Topic : Direct Method
Simulation : Graphical Simulation of the Method
Language : Mathcad 2001
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Abstract : This simulation illustrates the direct method of interpolation. Given n data points of y versus x, you are then required to find the value of y at a particular value of x using first, second, and third order interpolation. So one has to first pick the needed data points, and then use those to interpolate the data.

**INPUTS: Enter the following**

**Array of x data:**

$$x := \begin{pmatrix} 10 \\ 0 \\ 20 \\ 15 \\ 30 \\ 22.5 \end{pmatrix}$$

**Array of y data**

$$y := \begin{pmatrix} 227.04 \\ 0 \\ 517.35 \\ 362.78 \\ 901.67 \\ 602.97 \end{pmatrix}$$

**Value of x at which y is desired:**

$$x_{desired} := 16$$

**SOLUTION**

**This function considers the x and y data and selects the two points closest data points that bracket the desired value of x.**

```
firsttwo := | n ← rows(x)
            | comp ← vectorize(|x − x_desired|)
            | c ← min(comp)
            | for i ∈ 0..n − 1
            |    ci ← i  if  comp_i = c
            | if x_ci < x_desired
```

```
        q ← 0
        for i ∈ 0..n − 1
          if x_i > x_desired
            next_q ← x_i
            q ← q + 1
        b ← min(next)
        for i ∈ 0..n − 1
          bi ← i  if x_i = b
    if x_ci > x_desired
        q ← 0
        for i ∈ 0..n − 1
          if x_i < x_desired
            next_q ← x_i
            q ← q + 1
        b ← max(next)
        for i ∈ 0..n − 1
          bi ← i  if x_i = b
    (ci)
    (bi)
```

bi := firsttwo

**If more than two values are desired, the following function selects the subsequent values and puts all the values into a matrix, maintaining the original data order.**

```
selectxy(num) :=  n ← rows(x)
                  comp ← vectorize(|x − x_desired|)
                  for i ∈ 0..n − 1
                    A_{i,1} ← i
                    A_{i,0} ← comp_i
                  A ← csort(A,0)
                  for i ∈ 0..n − 1
                    A_{i,2} ← i
                  A ← csort(A,1)
                  d ← A^⟨2⟩
```

$$\text{if } d_{(bi_1)} \neq 1$$

$$\quad temp \leftarrow d_{(bi_1)}$$

$$\quad d_{(bi_1)} \leftarrow 1$$

$$\quad \text{for } i \in 0..n-1$$

$$\quad\quad d_i \leftarrow d_i + 1 \;\; \text{if } i \neq bi_0 \wedge i \neq bi_1 \wedge d_i \leq temp$$

$$xnew \leftarrow 0$$

$$ynew \leftarrow 0$$

$$\text{for } i \in 0..n-1$$

$$\quad xnew \leftarrow \text{stack}(xnew, x_i) \;\; \text{if } d_i \leq num - 1$$

$$\quad ynew \leftarrow \text{stack}(ynew, y_i) \;\; \text{if } d_i \leq num - 1$$

