SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

Convergence of Gauss-Seidel Method

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Introduction

This worksheet demonstrates the convergence of Gauss-Seidel method, an iterative technique used in solving a system of simultaneous linear equations.

Gauss-Seidel method is an advantageous approach to solving a system of simultaneous linear equations because it allows the user to control round-off error that is inherent to elimination methods such as Gaussian Elimination. However, this method is not without its pitfalls. Gauss-Seidel method is an iterative technique that relies on the convergence of solutions between subsequent iterations to generate a final answer. This convergence is ensured only if the coefficient matrix, $[A]_{nxn}$, is diagonally dominant, otherwise the iterations may or may not converge.

A diagonally dominant square matrix [A] is defined by the following:

$$\left|a_{i,i}\right| \geq \left(\sum_{j=1}^{n} \left|a_{i,j}\right|\right)_{i \neq j}$$ for all *i* and

$$\left|a_{i,i}\right| > \left(\sum_{j=1}^{n} \left|a_{i,j}\right|\right)_{i \neq j}$$ for at least one *i*

Fortunately, many physical systems that result in simultaneous linear equations have diagonally dominant coefficient matrices, or with the exchange of a few equations the coefficient matrix can become diagonally dominant. To learn more about diagonally dominant matrices as well as how to perform Gauss-Seidel method, click **here**.

The following simulation illustrates the convergence of Gauss-Seidel method.

Section 1: Input Data

The following are the input parameters for the simulation. The user may change those values that are highlighted only. Once entered, Mathcad will produce plots that demonstrate the convergence of each solution X_i as a function of the iteration number.
NOTE: The system of simultaneous equations must be a 4x4 linear system.

$ORIGIN := 1$

- The number of equations:

$n := 4$

- The *n*x*n* Coefficient matrix [A]:

$$A := \begin{pmatrix} 12 & 7 & 3 & 1 \\ 1 & 5 & 1 & 2 \\ 2 & 7 & -11 & 1 \\ 9 & 2 & 1 & 13 \end{pmatrix}$$

- The *n*x1 right hand side vector [RHS]:

$$RHS := \begin{pmatrix} 22 \\ 7 \\ -2 \\ 3 \end{pmatrix}$$

- The initial guess *n*x1 solution vector [X]:

$$X := \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

- The maximum number of iterations that are used to demonstrate convergence:

$maxit := 8$

Section 2: Exact Solution

The exact solution to the above system of equations can be found using Mathcad's built-
tools:

$$exactsoln := \mathrm{lsolve}(A, RHS)$$

$$exactsoln = \begin{pmatrix} 0.875 \\ 1.269 \\ 1.089 \\ -0.654 \end{pmatrix}$$

Section 3: Gauss-Seidel Procedure

The following procedure will use Gauss-Seidel method to calculate the value of the solution for the above system of equations using *maxit* iterations. It will then store each approximate solution, X_i , from each iteration in a matrix with *maxit* columns. The procedure will also calculate the maximum relative approximate error for each iteration and store it in an array. Thereafter, Mathcad will plot the solutions as a function of the iteration number.

Gauss-Seidel method equation:

$$x_i := \frac{\left[rhs_i - \sum_{i=1}^{n} \left[\left(a_{i,j} \right) \cdot x_j \right], i \neq j \right]}{a_{i,i}} \qquad \text{Equation (1.1)}$$

Local parameter names

Xprev = Previous guess of solution vector
epsmax = Array that stores maximum relative approximate error for each iteration
epsa = Array that stores absolute relative approximate error for each X_i
Xnew = New solution vector calculated at end of each iteration
Xstore = Matrix that stores all solutions where
$Xstore_i = X_i$
$Xstore_j$ = iteration number

