Differentiation of Continuous Functions

Effect of Significant Digits on Derivative of a Function

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Introduction

This worksheet demonstrates the use of Mathcad to illustrate the effect of significant digits numerical calculation of the Forward Difference Approximation of the first derivative of continuous functions.

Forward Difference Approximation of the first derivative uses a point *h* ahead of the given of *x* at which the derivative of *f(x)* is to be found.

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}$$

Section 1: Input

The following simulation approximates the first derivative of a function using Forward Di Approximation. The user inputs are

a) function, *f(x)*
b) point at which the derivative is to be found, *xv*
c) step size, *h*
d) lowest and highest number of significant digits user wants to use in the calcluation. The user should choose the lowest number to be at least 2.

The outputs include

a) exact value
c) true error and absolute relative true error as a function of significant digits.

Function *f(x)*

$f(x) := x \cdot \exp(2 \cdot x)$

Value of *x* at which *f '(x)* is desired, *xv*

$xv := 4.0$

Starting step size, *h*

$h := 0.5$

Lowest number of Significant Digits and Highest Number of Significant Digits

$nlow := 2$ $nhigh := 10$

This is the end of the user section. All the information must be entered before proceeding t
next section.

Section 2: Significant Digit Operators

The following functions modify standard arithmetic operators allowing computation with t appropriate number of significant digits. These redefined operators are then used in the Forward Difference Approximation method to generate a solution that was computed with number of significant digits specified.

$$\mathrm{sdscale}(sd,k) := \left|\begin{array}{l} m \leftarrow sd \quad \text{if } k = 0 \\ m \leftarrow sd - \mathrm{floor}(\log(|k|) + 1) \quad \text{otherwise} \\ q \leftarrow k\cdot 10^{m} \\ q \leftarrow \mathrm{floor}(q)\cdot 10^{-m} \\ q \end{array}\right.$$

$$\mathrm{add}(a,b) := \left|\begin{array}{l} c \leftarrow a + b \\ c \end{array}\right.$$

$$\mathrm{sub}(a,b) := \left|\begin{array}{l} c \leftarrow a - b \\ c \end{array}\right.$$

$$\mathrm{mul}(a,b) := \left|\begin{array}{l} c \leftarrow a\cdot b \\ c \end{array}\right.$$

$$\mathrm{div}(a,b) := \left|\begin{array}{l} c \leftarrow \dfrac{a}{b} \\ c \end{array}\right.$$

$$\mathrm{sdadd}(sd,a,b) := \left|\begin{array}{l} d \leftarrow \mathrm{sdscale}(sd,\mathrm{add}(\mathrm{sdscale}(sd,a),\mathrm{sdscale}(sd,b))) \\ d \end{array}\right.$$

$$\mathrm{sdsub}(sd,a,b) := \left|\begin{array}{l} d \leftarrow \mathrm{sdscale}(sd,\mathrm{sub}(\mathrm{sdscale}(sd,a),\mathrm{sdscale}(sd,b))) \\ d \end{array}\right.$$

$$\mathrm{sdmul}(sd,a,b) := \left|\begin{array}{l} d \leftarrow \mathrm{sdscale}(sd,\mathrm{mul}(\mathrm{sdscale}(sd,a),\mathrm{sdscale}(sd,b))) \\ d \end{array}\right.$$

$$\mathrm{sddiv}(sd,a,b) := \left|\begin{array}{l} d \leftarrow \mathrm{sdscale}(sd,\mathrm{div}(\mathrm{sdscale}(sd,a),\mathrm{sdscale}(sd,b))) \\ d \end{array}\right.$$

Section 3: Calculation

The following procedure estimates the solution of first derivate of an equation at a point *xv*.

f (x) = function
xv = value at which the solution is desired
h = starting step size value
sd = number of significant digits used in the calculation

$$\mathrm{FDD}(f, xv, h, sd) := \left| \begin{array}{l} \mathrm{deriv} \leftarrow \mathrm{sddiv}(sd, \mathrm{sdsub}(sd, f(\mathrm{sdadd}(sd, xv, h)), f(xv)), h) \\ \mathrm{deriv} \end{array} \right.$$

Section 4: Calculation

The exact value EV of the first derivative of the equation:

Given the function

$$f(x) \rightarrow x \cdot e^{2 \cdot x}$$

First, using the derivative command the solution is found. In the second step, the exact value of the derivative is shown

The solution of the first derivative is

$$f'(x) := \left(\frac{d}{dx} f(x)\right) \qquad f'(x) \rightarrow e^{2 \cdot x} + 2 \cdot x \cdot e^{2 \cdot x}$$