$$ynew \leftarrow \text{submatrix}(ynew, 1, num, 0, 0)$$

$$xnew \leftarrow \text{submatrix}(xnew, 1, num, 0, 0)$$

$$new \leftarrow \text{augment}(xnew, ynew)$$

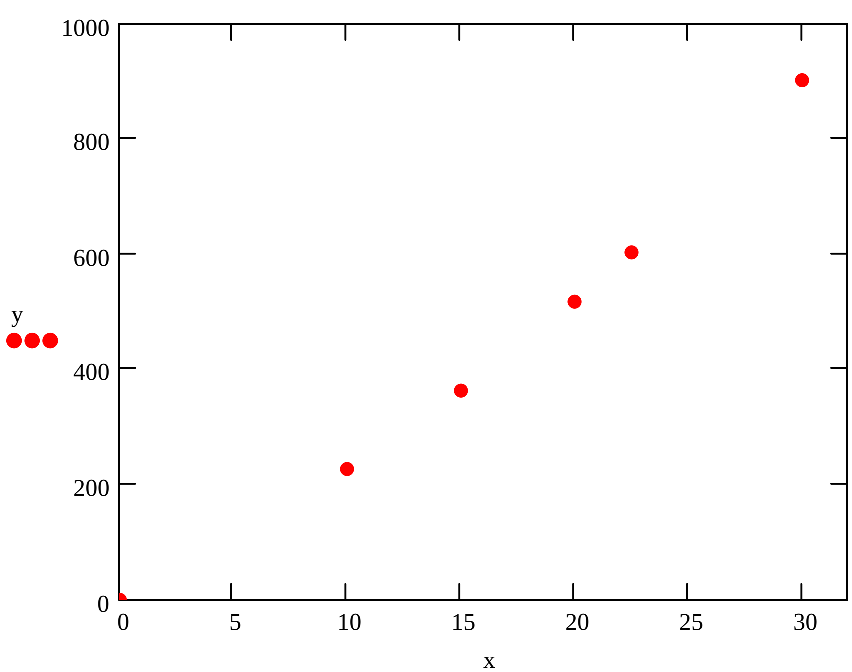
$$new$$

**These two functions use the above functions to assign the selected data to new variables.**

$$x_{sub}(n) := \text{submatrix}(\text{selectxy}(n), 0, \text{rows}(\text{selectxy}(n)) - 1, 0, 0)$$

$$y_{sub}(n) := \text{submatrix}(\text{selectxy}(n), 0, \text{rows}(\text{selectxy}(n)) - 1, 1, 1)$$

**Given y versus x data points**



**Linear Interpolation:**

**Pick two data points**

$x_s := x_{sub}(2)$

$y_s := y_{sub}(2)$

$$x_s = \begin{pmatrix} 20 \\ 15 \end{pmatrix}$$

$$y_s = \begin{pmatrix} 517.35 \\ 362.78 \end{pmatrix}$$

**Setting up equations to find coefficients of the linear interpolant**

$$\begin{pmatrix} 1 & x_{s_0} \\ 1 & x_{s_1} \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} y_{s_0} \\ y_{s_1} \end{pmatrix}$$

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} := \begin{pmatrix} 1 & x_{s_0} \\ 1 & x_{s_1} \end{pmatrix}^{-1} \cdot \begin{pmatrix} y_{s_0} \\ y_{s_1} \end{pmatrix}$$

**Coefficient of linear interpolant**

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} -100.93 \\ 30.914 \end{pmatrix}$$

**Linear interpolant**

$f(x) := a_0 + a_1 \cdot x$

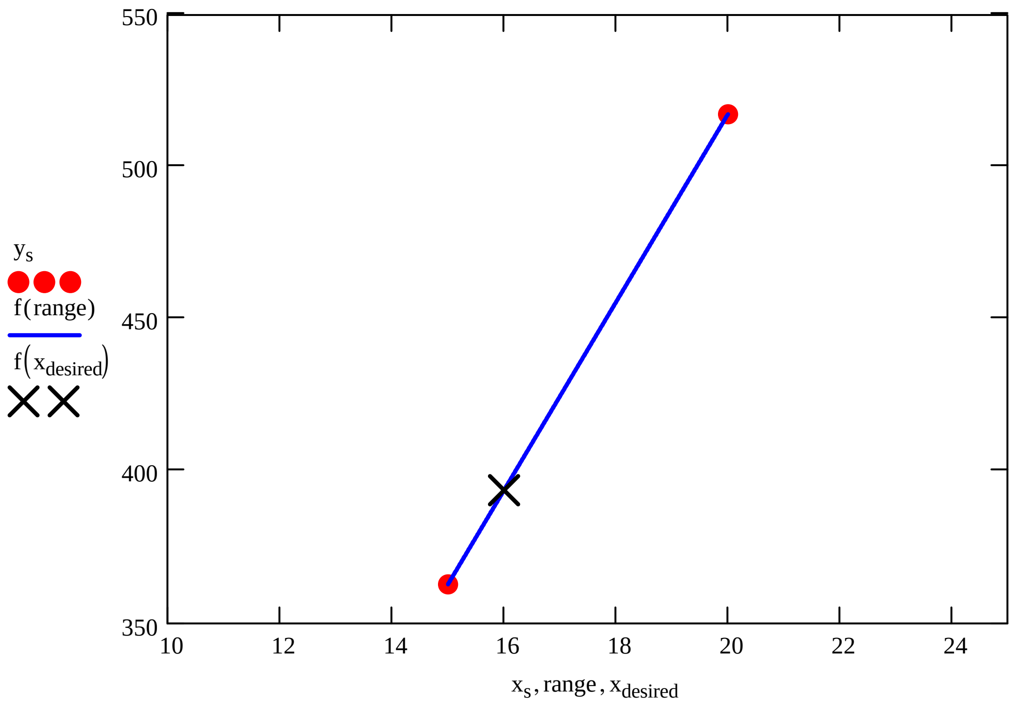
$$range := \min(x_s), \min(x_s) + \frac{\max(x_s) - \min(x_s)}{1000} .. \max(x_s)$$