Gauss-Seidel Procedure:

```
gauss_seidel := | Xprev ← X
                | for k ∈ 1 .. maxit
                |   | epsmax[k] ← 0
                |   | for i ∈ 1 .. n
                |   |   | sum ← 0.0
                |   |   | for j ∈ 1 .. n
                |   |   |   sum ← sum + A[i,j]·Xprev[j]  if  j ≠ i
                |   |   | Xnew[i] ← (RHS[i] − sum) / A[i,i]
                |   |   | epsa[i] ← |(Xnew[i] − Xprev[i]) / Xnew[i]|·100
                |   |   | epsmax[k] ← epsa[i]  if  epsmax[k] ≤ epsa[i]
                |   |   | Xprev[i] ← Xnew[i]
                |   |   | Xstore[i,k] ← Xnew[i]
                | (Xstore
                |  epsmax)
```

- Renaming the initial guess solution vector as *Xprev*
- Conducting *maxit* iterations
- Initializing *epsmax* to zero.
- Defining row operations
- Initializing series sum to zero.
- Defining column operations.
- Generating summation term
- Using Equation (1.1) to calculate new solution vector
- Calculating absolute relative appoximate error
- Finding max. epsa value for the iteration.
- Updating the previous guess.
- Storing each value of X_i for each iteration.
- Returning Xstore and epsmax

Section 4: Results

Extracting *Xstore* from the Gauss-Seidel procedure:

$$Xstore := gauss\_seidel_1$$

The stored solutions are now illustrated in the following matrix, where each X_i is placed in the i^{th} row and each column represents the iteration number from 1 to *maxit*:

$$Xstore = \begin{pmatrix} 0.333 & 1.22 & 1.017 & 0.839 & 0.847 & 0.877 & 0.881 & 0.876 \\ 0.733 & 1.066 & 1.327 & 1.311 & 1.265 & 1.261 & 1.268 & 1.27 \\ 0.8 & 1.066 & 1.133 & 1.099 & 1.083 & 1.086 & 1.09 & 1.089 \\ -0.174 & -0.86 & -0.765 & -0.637 & -0.633 & -0.654 & -0.658 & -0.655 \end{pmatrix}$$

Compare each X_i value as the number of iterations increases. Does the solution converge the exact solution?

The following matrix stores the maximum absolute relative approximate error after each iteration:

$$epsmax := gauss\_seidel_2$$

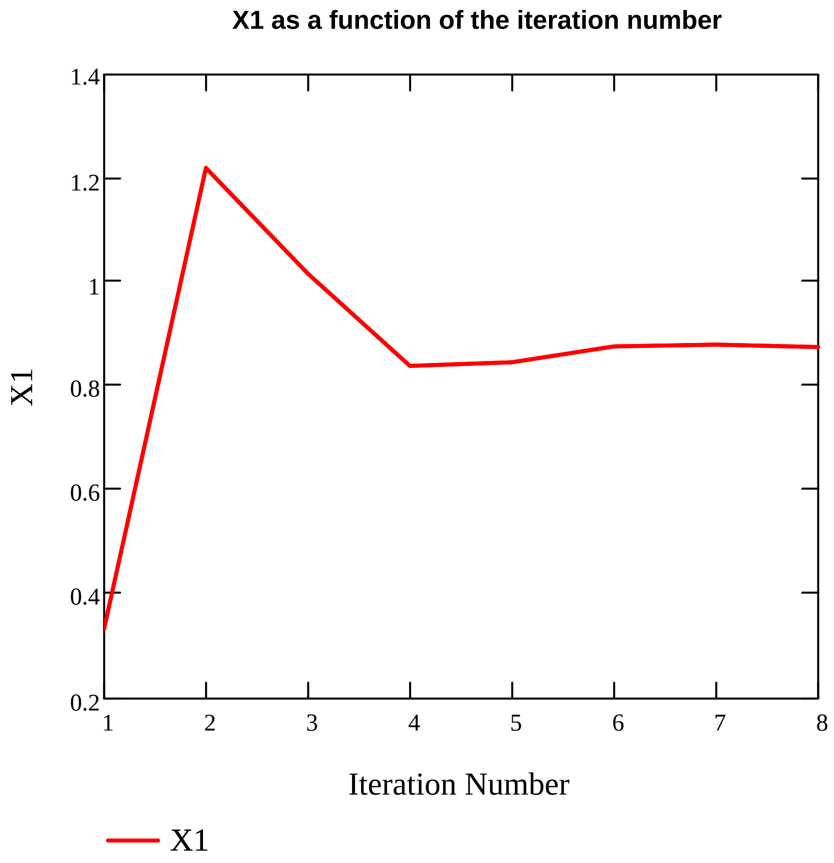
$$epsmax = \begin{pmatrix} 673.529 \\ 79.722 \\ 19.991 \\ 21.131 \\ 3.621 \\ 3.478 \\ 0.555 \\ 0.52 \end{pmatrix}$$

