SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

Effect of Significant Digits in Solution of Simultaneous Linear Equations

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Introduction

The number of significant digits used in numerical solutions of simultaneous linear equations influences the accuracy of the solution vector, especially if the coefficient matrix is nearly **singular**. In this worksheet, the reader can choose a system of equations and see the influence of significant digits on each element of the solution vector. To learn more about the effects of significant digits on the accuracy of the solution vector click **here**.

The following simulation uses **Naive Gaussian Elimination method** to demonstrate the effect that significant digits have on the accuracy of the solution.

Section 1: Input Data

The following are the input parameters to begin the simulation. This is the only section tha requires user input. The user can change the values that are highlighted only.

In the simulation, Naive Gaussian Elimination method is used to solve a set of simultaneou linear equations, [A] [X] = [RHS], where $[A]_{nxn}$ is the square coefficient matrix, $[X]_{nx1}$ is solution vector, and $[RHS]_{nx1}$ is the right hand side array. To demonstrate the effect that significant digits have on the accuracy of solution, Mathcad will return a list of solution ve that were calculated using the significant digits within the range of your choice. It will the each element of the solution vector as a function of the number of significant digits used.

***VERY IMPORTANT NOTE*:**

1. The system of simultaneous equations that you are inputting must be a 4x4 linear systen
2. To ensure proper representation of significant digits, please insert the number of decima digits displayed as a number greater than the range you are inputting by going to toolbar ->Format -> Result -> Number Format tab.

ORIGIN := 1

- *n*x*n* coefficient matrix, [AL]

$$AL := \begin{pmatrix} 12 & 7 & 3 & 2.5 \\ 1 & 5 & 1 & 6 \\ 13 & 12 & 4.001 & 9 \\ 4 & 8.03 & 2 & 3 \end{pmatrix}$$

- *n*x1 right hand side array, [RHS]

$$RHS := \begin{pmatrix} 22 \\ 7 \\ 29.001 \\ 5.43 \end{pmatrix}$$

- lower limit of range of significant digits, *lower_lim*

lower_lim := 3

- upper limit of range of significant digits, *upper_lim*

upper_lim := 10

- Number of equations, *n*

n := rows(AL)

Section 2: Significant Digit Arithmetic Functions

The following functions modify standard arithmetic operators allowing computation with the appropriate number of significant digits.
NOTE: To ensure proper representation of significant digits, please insert the number of decimal digits displayed as a number greater than the range you are inputting
by going to toolbar ->Format -> Result -> Number Format tab.

$$\text{sdscale}(sd, x) := \left|\begin{array}{l} \text{return } 0 \text{ if } x = 0 \\ n \leftarrow sd - (\text{floor}(\log(|x|)) + 1) \\ x \leftarrow x \cdot 10^{n} \\ \text{round}(x) \cdot 10^{-n} \end{array}\right.$$

$\text{add}(a, b) := a + b$

$\text{mul}(a, b) := a \cdot b$

$\text{sub}(a, b) := a - b$

$\text{div}(a, b) := a \div b$

$$\text{sd\_dyadic}(op, sd, x, y) := \left|\begin{array}{l} z \leftarrow op(\text{sdscale}(sd, x), \text{sdscale}(sd, y)) \\ \text{sdscale}(sd, z) \end{array}\right.$$

$\text{sdadd}(sd, x, y) := \text{sd\_dyadic}(add, sd, x, y)$

$\text{sdsub}(sd, x, y) := \text{sd\_dyadic}(sub, sd, x, y)$

$\text{sdmul}(sd, x, y) := \text{sd\_dyadic}(mul, sd, x, y)$

$\text{sddiv}(sd, x, y) := \text{sd\_dyadic}(div, sd, x, y)$

Section 3: Naive Gaussian Elimination Procedure

The following procedure utilizes the above defined significant digit operators in place of standard arithmetic operators.

- Forward Elimination

$$
\begin{array}{l}
\text{gauss\_naive}(n, A, B, dig) := \left| \begin{array}{l}
A \leftarrow A \\
X_n \leftarrow 0 \\
\text{for } k \in 1..n-1 \\
\quad \text{for } i \in k+1..n \\
\quad\quad \left| \begin{array}{l}
\text{multiplier} \leftarrow \text{sddiv}\left(dig, A_{i,k}, A_{k,k}\right) \\
\text{for } j \in k+1..n \\
\quad A_{i,j} \leftarrow \text{sdsub}\left(dig, A_{i,j}, \text{sdmul}\left(dig, \text{multiplier}, A_{k,j}\right)\right) \\
B_i \leftarrow \text{sdsub}\left(dig, B_i, \text{sdmul}\left(dig, \text{multiplier}, B_k\right)\right)
\end{array} \right. \\
X_n \leftarrow \text{sddiv}\left(dig, B_n, A_{n,n}\right) \\
\text{for } i \in n-1, n-2..1 \\
\quad \left| \begin{array}{l}
\text{sum} \leftarrow 0 \\
\text{for } j \in i+1..n \\
\quad \text{sum} \leftarrow \text{sdadd}\left(dig, \text{sum}, \text{sdmul}\left(dig, A_{i,j}, X_j\right)\right) \\
X_i \leftarrow \text{sddiv}\left(dig, \text{sdsub}\left(dig, B_i, \text{sum}\right), A_{i,i}\right)
\end{array} \right. \\
X
\end{array} \right.
\end{array}
$$