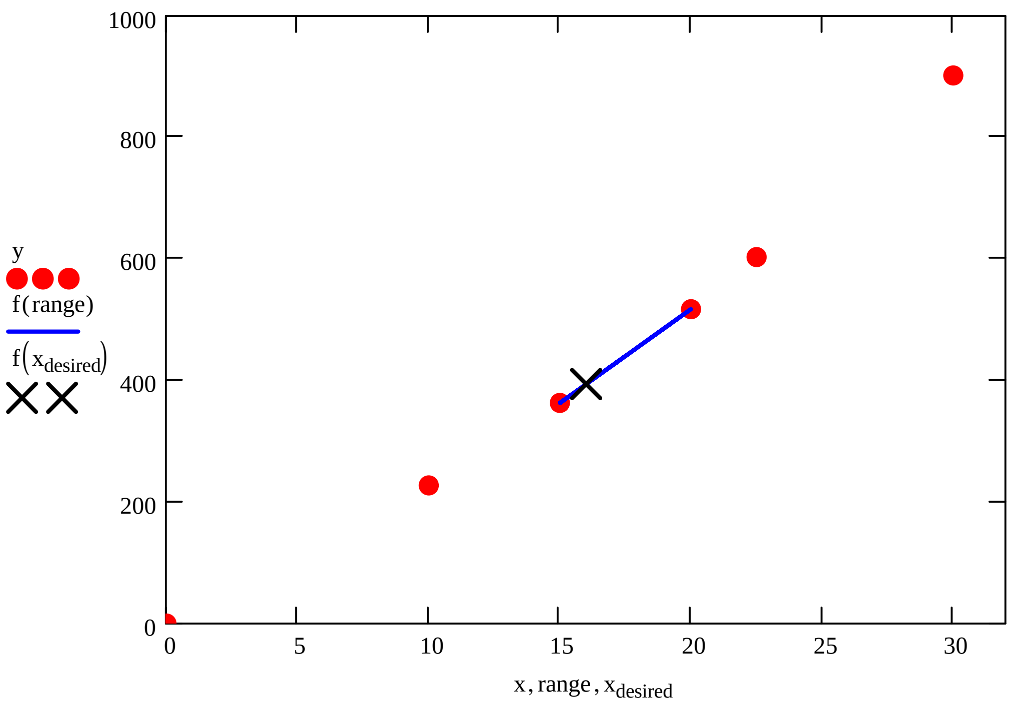
**Value of function at desired point**

$f(x_{desired}) = 393.694$

$results_{0,0} := f(x_{desired})$

**Linear Interpolation**



**Quadratic Interpolation (Second order polynomial):**

**Pick three data points**

$x_s := x_{sub}(3)$

$y_s := y_{sub}(3)$

$$x_s = \begin{pmatrix} 10 \\ 20 \\ 15 \end{pmatrix}$$

$$y_s = \begin{pmatrix} 227.04 \\ 517.35 \\ 362.78 \end{pmatrix}$$

**Setting up equations to find coefficients of the quadratic interpolant.**

$$\begin{bmatrix} 1 & x_{s_0} & \left(x_{s_0}\right)^2 \\ 1 & x_{s_1} & \left(x_{s_1}\right)^2 \\ 1 & x_{s_2} & \left(x_{s_2}\right)^2 \end{bmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_{s_0} \\ y_{s_1} \\ y_{s_2} \end{pmatrix}$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} := \begin{bmatrix} 1 & x_{s_0} & \left(x_{s_0}\right)^2 \\ 1 & x_{s_1} & \left(x_{s_1}\right)^2 \\ 1 & x_{s_2} & \left(x_{s_2}\right)^2 \end{bmatrix}^{-1} \cdot \begin{pmatrix} y_{s_0} \\ y_{s_1} \\ y_{s_2} \end{pmatrix}$$