Does the maximum absolute relative approximate error approach zero as the iteration nu increases?

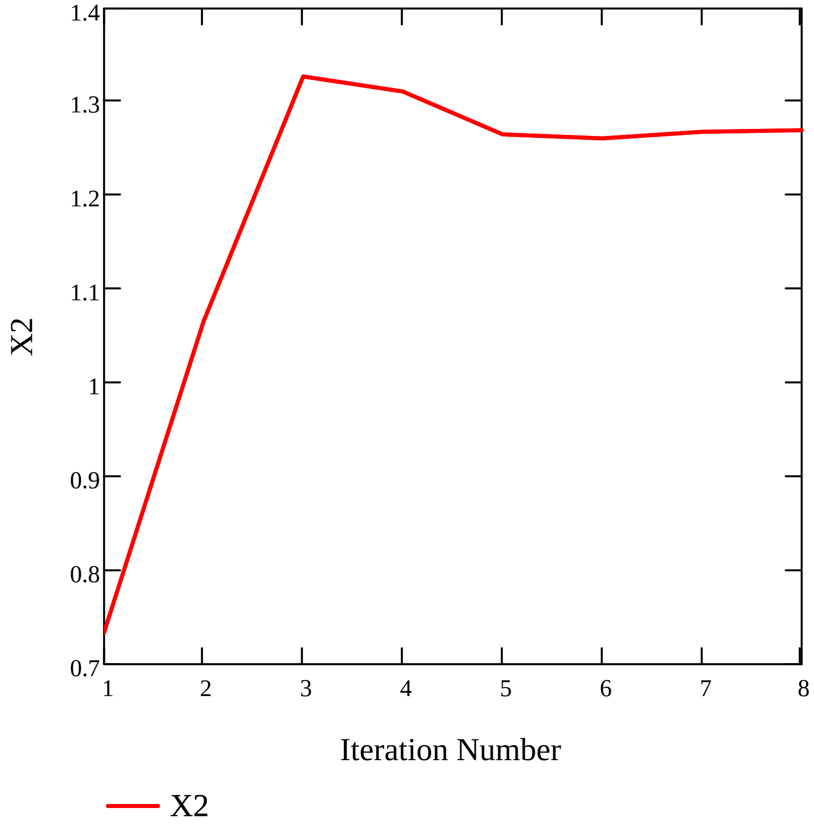
Section 5: Convergence Graphs

The following graphs plot the value of X_i as a function of the iteration number to demonstrate the convergence of each solution.

