Comparing Methods of First Derivative Approximation

Forward, Backward and Central Divided Difference Comparison

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Introduction

This worksheet demonstrates the use of Mathcad to compare the approximation of first order derivatives using three different methods. Each method uses a point *h* ahead, or a point *h* behind, or both of the given value of *x* at which the first derivative of *f(x)* is to be found.

Forward Difference Approximation *(FDD)*

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}$$

Backward Difference Approximation *(BDD)*

$$f'(x) \approx \frac{f(x)-f(x-h)}{h}$$

Central Difference Approximation *(CDD)*

$$f'(x) \approx \frac{f(x+h)-f(x-h)}{2\cdot h}$$

Section 1: Input

The following simulation approximates the first derivative of a function using different methods of approximation *(FDD,BDD,CDD)*. The user inputs are

a) function, *f(x)*
b) point at which the derivative is to be found, *xv*
c) starting step size, *h*
d) number of times user wants to halve the starting step size, *n*

The outputs include

a) approximate values of the first derivative at the point and initial step size given using different types of approximation
b) exact value
c) absolute relative true error, absolute relative approximate error, and least correct number of significant digits in the solution as a function of step size.

Function *f(x)*

$$f(x) := \exp(2\cdot x)$$

Value of x at which $f'(x)$ is desired, xv

$xv := 4.0$

Starting step size, h

$h := 0.2$

Number of times starting step size is halved

$n := 12$

This is the end of the user section. All the information must be entered before proceeding to the next section.

Section 2: Procedure

The following procedure estimates the solution of first derivate of an equation at a point *xv*.

f (x) = function
xv = value at which the solution is desired
h = starting step size value
n = number of times starting step size is halved

Forward Divided Difference Procedure

$$\mathrm{FDD}(f, xv, h) := \left| \begin{array}{l} \mathrm{deriv} \leftarrow \dfrac{(f(xv + h) - f(xv))}{h} \\ \mathrm{deriv} \end{array} \right.$$

Backward Divided Difference Procedure

$$\mathrm{BDD}(f, xv, h) := \left| \begin{array}{l} \mathrm{deriv} \leftarrow \dfrac{(f(xv) - f(xv - h))}{h} \\ \mathrm{deriv} \end{array} \right.$$

Central Divided Difference Procedure

$$\mathrm{CDD}(f, xv, h) := \left| \begin{array}{l} \mathrm{deriv} \leftarrow \dfrac{(f(xv + h) - f(xv - h))}{2 \cdot h} \\ \mathrm{deriv} \end{array} \right.$$

Section 3: Calculation

The exact value EV of the first derivative of the equation:

Given the function

$$f(x) \rightarrow e^{2\cdot x}$$

First, using the derivative command the solution is found. In the second step, the exact value of the derivative is shown

The solution of the first derivative is

$$f'(x) := \left(\frac{d}{dx} f(x)\right) \qquad f'(x) \rightarrow 2\cdot e^{2\cdot x}$$

The exact solution of the first derivative is

$$EV := f'(xv)$$

$$EV = 5961.915974$$

The next loop calculates the following:
Av: Approximate value of the first derivative using various first derivative approximation methods by calling the procedures "*FDD*", "*BDD*", and "*CDD*"
Ev: Exact value of the first derivative
ε_t: Absolute relative true percentage error
ε_a: Absolute relative approximate percentage error
n_{sig}: Least number of correct significant digits in an approximation

```
table1 := for i ∈ 0 .. n − 1
            | N_i ← 2^i
            | H_i ← h / N_i
            | AVFDD_i ← FDD(f, xv, H_i)
            | AVBDD_i ← BDD(f, xv, H_i)
            | AVCDD_i ← CDD(f, xv, H_i)
            | augment(H, AVFDD, AVBDD, AVCDD)
```

$$
\begin{aligned}
\text{table2} := \;& \text{for } i \in 0..n-1 \\
& \left|\;
\begin{aligned}
& N_i \leftarrow 2^i \\
& H_i \leftarrow \frac{h}{N_i} \\
& AVFDD_i \leftarrow FDD(f, xv, H_i) \\
& AVBDD_i \leftarrow BDD(f, xv, H_i) \\
& AVCDD_i \leftarrow CDD(f, xv, H_i) \\
& \varepsilon FDD_{t_i} \leftarrow \left| \frac{EV - AVFDD_i}{EV} \cdot 100 \right| \\
& \varepsilon BDD_{t_i} \leftarrow \left| \frac{EV - AVBDD_i}{EV} \cdot 100 \right| \\
& \varepsilon CDD_{t_i} \leftarrow \left| \frac{EV - AVCDD_i}{EV} \cdot 100 \right| \\
& \text{augment}(H, \varepsilon FDD_t, \varepsilon BDD_t, \varepsilon CDD_t)
\end{aligned}
\right.
\end{aligned}
$$