**Coefficient of linear interpolant**

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 12.05 \\ 17.733 \\ 0.3766 \end{pmatrix}$$

$f_{prev} := f\left(x_{desired}\right)$

$f(x) := a_0 + a_1 \cdot x + a_2 \cdot x^2$

**Value of function at desired point**

$f(x_{desired}) = 392.1876$

$results_{0,1} := f(x_{desired})$

$f_{new} := f(x_{desired})$

**Absolute percentage relative approximate error**

$$\varepsilon_a := \left| \frac{f_{new} - f_{prev}}{f_{new}} \right| \cdot 100$$

$\varepsilon_a = 0.3841$

$results_{1,1} := \varepsilon_a$

**Number of significant digits at least correct in the solution**

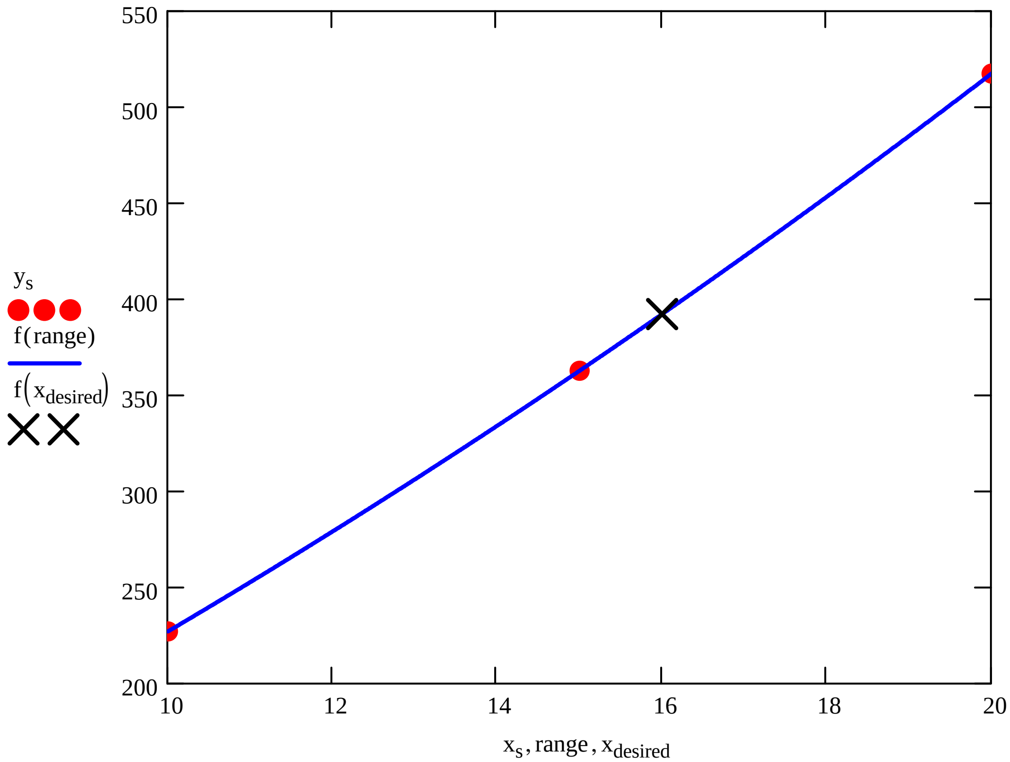
$$sigdigits := \begin{cases} 0 & \text{if } \left|\varepsilon_a\right| \le 0 \\ trunc\left(\left(2 - \log\left(\left|\frac{\left|\varepsilon_a\right|}{0.5}\right|\right)\right)\right) & \text{otherwise} \end{cases}$$