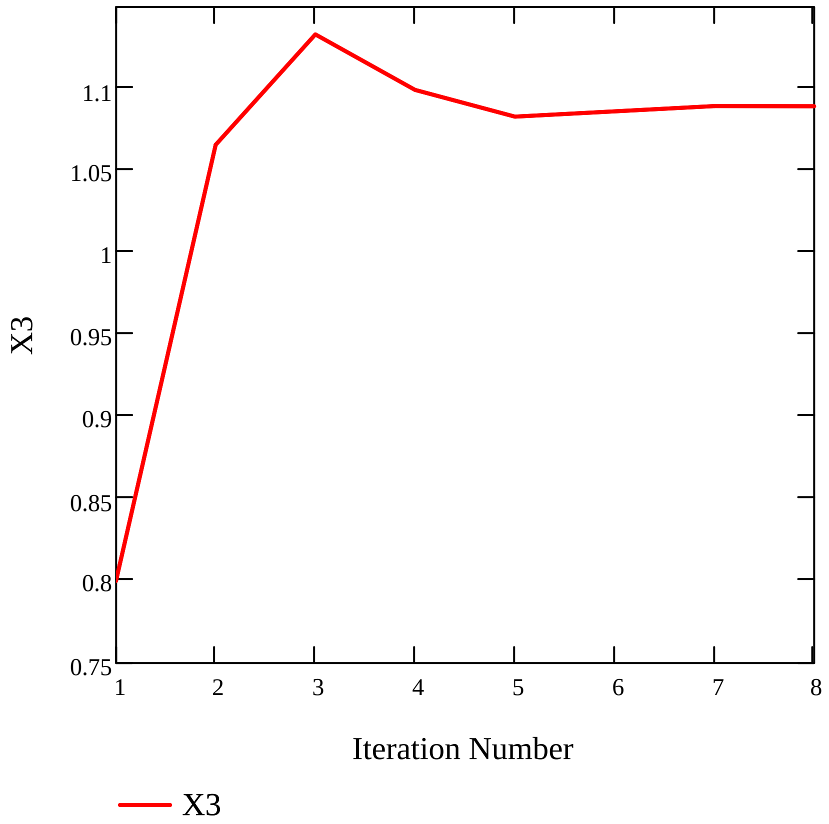
$k := 1 .. maxit$



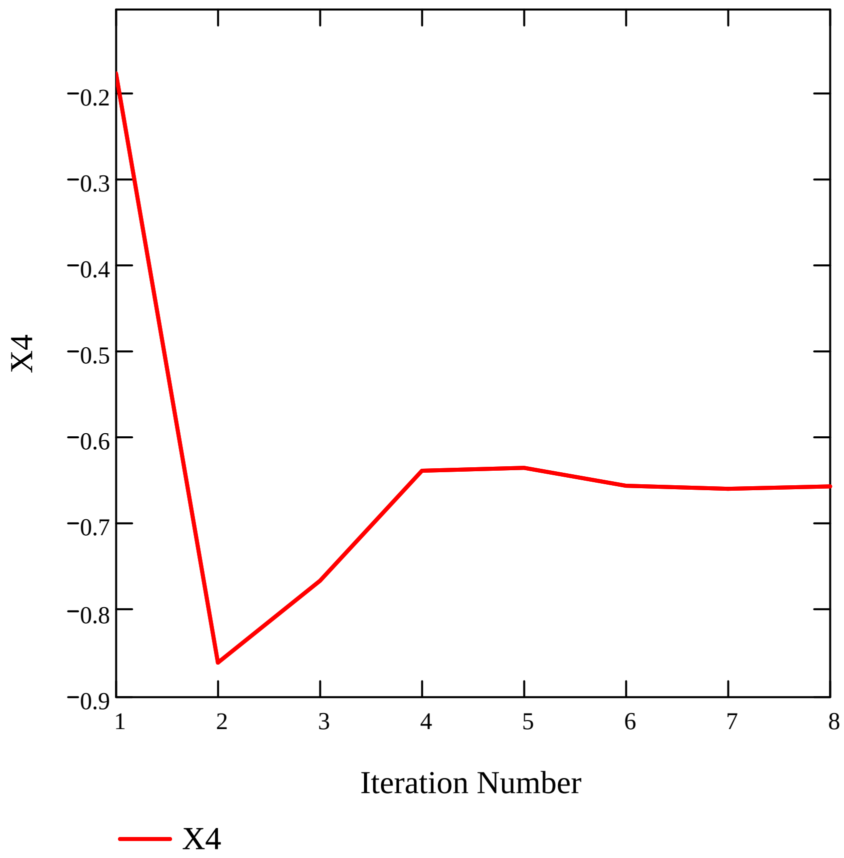
X2 as a function of the iteration number