- Back substitution

Section 4: Results

In this section, the procedure for Naive Gaussian Elimination is called for different numbers of significant digits within the *lower_lim* to *upper_lim* significant digit range. Each row i contains a solution vector [X], where the i^{th} row vector was calculated with i significant digits. Each column j contains the j^{th} element of the solution vector. Compare the values within each column to the corresponding element in the exact solution found below to see how roundoff error effects the solution to the system of simultaneous linear equations.

$k := lower\_lim .. upper\_lim$

$X_k := gauss\_naive(n, AL, RHS, k)$

$k =$	$(X_k)_1 =$	$(X_k)_2 =$	$(X_k)_3 =$	$(X_k)_4 =$
3	11	9.86	-60.8	1.25
4	25.48	27.33	-158.9	0.642
5	30.521	33.387	-193.01	0.427
6	31.164	34.158	-197.357	0.401
7	31.229	34.237	-197.801	0.398
8	31.236	34.245	-197.846	0.398
9	31.237	34.246	-197.85	0.398
10	31.237	34.246	-197.851	0.398

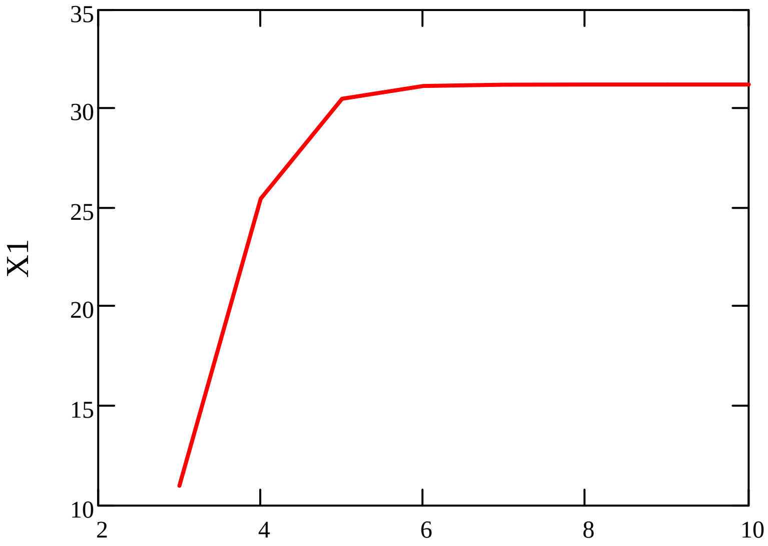
The exact solution to the system of equations is calculated below with Mathcad's built-in tools.

$exactsltn := lsolve(AL, RHS)$

$$exactsltn = \begin{pmatrix} 31.236588037 \\ 34.245614418 \\ -197.850870572 \\ 0.397701741 \end{pmatrix}$$

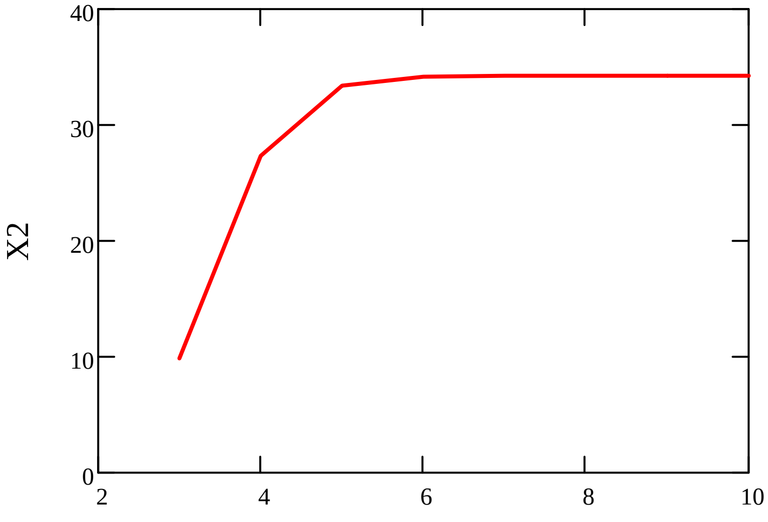
The following graphs demonstrate the effect that the number of significant digits has on the solution. Each element of the solution vector is plotted as a function of the number of significant digits used.