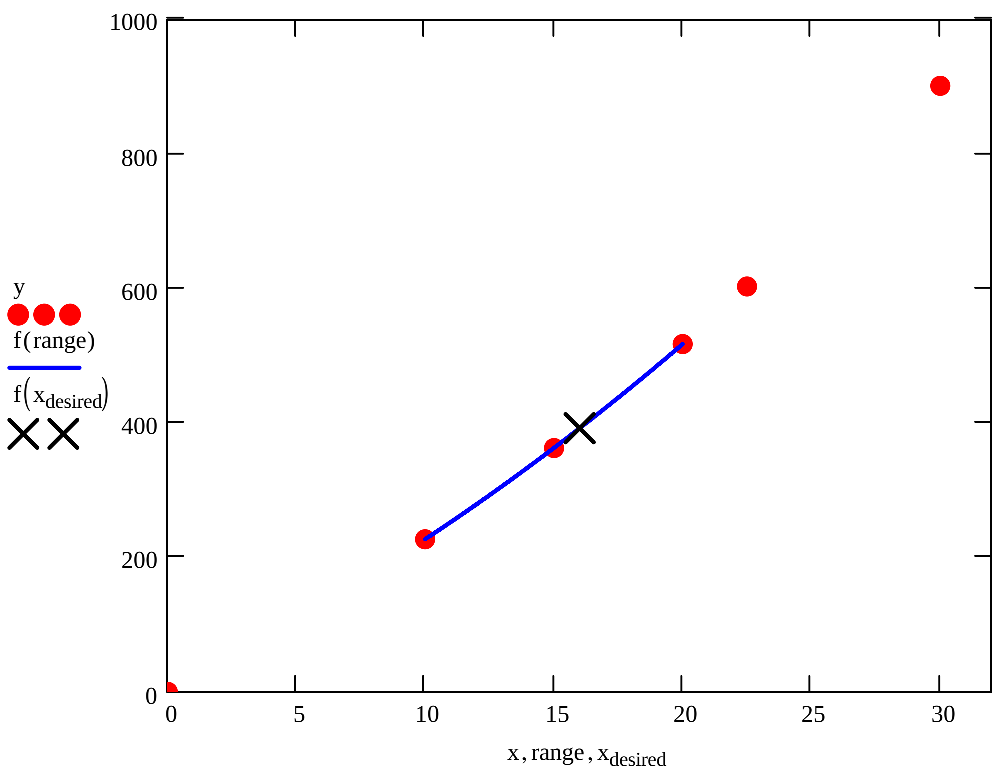
$sigdigits = 2$

$results_{2,1} := sigdigits$

$$range := \min(x_s), \min(x_s) + \frac{\max(x_s) - \min(x_s)}{1000} .. \max(x_s)$$