X3 as a function of the iteration number

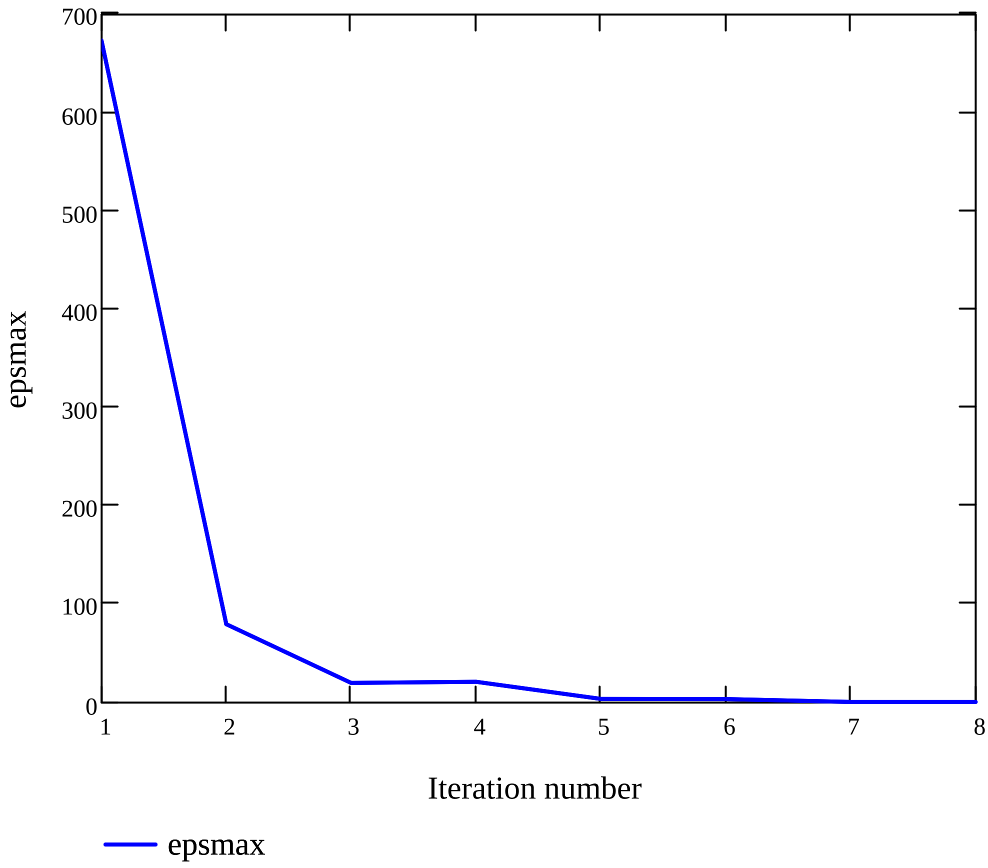


X4 as a function of the iteration number



The following graph plots the maximum absolute relative approximate error as a functi
iteration number.

Value of the maximum absolute relative approximate error as a function of iteration number



References

Autar Kaw, *Holistic Numerical Methods Institute, http://numericalmethods.eng.usf.edu/mws,* See
How does Gauss-Seidel method work?

Conclustion

Mathcad helped us to study the convergence of Gauss-Seidel method.

Question: Solve a set of equations for which the coefficient matrix is not diagonally dom
For example,

$$5x_1 + 6x_2 + 7x_3 + 1x_4 = 18$$
$$6x_1 + 3x_2 + 9x_3 + 2x_4 = 18$$
$$7x_1 + 9x_2 + 2x_3 + 6x_4 = 29$$
$$3x_1 + 5x_2 + 1x_3 + 2x_4 = 26$$

Choose an intial solution vector guess of [2, 5, 7, 3]. Does the solution converge? Now choose [0.99, 0.995, 0.997, 0.989] as the initial guess of the solution vector. Does the solution converge now?