The exact solution of the first derivative is

$$EV := f'(xv)$$

$$EV = 26828.621883$$

The next loop calculate the following:
Av: Approximate value of the first derivative using Forward Difference Approximation by calling the procedure "FDD"
Et: True error
ε_t: Absolute relative true percentage error
Ea: Approximate error
ε_a: Absolute relative approximate percentage error

$$\begin{aligned} \text{table1} := \; & \text{for } i \in \text{nlow} .. \text{nhigh} \\ & \left| \begin{aligned} & \text{Digits}_i \leftarrow i \\ & AV_i \leftarrow \text{FDD}(f, xv, h, i) \\ & E_{t_i} \leftarrow EV - AV_i \\ & \varepsilon_{t_i} \leftarrow \left| \frac{E_{t_i}}{EV} \cdot 100 \right| \\ & \text{augment}\left(\text{Digits}, AV, E_t, \varepsilon_t\right) \end{aligned} \right. \end{aligned}$$

The loop calculates the approximate value of the first derivative, the corresponding true error and relative true error as a function of the number of significant digits used in the calculations.

Section 5: Table of Values

The next table shows the approximate value, true error, and the absolute relative true percentage error as a function of the number of significant digits used in the calculations.

table1 =

	0 (Digits)	1 (AV)	2 (E_t)	3 (ε_t)
0	$0 \cdot 10^0$	$0 \cdot 10^0$	$0 \cdot 10^0$	$0 \cdot 10^0$
1	$0 \cdot 10^0$	$0 \cdot 10^0$	$0 \cdot 10^0$	$0 \cdot 10^0$
2	$2 \cdot 10^0$	$5 \cdot 10^4$	$-2.317 \cdot 10^4$	$8.637 \cdot 10^1$
3	$3 \cdot 10^0$	$4.9 \cdot 10^4$	$-2.217 \cdot 10^4$	$8.264 \cdot 10^1$
4	$4 \cdot 10^0$	$4.908 \cdot 10^4$	$-2.225 \cdot 10^4$	$8.294 \cdot 10^1$
5	$5 \cdot 10^0$	$4.908 \cdot 10^4$	$-2.225 \cdot 10^4$	$8.294 \cdot 10^1$
6	$6 \cdot 10^0$	$4.908 \cdot 10^4$	$-2.225 \cdot 10^4$	$8.294 \cdot 10^1$
7	$7 \cdot 10^0$	$4.908 \cdot 10^4$	$-2.225 \cdot 10^4$	$8.294 \cdot 10^1$
8	$8 \cdot 10^0$	$4.908 \cdot 10^4$	$-2.225 \cdot 10^4$	$8.294 \cdot 10^1$
9	$9 \cdot 10^0$	$4.908 \cdot 10^4$	$-2.225 \cdot 10^4$	$8.294 \cdot 10^1$
10	$1 \cdot 10^1$	$4.908 \cdot 10^4$	$-2.225 \cdot 10^4$	$8.294 \cdot 10^1$

Section 5: Graphs

The following graphs show the approximate solution, true error and absolute relative true error as a function of the number of significant digits used.