Value of X_1 as a function of the number of significant digits used



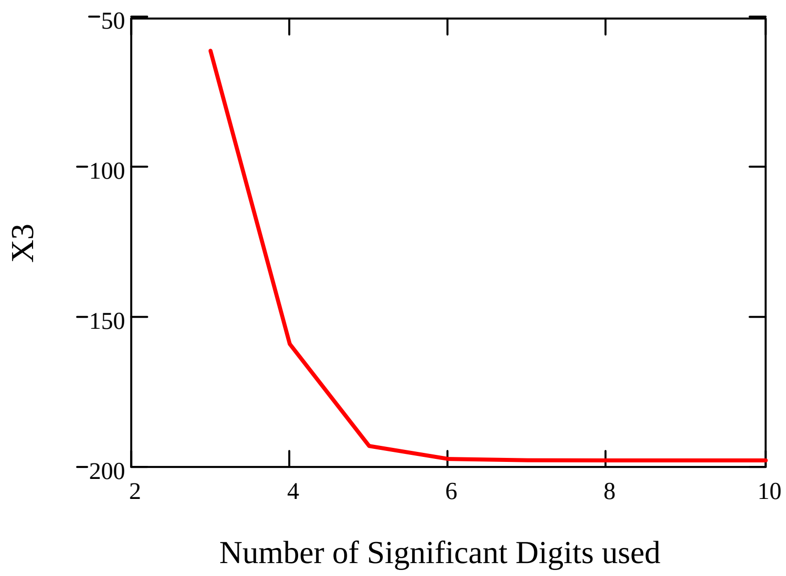
Number of Significant Digits used

Value of X_2 as a function of the number of significant digits used

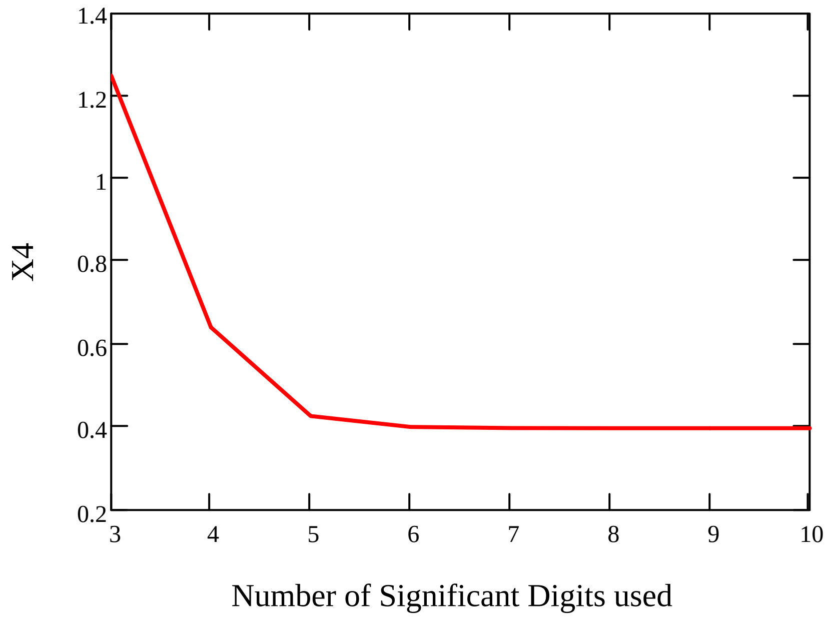


Number of Significant Digits used

Value of X_3 as a function of the number of significant digits used



Value of X_4 as a function of the number of significant digits used



References

Autar Kaw, *Holistic Numerical Methods Institute*, *http://numericalmethods.eng.usf.edu/* , See
[**Introduction to Systems of Equations**]
[**Effect of Significant Digits on Solution of Equations**]
[**How does Gaussian Elimination work?**]

Conclusion

Mathcad helped us apply our knowledge of Naive Gaussian Elimination to study the effect of significant digits on the solution of a set of simultaneous linear equations.

Question 1: Choose a set of equations for which the coefficient matrix is nonsingular. For example

$$\begin{aligned} 5x + 6y + 9z &= 29 \\ 6x + 9y + 2z &= 19 \\ 11x + 9y + 5z &= 30 \end{aligned}$$

See how the number of significant digits makes a difference in the solution vector.

Question 2: Choose a set of equations for which the coefficient matrix is nearly singular. For example

$$\begin{aligned} 5x + 6y + 9z &= 29 \\ 6x + 9y + 2z &= 19 \\ 11x + 15y + 11.001z &= 49.002 \end{aligned}$$

See if the number of significant digits makes a difference in the solution vector.

Question 3: One of the classical problems to show the effect of significant digits on solutions of simultaneous linear equations is with a Hilbert matrix as the coefficient matrix. A matrix $[H]_{nxn}$ is called the n^{th} Hilbert matrix if

$$h_{ij} = \frac{1}{i + j - 1}$$

For example, a 4x4 Hilbert matrix is

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}$$