$$
\begin{aligned}
\text{table3} := \;& \text{for } i \in 0..n-1 \\
& \left|\;
\begin{aligned}
& N_i \leftarrow 2^i \\
& H_i \leftarrow \frac{h}{N_i} \\
& AVFDD_i \leftarrow FDD(f, xv, H_i) \\
& AVBDD_i \leftarrow BDD(f, xv, H_i) \\
& AVCDD_i \leftarrow CDD(f, xv, H_i) \\
& \varepsilon FDD_{t_i} \leftarrow \left| \frac{EV - AVFDD_i}{EV} \cdot 100 \right| \\
& \varepsilon BDD_{t_i} \leftarrow \left| \frac{EV - AVBDD_i}{EV} \cdot 100 \right| \\
& \varepsilon CDD_{t_i} \leftarrow \left| \frac{EV - AVCDD_i}{EV} \cdot 100 \right| \\
& \text{if } i > 0 \\
& \quad \left|\;
\begin{aligned}
& \varepsilon FDD_{a_i} \leftarrow \left| \frac{AVFDD_i - AVFDD_{i-1}}{AVFDD_i} \right| \cdot 100 \\
& \varepsilon BDD_{a_i} \leftarrow \left| \frac{AVBDD_i - AVBDD_{i-1}}{AVBDD_i} \right| \cdot 100
\end{aligned}
\right.
\end{aligned}
\right.
\end{aligned}
$$

$$\varepsilon CDD_{a_i} \leftarrow \left| \frac{AVCDD_i - AVCDD_{i-1}}{AVCDD_i} \right| \cdot 100$$

$$\text{augment}\left(H, \varepsilon FDD_a, \varepsilon BDD_a, \varepsilon CDD_a\right)$$

$$\text{table4} := \text{for } i \in 0 .. n-1$$

$$N_i \leftarrow 2^i$$

$$H_i \leftarrow \frac{h}{N_i}$$

$$AVFDD_i \leftarrow FDD\left(f, xv, H_i\right)$$

$$AVBDD_i \leftarrow BDD\left(f, xv, H_i\right)$$

$$AVCDD_i \leftarrow CDD\left(f, xv, H_i\right)$$

$$\varepsilon FDD_{t_i} \leftarrow \left| \frac{EV - AVFDD_i}{EV} \cdot 100 \right|$$

$$\varepsilon BDD_{t_i} \leftarrow \left| \frac{EV - AVBDD_i}{EV} \cdot 100 \right|$$

$$\varepsilon CDD_{t_i} \leftarrow \left| \frac{EV - AVCDD_i}{EV} \cdot 100 \right|$$

$$\text{if } i > 0$$

$$\varepsilon FDD_{a_i} \leftarrow \left| \frac{AVFDD_i - AVFDD_{i-1}}{AVFDD_i} \right| \cdot 100$$

$$\varepsilon BDD_{a_i} \leftarrow \left| \frac{AVBDD_i - AVBDD_{i-1}}{AVBDD_i} \right| \cdot 100$$

$$\varepsilon CDD_{a_i} \leftarrow \left| \frac{AVCDD_i - AVCDD_{i-1}}{AVCDD_i} \right| \cdot 100$$

$$nFDD_{sig_i} \leftarrow \text{floor}\left(\left(2 - \log\left(\frac{\varepsilon FDD_{a_i}}{0.5}\right)\right)\right) \quad \text{if } 0 < \varepsilon FDD_{a_i} < 5$$

$$nFDD_{sig_i} \leftarrow 0 \quad \text{otherwise}$$

$$nBDD_{sig_i} \leftarrow \text{floor}\left(\left(2 - \log\left(\frac{\varepsilon BDD_{a_i}}{0.5}\right)\right)\right) \quad \text{if } 0 < \varepsilon BDD_{a_i} < 5$$

$$nBDD_{sig_i} \leftarrow 0 \quad \text{otherwise}$$

$$
\left|\begin{array}{l}
\left|\begin{array}{l}
nCDD_{sig_i} \leftarrow \mathrm{floor}\left(\left(2 - \log\left(\frac{\varepsilon CDD_{a_i}}{0.5}\right)\right)\right) \ \text{ if } \ 0 < \varepsilon CDD_{a_i} < 5 \\
nCDD_{sig_i} \leftarrow 0 \ \text{ otherwise}
\end{array}\right. \\
\mathrm{augment}\left(H, nFDD_{sig}, nBDD_{sig}, nCDD_{sig}\right)
\end{array}\right.
$$

The four loops halves the value of the starting step size *n* times. Each time, the approximate value of the derivative is calculated and saved in a vector according the approximation method used. The approximate error is calculated after at least two approximate values of the derivative have been saved. The number of significant digits is calculated and written as the lowest real number. If the number of significant digits calculated is less than zero, then it is shown as zero.