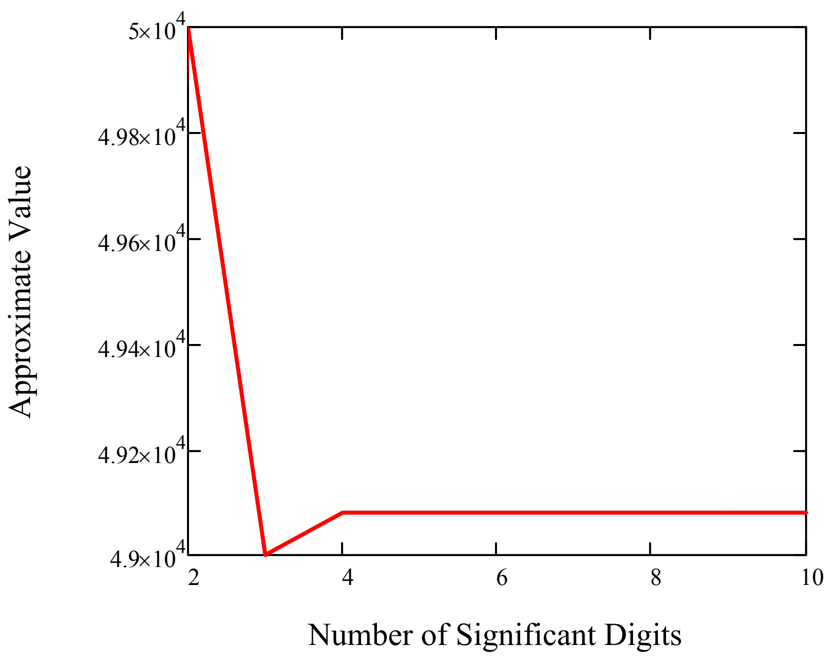
$a := \mathrm{FDD}(f, xv, h, nlow)$

$b := \mathrm{FDD}(f, xv, h, nlow + 1)$

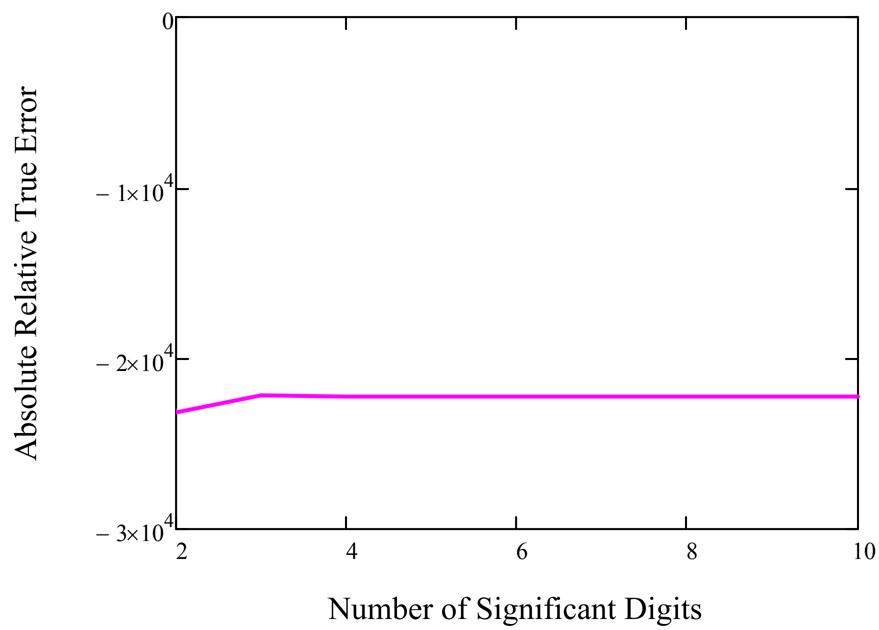
Override the default axis limits on the graphs to visualize the desired range:

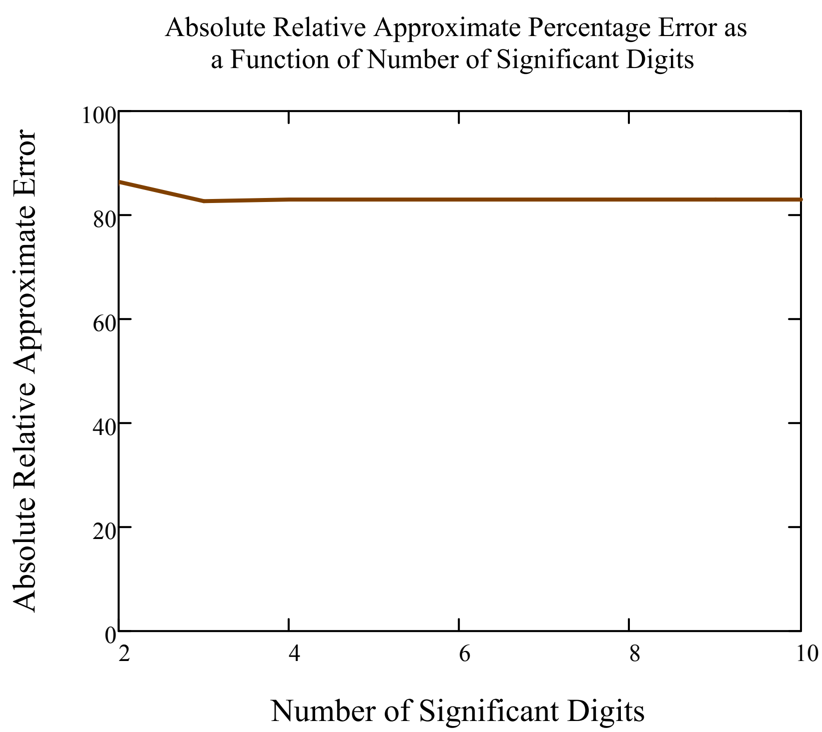
Click in the plot to select it. Click on the number closest to the axis limit you wish to change and type a new number or expression. Click outside the plot region to redisplay the plot using the new numbers specified as axis limits.

Approximate Solution of the First Derivative using Forward Difference Approximation as a Function of Number of Significant Digits



Absolute Relative True Percentage Error as a Function of Number of Significant Digits


0
− 1×10^4
− 2×10^4
− 3×10^4
2
4
6
8
10
Absolute Relative True Error
Number of Significant Digits


Absolute Relative Approximate Percentage Error as
a Function of Number of Significant Digits
100
80
60
40
20
0
2
4
6
8
10
Absolute Relative Approximate Error
Number of Significant Digits

References

Numerical Differentiation of Continuous Functions.
http://numericalmethods.eng.usf.edu/mws/gen/02dif

Questions

1. The velocity of a rocket is given by

$$v(t) = 2000 \cdot ln \frac{140000}{140000 - 2100 \cdot t} - 9.8 \cdot t$$

Use Forward Divided Difference method with a step size of 0.25 to find the acceleration at $t=5s$ using different number of significant digits.

Conclusions

The effect of significant digits on the calculation of the first derivative using Forward Difference approximation is studied.

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