**Quadratic polynomial interpolation**



**Cubic Interpolation (Third order polynomial interpolation)**

**Choose four data points.**

$$x_s := x_{sub}(4)$$

$$y_s := y_{sub}(4)$$

$$x_s = \begin{pmatrix} 10 \\ 20 \\ 15 \\ 22.5 \end{pmatrix}$$

$$y_s = \begin{pmatrix} 227.04 \\ 517.35 \\ 362.78 \\ 602.97 \end{pmatrix}$$

**Setting up equations to find the coefficients of the third order polynomial**

$$\begin{bmatrix} 1 & x_{s_0} & \left(x_{s_0}\right)^2 & \left(x_{s_0}\right)^3 \\ 1 & x_{s_1} & \left(x_{s_1}\right)^2 & \left(x_{s_1}\right)^3 \\ 1 & x_{s_2} & \left(x_{s_2}\right)^2 & \left(x_{s_2}\right)^3 \\ 1 & x_{s_3} & \left(x_{s_3}\right)^2 & \left(x_{s_3}\right)^3 \end{bmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} y_{s_0} \\ y_{s_1} \\ y_{s_2} \\ y_{s_3} \end{pmatrix}$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} := \begin{bmatrix} 1 & x_{s_0} & \left(x_{s_0}\right)^2 & \left(x_{s_0}\right)^3 \\ 1 & x_{s_1} & \left(x_{s_1}\right)^2 & \left(x_{s_1}\right)^3 \\ 1 & x_{s_2} & \left(x_{s_2}\right)^2 & \left(x_{s_2}\right)^3 \\ 1 & x_{s_3} & \left(x_{s_3}\right)^2 & \left(x_{s_3}\right)^3 \end{bmatrix}^{-1} \cdot \begin{pmatrix} y_{s_0} \\ y_{s_1} \\ y_{s_2} \\ y_{s_3} \end{pmatrix}$$

**Coefficients of the third order polynomial**

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -4.254 \\ 21.26553 \\ 0.13204 \\ 5.43467 \times 10^{-3} \end{pmatrix}$$

$f_{prev} := f_{new}$

$f(x) := a_0 + a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3$

**Value of function at denied point**

$f(x_{desired}) = 392.05717$

$f_{new} := f(x_{desired})$

$results_{0,2} := f(x_{desired})$

**Absolute relative approximate error**

$$\varepsilon_a := \left| \frac{f_{new} - f_{prev}}{f_{new}} \right| \cdot 100$$

$\varepsilon_a = 0.03327$

$results_{1,2} := \varepsilon_a$

**Number of significant digits at least correct in the solution**

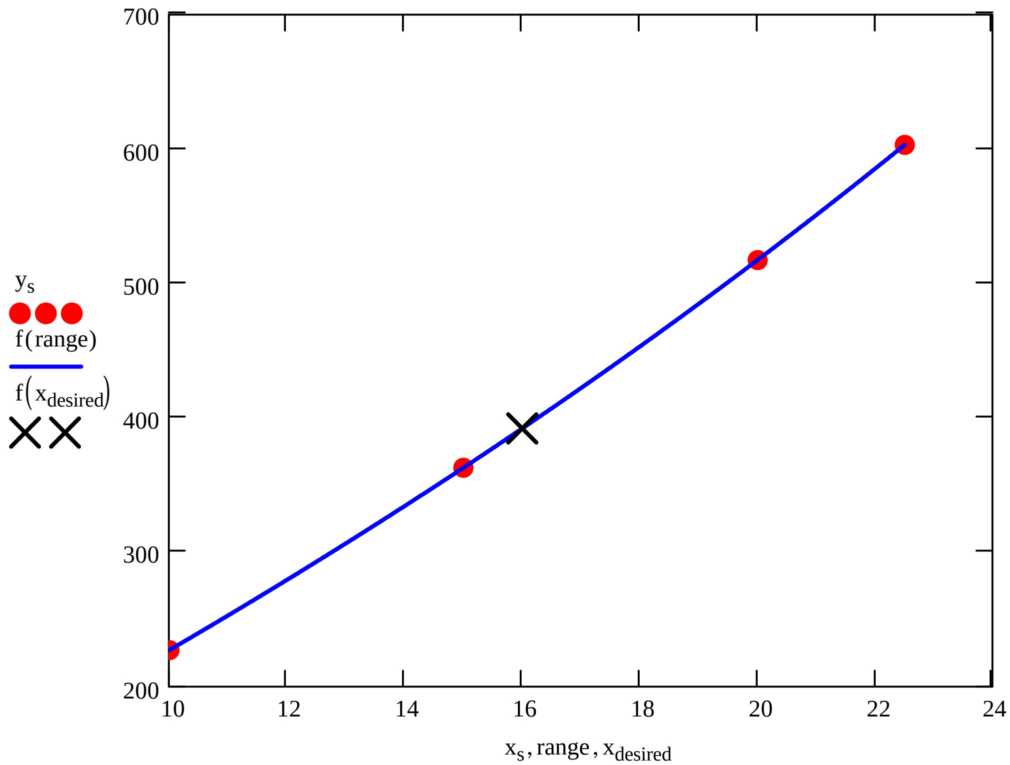
$$sigdigits := \begin{cases} 0 & \text{if } |\varepsilon_a| \le 0 \\ \operatorname{trunc}\left(\left(2 - \log\left(\left|\frac{|\varepsilon_a|}{0.5}\right|\right)\right)\right) & \text{otherwise} \end{cases}$$