Section 4: Table of Values

The next tables show the step size value, approximate value, true error, the absolute relative true percentage error, the approximate error, the absolute relative approximate percentage error and the least number of correct significant digits in an approximation as a function of the step size value.

table1 =

	H	AVFDD	AVBDD	AVCDD
	0	1	2	3
0	0.2	7330.5438	4913.8105	6122.1771
1	0.1	6599.9232	5403.5601	6001.7416
2	0.05	6270.2018	5673.5132	5971.8575
3	0.025	6113.4794	5815.3215	5964.4004
4	0.0125	6037.0649	5888.0092	5962.537
5	0.0063	5999.3337	5924.8088	5962.0712
6	0.0031	5980.5858	5943.3237	5961.9548
7	0.0016	5971.2412	5952.6102	5961.9257
8	$7.8125\cdot10^{-4}$	5966.5761	5957.2607	5961.9184
9	$3.9063\cdot10^{-4}$	5964.2455	5959.5877	5961.9166
10	$1.9531\cdot10^{-4}$	5963.0806	5960.7517	5961.9161
11	$9.7656\cdot10^{-5}$	5962.4982	5961.3338	5961.916

table2 =

	H	εFDD_t	εBDD_t	εCDD_t
	0	1	2	3
0	0.2	22.9562	17.58	2.6881
1	0.1	10.7014	9.3654	0.668
2	0.05	5.1709	4.8374	0.1668
3	0.025	2.5422	2.4588	0.0417
4	0.0125	1.2605	1.2396	0.0104
5	0.0063	0.6276	0.6224	0.0026
6	0.0031	0.3132	0.3118	$6.5104\cdot10^{-4}$
7	0.0016	0.1564	0.1561	$1.6276\cdot10^{-4}$
8	$7.8125\cdot10^{-4}$	0.0782	0.0781	$4.069\cdot10^{-5}$
9	$3.9063\cdot10^{-4}$	0.0391	0.0391	$1.0172\cdot10^{-5}$
10	$1.9531\cdot10^{-4}$	0.0195	0.0195	$2.543\cdot10^{-6}$
11	$9.7656\cdot10^{-5}$	0.0098	0.0098	$6.3594\cdot10^{-7}$

table3 =

	H	εFDD_a	εBDD_a	εCDD_a
	0	1	2	3
0	0.2	0	0	0
1	0.1	11.0701	9.0635	2.0067
2	0.05	5.2585	4.7581	0.5004
3	0.025	2.5636	2.4385	0.125
4	0.0125	1.2658	1.2345	0.0313
5	0.0063	0.6289	0.6211	0.0078
6	0.0031	0.3135	0.3115	0.002
7	0.0016	0.1565	0.156	$4.8828 \cdot 10^{-4}$
8	$7.8125 \cdot 10^{-4}$	0.0782	0.0781	$1.2207 \cdot 10^{-4}$
9	$3.9063 \cdot 10^{-4}$	0.0391	0.039	$3.0518 \cdot 10^{-5}$
10	$1.9531 \cdot 10^{-4}$	0.0195	0.0195	$7.6295 \cdot 10^{-6}$
11	$9.7656 \cdot 10^{-5}$	0.0098	0.0098	$1.9071 \cdot 10^{-6}$

