Differentiation of Discrete Functions

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Introduction

This worksheet demonstrates the use of Mathcad to illustrate the differentiation of discrete functions using:

a) Forward Divided Difference Method to find the derivative of a function given at discrete $n+1$ data points (x_1,y_1), (x_2,y_2),...,(x_n,y_n), the value of the $f'(x)$ for $x_i \le x \le x_{i+1}$, $i=1,\ldots,n-1$, is given by

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

b) By differentiation of a second order interpolated polynomial, and

c) By differentiation of a polynomial using all data points.

Section 1: Input

The following simulation finds the approximate value of the first derivative of a discrete function using Forward Difference Divided Difference. The user inputs are

a) data with *x* and *y* values. At leats 3 data points are requiered.

b) point at which the derivative is to be found, *xv*

The outputs include

a) approximate value of the derivative at the given point using Forward Divided Difference

b) approximate value of the derivative at the given point by differentiating a second order polynomial found using the three closest values to *xv*, and that bracket *xv*.

c) approximate value of the derivative at the given point using all data points and differentiating a $n\text{-}1^{th}$ order polynomial that goes through the n data points

Data points, *y* vs. *x*

$x := (10 \;\; 15 \;\; 20 \;\; 21 \;\; 25)$

$y := (100 \;\; 225 \;\; 400 \;\; 441 \;\; 625)$

Value of *x* at which $f'(x)$ is desired, *xv*

$xv := 11$

This is the end of the user section. All the information must be entered before proceeding to the next section.

Section 2: Calculation

The following loop estimates the first derivate of a discrete function at a point *xv* using Forward Divided Difference Method. *n* is the number of data points. The loop checks if the value at which the solution is desired is between *x[1]* and *x[n]*. If the value is between *x[1]* and *x[n]*, then the slope from the closest points that bracket the value is found. The value of the slope is the value of the derivative at that point.

$n := \text{cols}(x) - 1$

```
Straight_Line := | for i ∈ 0..n − 1                                          if x[0,0] ≤ xv < x[0,n−1]
                 |   AV ← (y[0,i+1] − y[0,i]) / (x[0,i+1] − x[0,i])   if x[0,i] ≤ xv < x[0,i+1]
                 | "Point where derivative was requested is outside the domain of x"   otherwise
```

$$\text{Straight\_Line} := \begin{cases} \text{for } i \in 0..n-1 & \text{if } x_{0,0} \le xv < x_{0,n-1} \\ \quad AV \leftarrow \dfrac{y_{0,i+1} - y_{0,i}}{\left(x_{0,i+1} - x_{0,i}\right)} \quad \text{if } x_{0,i} \le xv < x_{0,i+1} & \\ \text{"Point where derivative was requested is outside the domain of x"} & \text{otherwise} \end{cases}$$

The next method takes the three closest points to the given value to find a second order polynomial and differentiates it to find the value of *f'(x)* at *x=xv*.

$$\text{data1} := \begin{pmatrix} x_{0,0} & y_{0,0} \\ x_{0,1} & y_{0,1} \\ x_{0,2} & y_{0,2} \end{pmatrix} \qquad \text{data2} := \begin{pmatrix} x_{0,n-2} & y_{0,n-2} \\ x_{0,n-1} & y_{0,n-1} \\ x_{0,n} & y_{0,n} \end{pmatrix}$$

```
data3 := | for i ∈ 0..n − 2                                                        if x[0,1] ≤ xv < x[0,n
         |   | a ← i
         |   | for j ∈ 0..2                                                        if (x[0,a] ≤ xv < x[0,a+1])
         |   |   | B[j,0] ← x[0,i]    if |x[0,a+2] − xv| ≤ |xv − x[0,a−1]|
         |   |   | B[j,0] ← x[0,i−1]  otherwise
         |   |   | B[j,1] ← y[0,i]    if (|x[0,a+2] − xv| ≤ |xv − x[0,a−1]|)
         |   |   | B[j,1] ← y[0,i−1]  otherwise
         |   |   | i ← i + 1
         | augment(B)
```

```
data := | A ← data1  if x[0,0] ≤ xv < x[0,1]
        | A ← data2  if x[0,n−1] ≤ xv < x[0,n]
        | A ← data3  otherwise
        | A
```

$data := csort(data, 0)$ $X := data^{\langle 0 \rangle}$ $Y := data^{\langle 1 \rangle}$

Spline coefficients:

$S := lspline(X, Y)$

Fitting function:

$fit(a) := interp(S, X, Y, a)$

$dfit(x) := \frac{d}{dx} fit(x)$

$dfit(xv) = 22.8$

Next, all data points are going to be used to find a *n-1* order polynomial. At the end, the polynomial is going to be differentiated and the value at which the derivative is wished to be found is going to be found.

```
datainterp := | for i ∈ 0 .. n
              |   | B[i,0] ← x[0,i]
              |   | B[i,1] ← y[0,i]
              | augment(B)
```

$datainterp := csort(datainterp, 0)$ $X := datainterp^{\langle 0 \rangle}$ $Y := datainterp^{\langle 1 \rangle}$

Spline coefficients:

$S2 := cspline(X, Y)$ $S3 := lspline(X, Y)$

Fitting function:

$fit2(b) := interp(S2, X, Y, b)$ $fit3(b) := interp(S3, X, Y, b)$

$dfit2(x) := \frac{d}{dx} fit2(x)$

$dfit2(xv) = 22$

Section 3: Table of Values

The next table shows the approximate values of the derivative at the given point using Forward Divic Difference, second and *n-1* order polynomials.

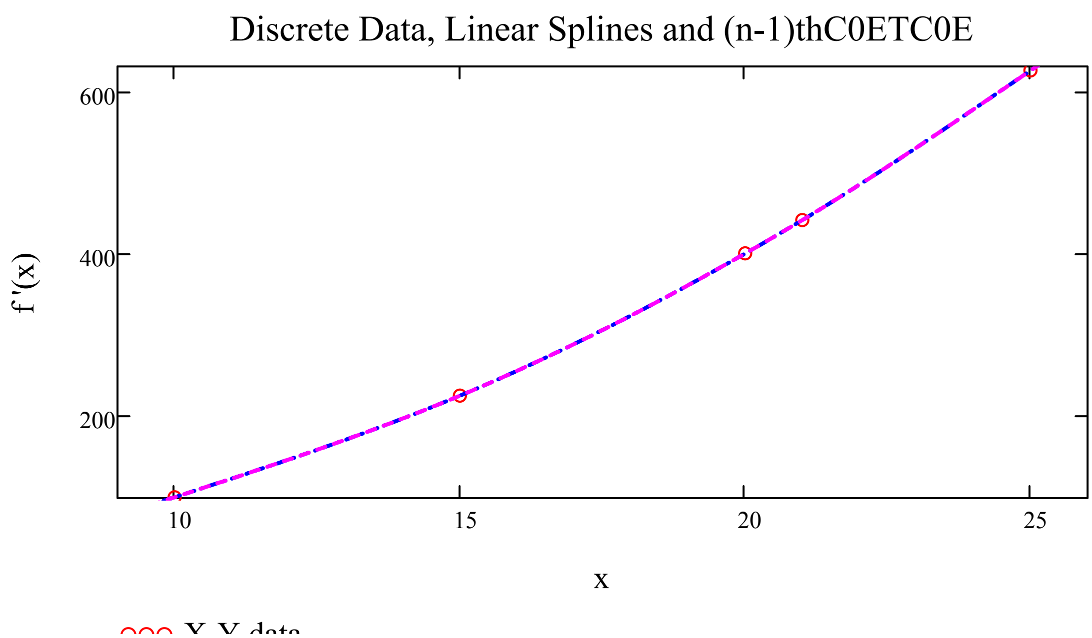
$$a := \begin{pmatrix} . & \text{"Straight Line"} & \text{"Second Order"} & \text{"n-1th Order"} \\ \text{"f ' (x)"} & Straight\_Line & dfit(xv) & dfit2(xv) \end{pmatrix}$$

a =

	0	1	2	3
0	0	"Straight Line"	"Second Order"	"n-1th Order"
1	"f ' (x)"	25	22.8	22

Section 4: Graphs

The following graph shows the the discrete data, linear splines and the *n-1th* order polynomial.



X-Y data
Linear splines
n-1 Order Polynomial

References

Numerical Differentiation of Continuous Functions. See http://numericalmethods.eng.usf.edu/mws/gen/02dif/

Questions

1. The thermal expansion coefficient of steel is a function of temperature. Find the rate of change of t thermal expansion coefficient as a function of temperature at T=-200 F. Is this rate of change at T=-200 F more or less than that at T=50 F? Use Forward Divided Difference to answer this question.

2. The distance traveled by a rocket is given as a function of time

t, s	0	10	20	30	40
x, miles	0	16	28	39	53

Find the rocket velocity and acceleration at t=25s using numerical differentiation. Use all three methods illustrated in the worksheet.

Conclusions

The more data points taken to obtain the first derivative of a discrete function, more accurate the approximate value is. However, the more data points taken may result in oscillatory behavior generally observed with higher order interpolation. See http://numericalmethods.eng.usf.edu/mws/gen/05inp/mws_gen_inp_spe_higherorder.pdf

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