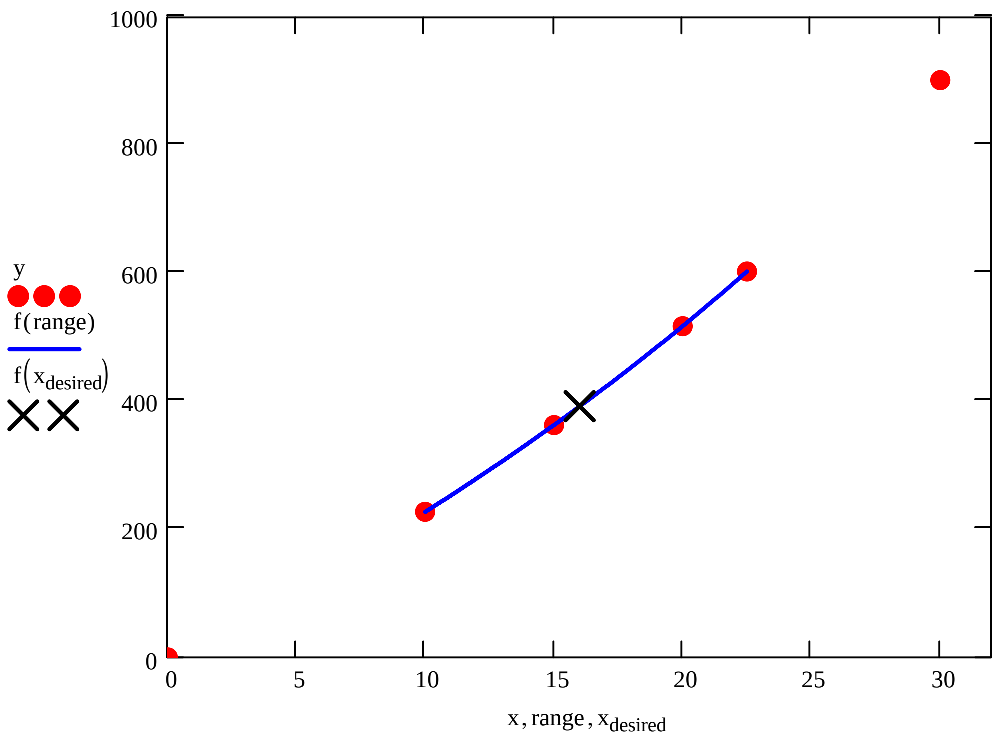
$sigdigits = 3$

$results_{2,2} := sigdigits$

$$range := \min(x_s), \min(x_s) + \frac{\max(x_s) - \min(x_s)}{1000} .. \max(x_s)$$

**Cubic interpolation**



## Summary of Direct Interpolation

results =

| | First Order | Second Order | Third Order | |
|---|---|---|---|---|
| | 393.694 | 392.1876 | 392.05717 | **Interpolated Value** |
| | 0 | 0.3841 | 0.03327 | **Absolute relative approximate error** |
| | 0 | 2 | 3 | **Number of significant digits** |