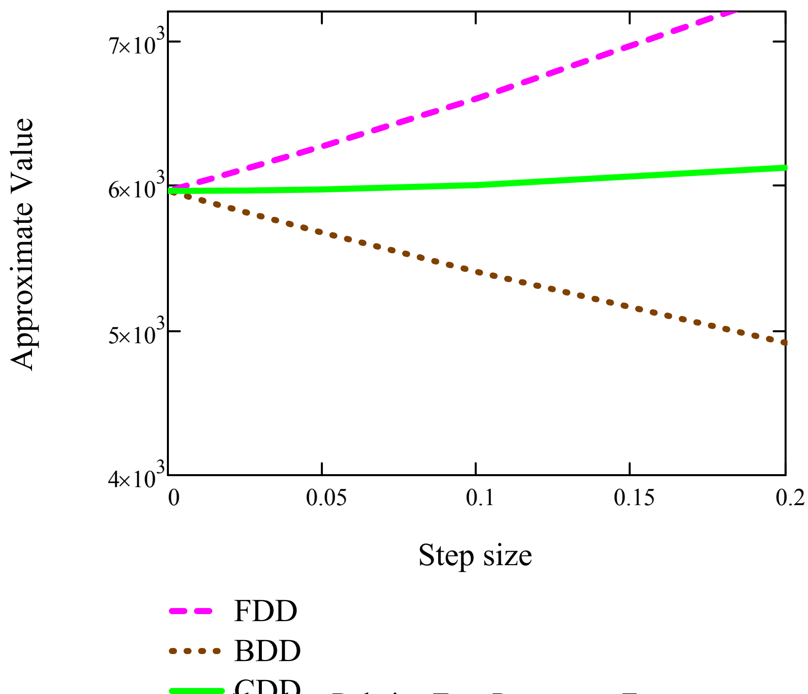
table4 =

	H	$nFDD_{sig}$	$nBDD_{sig}$	$nCDD_{sig}$
	0	1	2	3
0	0.2	0	0	0
1	0.1	0	0	1
2	0.05	0	1	1
3	0.025	1	1	2
4	0.0125	1	1	3
5	0.0063	1	1	3
6	0.0031	2	2	4
7	0.0016	2	2	5
8	$7.8125 \cdot 10^{-4}$	2	2	5
9	$3.9063 \cdot 10^{-4}$	3	3	6
10	$1.9531 \cdot 10^{-4}$	3	3	6
11	$9.7656 \cdot 10^{-5}$	3	3	7

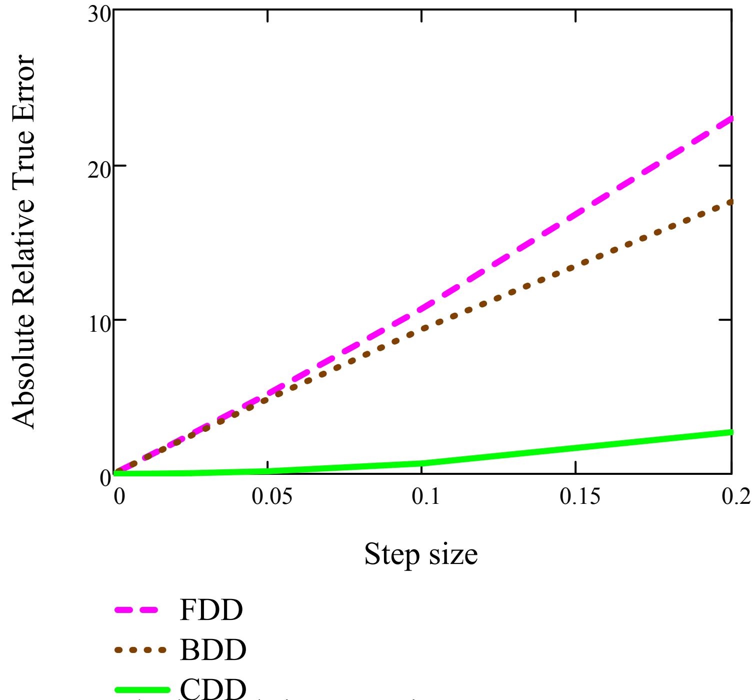
Section 5: Graphs

The following graphs show the approximate solution, absolute relative true error, absolute relative approximate error and least number of significant digits as a function of step size.

