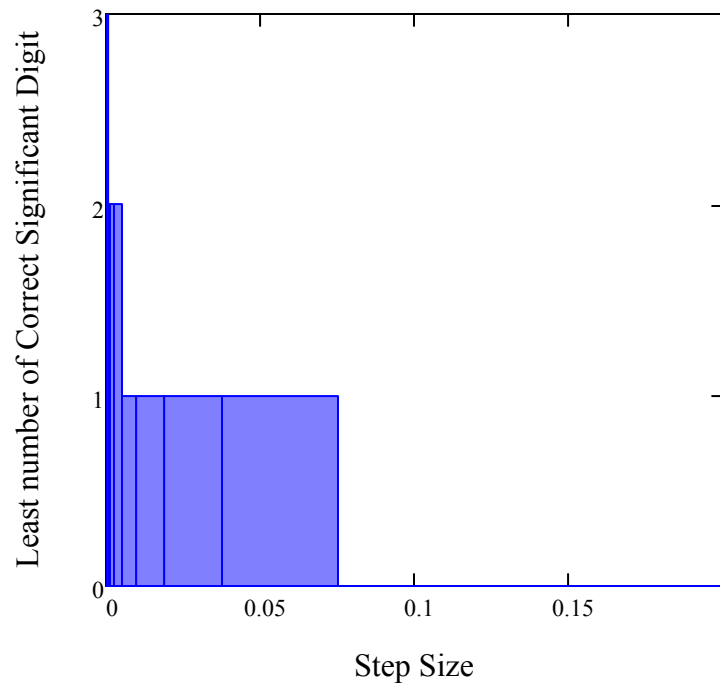
Absolute Relative True Percentage Error as a Function of Step Size



Absolute Relative Approximate Percentage Error as a Function of Step Size



Least Significant Digits Correct as a Function of Step Size



References

Numerical Differentiation of Continuous Functions. See

http://numericalmethods.eng.usf.edu/mws/gen/02dif/mws_gen_dif_txt_continuous.pdf

Questions

1. The velocity of a rocket is given by

$$v(t) = 2000 \cdot \ln \frac{140000}{140000 - 2100 \cdot t} - 9.8 \cdot t$$

Use Backward Divided Difference method with a step size of 0.25 to find the acceleration at $t=5s$. Compare with the exact answer and study the effect of the step size.

2. Look at the true error vs. step size data for problem # 1. Do you see a relationship between the value of the true error and step size ? Is this coincidental?

3. Choose a step size of $h=10^{-10}$ in problem #1. Keep halving the step size. Does the approximate value get closer to the exact result or does the result seem odd?

Conclusions

To obtain more accurate values of the first derivative using Backward Difference Approximation, the step size needs to be small. As the spreadsheet shows, the smaller the step size value is, the approximation is closest to the exact value. However, too small a step size can result in noticeable round-off errors, and hence giving highly inaccurate results.

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