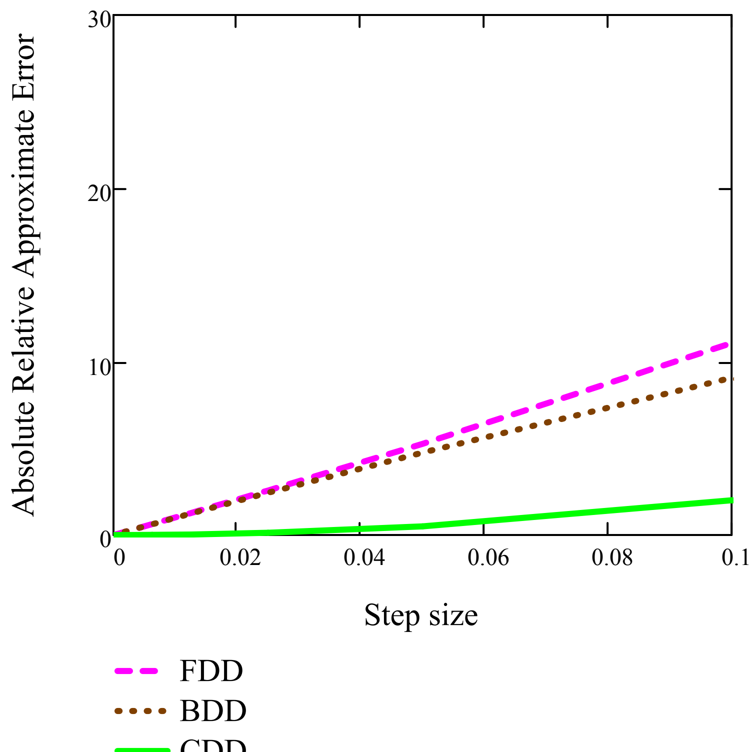
Approximate Solution of the First Derivative of a Function using different Methods of Approximation as a Function of Step Size



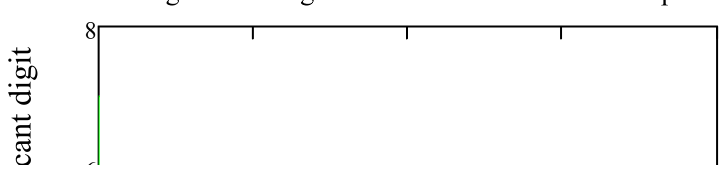
Absolute Relative True Percentage Error as a Function of Step Size

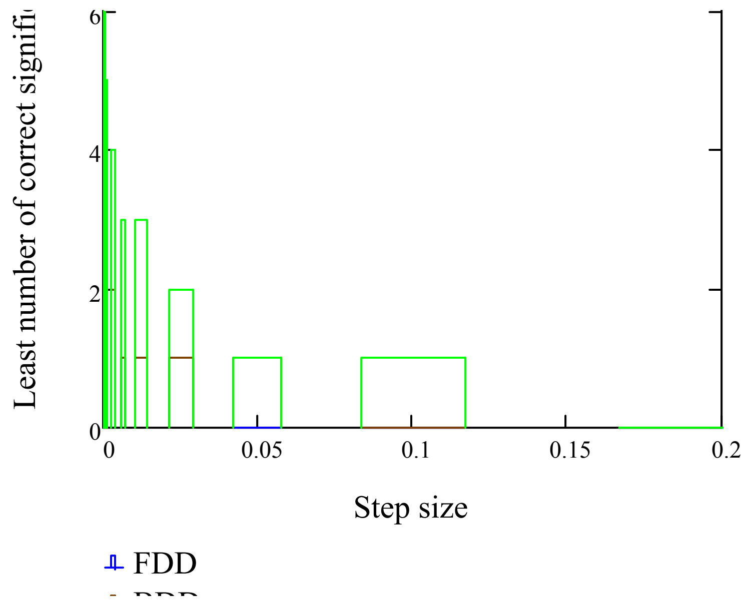


Absolute Relative Approximate Percentage Error as
a Function of Step Size



Least Significant Digits Correct as a Function of Step Size




Least number of correct signific
6
4
2
0
0
0.05
0.1
0.15
0.2
Step size
FDD

BDD

CDD

References

Numerical Differentiation of Continuous Functions. See http://numericalmethods.eng.usf.edu/mws/gen/02dif/mws_gen_dif_txt_continuous.pdf

Questions

1. The velocity of a rocket is given by

$$v(t) = 2000 \cdot ln \frac{140000}{140000 - 2100 \cdot t} - 9.8 \cdot t$$

Use the three different methods with a step size of 0.25 to find the acceleration at $t=5s$. Compare with the exact answer and study the effect of the step size.

2. Look at the true error vs. step size data for problem # 1. Do you see a relationship between the value of the true error and step size ? Is this concidential? Is it similar for Forward and Backward Divided Difference? Is it different for Central Divided Difference method?

3. Choose a step size of $h=10^{-10}$ in problem # 1. keep halving the step size. Do the approximate values get closer to the exact result or do they seem odd? Is it similar for Forward and Backward Divided Difference? Is it different for Central Divided Difference method? Why?

Conclusions

The worksheet shows the nature of accuracy of the three different methods of finding the first derivative of a continuous function. Forward and Backward Divided Difference methods exhibit similar accuraciees as they are first order accurate, while central divided difference shows more accuracy as it is second order accurate.